Solution Hints for Spring 95 Exam

1. The idea is to consider $F(t) = t(f(t) - g(t))$. Then for all $n = 0, 1, 2, \ldots$, (including $n = 0$)

$$\int_0^1 F(t)t^n \, dt = 0.$$

Then one shows, in the standard way that $F(t) = 0$ as follows: By separating F into real and imaginary parts it suffices to assume F is real valued. Fix $\epsilon > 0$ and let $P(t)$ be a polynomial such that $\sup_{[0,1]}|F(t) - P(t)| < \epsilon/(1 + \int_0^1 |F(x)| \, dx)$. Then

$$\begin{aligned} \int_0^1 F(t)^2 \, dt &= \int_0^1 |F(t) - P(t)| \cdot |F(t)| \, dt + \int_0^1 P(t)F(t) \, dt \\ &\leq \epsilon + 0. \end{aligned}$$

(The last integral is 0 because the integral of F against any power of t (including the 0^{th}) is 0.) Since f, g are continuous at 0, we must have $f(t) - g(t) = 0$ on $[0, 1]$.

2. a. To show $\log(1 + x) \leq x$, let $f(x) = \log(1 + x) - x$. Then $f'(x) = \frac{-x}{1+x}$ has only one zero on $[-1/2, \infty)$ at which f has a global maximum value of 0. So $\log(1 + x) \leq x$ on $[-1/2, \infty)$. Similarly to show $x - 2x^2 \leq \log(1 + x)$, let $g(x) = x - 2x^2 - \log(1 + x)$. Then $g'(x) = \frac{-x(3+4x)}{1+x}$ has only one zero on $[-1/2, \infty)$ at which g has a global maximum value of 0.

b. Since $\sum_1^\infty a_n$ converges, we can choose N such that $n \geq N$ implies $|a_n| \leq 1/2$. For $n \geq N$ we can use the estimates in part a to find

$$a_n - 2a_n^2 \leq \log(1 + a_n) \leq a_n.$$

Noting that $a_n \leq 1/2$ we have $a_n^2 \leq |a_n|$, then $a_n - a_n^3 \geq -|a_n| - 2|a_n|$ and so

$$-3|a_n| \leq \log(1 + a_n) \leq |a_n| \leq 3|a_n|,$$

i.e., $|\log(1 + a_n)| \leq 3|a_n|$. Thus the series $\sum_{n=1}^\infty \log(1 + a_n)$ converges (absolutely) by comparison, and

$$\lim_{M \to \infty} \prod_{n=1}^{M} (1 + a_n) = e^{\lim_{M\to\infty} \sum_{n=1}^\infty \log(1+a_n)}$$

exists and is not zero (because the sum converges to some finite number).

3. Suppose not. Since $|f|$ is a continuous function on a compact space there is an $x_1 \in X$ such that $|f(x_1)| = c > 0$ is the minimum value of f. According to the statement of the problem we can choose $x_o \in X$ such that $|f(x_o)| \leq \frac{1}{2}|f(x_1)| = \frac{c}{2}$. This contradicts the assumption that c is the minimum value of f.

4. a. Suppose $\lim\inf_{n\to\infty} na_n = \epsilon > 0$. Then there is an N such that $n > N$ implies $na_n \geq \epsilon/2$ (if not, then there are infinitely many n_k such that $n_k a_{n_k} < \epsilon/2$ and we could find a subsequence converging to a number less than ϵ. C.f. Rudin, Principles of Mathematical Analysis, 3.17 b.) Then $a_n \geq \epsilon/2n$ for $n > N$, contradicting the fact that the sum of the a_n converges.

b. For example, let $a_n = 1/n^2$ if n is not a perfect square, and $a_n = 1/k^2$ if $n = k^2$. Then $\lim\sup na_n = 1$ and the sum of the a_n still converges. (See Kaczor and Nowak, "Problems i Mathematical Analysis I," Problem 3.2.36)

c. (See ibid, Problem 3.2.35). We have (by the monotonicity of the a_n's)

$$\sum_{k=n+1}^{2n} a_k \geq na_{2n} = \frac{1}{2}(2na_{2n})$$

$$\sum_{k=n+1}^{2n+1} a_k \geq (n + 1)a_{2n+1} = \frac{n + 1}{2n + 1}(2n + 1)a_{2n+1}.$$

Since both sums go to zero, $\lim_{n\to\infty} 2na_{2n} = \lim_{n\to\infty}(2n+1)a_{2n+1} = 0$. It follows that $\lim_{n\to\infty} na_n = 0$.

5. The idea is to "evaluate" or "estimate" $f(\frac{T}{2})$ in terms of f, f', and f'' in two different ways. For the first way use Taylor's Theorem at 0: There exists a $c_1 \in (0, \frac{T}{2})$ such that

$$f\left(\frac{T}{2}\right) = f(0) + f'(0)\frac{T}{2} + \frac{f''(c_1)}{2}\left(\frac{T}{2}\right)^2.$$

The second way to evaluate using Taylor's Theorem at T: there exists a $c_2 \in (\frac{T}{2}, T)$ such that

$$f\left(\frac{T}{2}\right) = f(T) + f'(T)\frac{T}{2} + \frac{f''(c_2)}{2}\left(\frac{T}{2}\right)^2.$$

Subtracting gives

$$0 = f(T) - f(0) + \left(\frac{f''(c_2) - f''(c_1)}{2}\right)\frac{T^2}{4}$$

Estimating $|f''(c_2) - f''(c_1)| \leq |f''(c_2)| + |f''(c_1)| \leq 2$ gives

$$|f(T) - f(0)| \leq \frac{T^2}{4}.$$

Using $f(T) - f(0) = 1$ gives the desired result. (Well, almost—it shows $T \geq 2$, not $T > 2$. I don't know how to rule out the possibility that T could be equal to 2 unless the hypothesis is changed to $|f''(x)| < 1$).

(Note: as one of our colleagues pointed out in class by example, we can only prove $T \geq 2$ under the given hypothesis. If we assume f'' is continuous then we can get $T > 2$).

6. a. By the mean value theorem we have $g(\frac{x}{n}) - g(0) = g'(c)\frac{x}{n}$ for some $c \in (0, \frac{x}{n})$. Since $g(0) = 0$ and $|g'(c)| \leq M$ for all possible values of c, we have $|g(\frac{x}{n})| \leq M\frac{|x|}{n}$. Thus the series

$$\sum_{n=1}^{\infty} \frac{1}{n} g\left(\frac{x}{n}\right)$$

converges uniformly on any bounded interval by comparison with $\sum \frac{1}{n^2}$. If a series of continuous functions converges uniformly on an interval, then the sum is continuous on that interval.

b. Differentiating each term individually gives the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} g'\left(\frac{x}{n}\right). \qquad (*)$$

Since $|g'(\frac{x}{n})|$ is bounded (independent of n or x), this series converges uniformly (on all of $\mathbf{R}$). Therefore the original series is differentiable with $(*)$ as its derivative.