Solution Hints for Spring 97 Exam

Revised 7/03

1. See Kaczor and Nowak, V.I, problem 3.2.2e. Here's a very slightly different solution. The idea is that $(\log n)^{\log\log n} < n$ for n sufficiently large, so the series diverges by comparison with the harmonic series. Using L'Hôpital's rule we can show that $\lim_{x\to\infty} \frac{\log x}{\sqrt{x}} = 0$. So there exists an $M > 0$ such that for all $x > M$, we have $0 < \log x < \sqrt{x}$ (or one can use calculus to show the same thing). Then $(\log x)^2 < x$ (again you could show this directly using calculus), and so for $n > 10^M$, $(\log\log n)^2 < \log n$. Since the exponential function is increasing, $10^{(\log\log n)^2} = (\log n)^{\log\log n} < n$. Then for $n > 10^M$, $\frac{1}{(\log n)^{\log\log n}} > \frac{1}{n}$, and the series diverges by comparison with the harmonic series $\sum_{10^M}^{\infty} \frac{1}{n}$.

2. (It's well known that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$, so if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exits, then clearly $\lim_{n\to\infty} \sqrt[n]{a_n}$ does also. But I think this problem is asking for a direct proof.)

   First assume $0 < L < \infty$ and fix $\epsilon > 0$ small enough that $L - \epsilon > 0$ but otherwise arbitrary. Choose M such that $n > M$ implies $|\frac{a_{n+1}}{a_n} - L| < \epsilon$. Then $(L-\epsilon)a_M < a_{M+1} < (L+\epsilon)a_M$ and, inductively, for $n > M$ we have
$$0 < (L-\epsilon)^{n-M} a_M < a_n < (L+\epsilon)^{n-M} a_M$$
   and so, for all $n > M$,
$$(L-\epsilon)^{1-M/n}(a_M)^{1/n} < (a_n)^{1/n} < (L+\epsilon)^{1-M/n}(a_M)^{1/n}.$$
   Since $\lim_{n\to\infty}(L-\epsilon)^{1-M/n}(a_M)^{1/n} = L - \epsilon$ and $\lim_{n\to\infty}(L+\epsilon)^{1-M/n}(a_M)^{1/n} = L + \epsilon$, we have
$$L - \epsilon \le \liminf_{n\to\infty}(a_n)^{1/n} \le \limsup_{n\to\infty}(a_n)^{1/n} \le L + \epsilon.$$
   Since ϵ can be arbitrarily small, we must have $\liminf_{n\to\infty}(a_n)^{1/n} = \limsup_{n\to\infty}(a_n)^{1/n} = L$, so $\lim_{n\to\infty}(a_n)^{1/n}$ exists and equals L.

   If $L = 0$, then since all $a_n > 0$ we can repeat the above argument with $L - \epsilon$ replaced by 0. If $L = \infty$ (it's not clear in the statement of the problem whether L is supposed to be a real number or an "extended" real number), then the same type of argument works—For any $C > 0$ there exists an M such that $n > M$ implies $M < \frac{a_{n+1}}{a_n}$. Then one finds as above that for $n > M$, $C^{1-M/n} \le (a_n)^{1/n}$. Then $C < \liminf_{n\to\infty}(a_n)^{1/n}$ and since $C > 0$ was arbitrary, $\liminf_{n\to\infty}(a_n)^{1/n} = \infty$. So $\lim_{n\to\infty}(a_n)^{1/n} = \infty$.

   Consider the sequence $a_n = \frac{1}{n^2}$ if n is not a perfect square, and $a_n = \frac{1}{n}$ if $n = k^2$ for some positive integer k. By taking a subsequence where n is a perfect square and $n+1$ is not, we see $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. By taking a subsequence where $n+1$ is a perfect square and n is not, we see $\limsup_{n\to\infty} a_n = \infty$. So $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist. On the other hand, $\lim_{n\to\infty}(\frac{1}{n})^{1/n} = \lim_{n\to\infty}(\frac{1}{n^2})^{1/n} = 1$, so $\lim_{n\to\infty}(a_n)^{1/n} = 1$.

3. Since f/t is continuous on on $\{1/2 \le |t| \le 1\}$, it suffices to show the integral over $\{\epsilon < |t| \le \frac{1}{2}\}$ converges as $\epsilon \to 0$. Define a function h on by
$$h(t) = \begin{cases} \dfrac{f(t) - f(0)}{t} & \text{if } t \ne 0 \\ f'(0) & \text{if } t = 0. \end{cases}$$
   Since f is differentiable at 0, h is continuous on $\{|t| \le \frac{1}{2}\}$. Then
$$\int_{\epsilon<|t|\le\frac{1}{2}} \frac{f(t)}{t}\,dt = \int_{\epsilon<|t|\le\frac{1}{2}} h(t)\,dt + \int_{\epsilon<|t|\le\frac{1}{2}} \frac{f(0)}{t}\,dt.$$

Since h is continuous on $[-1/2, 1/2]$ the first integral on the right converges as $\epsilon \to 0$. The second integral on the right is identically 0 for all $\epsilon > 0$.

4. The lower Riemann sum for the function $f(x) = \frac{1}{(1+x)^2}$ corresponding to the partition of $[0,1]$ into n intervals of equal length is

$$\sum_{k=1}^{n} \frac{1}{(1+\frac{k}{n})^2} \cdot \frac{1}{n} = n \sum_{k=1}^{n} \frac{1}{(n+k)^2} = n \sum_{k=n+1}^{2n} \frac{1}{k^2}.$$

Thus

$$\lim_{n\to\infty} n \sum_{k=n+1}^{2n} \frac{1}{k^2} = \int_0^1 \frac{1}{(x+1)^2}\, dx = \frac{1}{2}.$$

5. Hölder's inequality says that if g, h are measurable on $[a,b]$ and p, $q \in (1,\infty)$ are positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \int_a^b gh \right| \le \left(\int_a^b |g|^p \right)^{1/p} \left(\int_a^b |h|^q \right)^{1/q}.$$

If $f : [0,1] \to \mathbf{C}$ satisfies $\int_0^1 |f|^8 < \infty$, then applying Hölder's inequality with $[a,b] = [0,1]$, $g = |f|^8$, $h = 1$, $p = 8/3$ and $q = 8/5$ gives

$$\int_0^1 |f|^3 \le \left(\int_0^1 |f|^8 \right)^{3/8} \left(\int_0^1 1 \right)^{5/8} < \infty.$$

For an example of a function $f : \mathbf{R} \to \mathbf{C}$ with $\int_{\mathbf{R}} |f|^8 < \infty$ and $\int_{\mathbf{R}} |f|^3 = \infty$, consider $f(x) = 1/(|x|+1)^{1/3}$. Then $|f(x)|^8 < 1/(|x|+1)^2$, so $\int_{\mathbf{R}} |f|^8 < \infty$, but $\int_{\mathbf{R}} |f(x)|^3 > \int_0^\infty \frac{1}{x+1} = \infty$.

6. It suffices to show that $f_n(x) = \frac{x^2+3x^{2n+4}}{1+3x^{2n}}$ converges uniformly to a function f on the sets $A_1 = \{|x| \le 1\}$, $A_2 = \{1 \le |x| \le 2\}$, $A_3 = \{|x| \ge 2\}$ separately, for then given $\epsilon > 0$ we have M_1, M_2, M_3 such that $n \ge M_i$ implies that for all $x \in A_i$, we have $|f_n(x) - f(x)| \le \epsilon$, and letting $M = \max(M_1, M_2, M_3)$ we have for all $x \in \mathbf{R}$ that $n \ge M$ implies $|f_n(x) - f(x)| \le \epsilon$.

The main tool will be Dini's Theorem: If $\{f_n\}_{n=1}^\infty$ is a sequence of continuous, real valued functions on a compact metric space A and $f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots$ for all $x \in A$ and f_n converges pointwise on A to a continuous function f, then f_n converges uniformly on A to f. The same conclusion is true if the inequalities are all reversed. The f_n don't have to be non-negative.

By long division we have $f_n(x) = x^4 + \frac{x^2-x^4}{1+3x^{2n}}$. Let $f(x) = \begin{cases} x^2, & |x| < 1 \\ x^4, & |x| \ge 1 \end{cases}$. f is a continuous function on $\mathbf{R}$. We will show f_n converges uniformly to f on each of the sets A_i above. On the compact set A_1 we have $f_n(1) = 1 = f(1)$ for all n, and for $|x| < 1$ we have $\lim_{n\to\infty} x^{2n} = 0$ so

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^4 + \frac{x^2 - x^4}{1+3x^{2n}} = x^2.$$

So f_n converges pointwise on A_1 to f. Since $|x| \le 1$ we have $x^{2n} \ge x^{2(n+1)}$ and $x^2 - x^4 \ge 0$ so

$$f_n(x) = x^4 + \frac{x^2 - x^4}{1+3x^{2n}} \le x^4 + \frac{x^2 - x^4}{1+3x^{2(n+1)}} = f_{n+1}(x).$$

By Dini's Theorem f_n converges uniformly go f on A_1.

On the compact set A_2 we have $f_n(1) = f(1)$ for all n, and for $|x| > 1$ we have $\lim_{n\to\infty} x^{2n} = \infty$ so for $x \in A_2$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^4 + \frac{x^2 - x^4}{1+3x^{2n}} = x^4.$$

Thus f_n converges pointwise to f on A_2. On A_2 we have $|x| \geq 1$ so $x^{2n} \leq x^{2(n+1)}$ and $x^2 - x^4 \leq 0$ so

$$f_n(x) = x^4 + \frac{x^2 - x^4}{1 + 3x^{2n}} \leq x^4 + \frac{x^2 - x^4}{1 + 3x^{2(n+1)}} = f_{n+1}(x).$$

By Dini's Theorem f_n coverges uniformly to f on A_2.

On the non-compact set A_3 we can't use Dini's Theorem. We show directly that f_n converges uniformly to f. For $|x| \geq 2$ we have $x^2 < x^4$ and

$$|f_n(x) - f(x)| = \left|\frac{x^2 - x^4}{1 + 3x^{2n}}\right| \leq \left|\frac{x^2 - x^4}{3x^{2n}}\right| \leq \frac{2x^4}{3x^{2n}} \leq \frac{1}{3 \cdot 2^{2n-4}}$$

for all $n > 2$. Given $\epsilon > 0$ we can choose N such that for all $n > N$ implies $\frac{1}{3 \cdot 2^{2n-4}} \leq \epsilon$. This shows f_n converges to f uniformly on A_3.