Math 787.03 Mock Exam 1, Summer, 2003

July 22, 2003

Four complete solutions are sufficient to pass. Please use only one side of every sheet, and use distinct sheets for distinct problems.

1. If a, b, c are positive real numbers and $a + b + c = 1$, then show that
$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8.$$

2. Let $f_n \colon \mathbf{R} \to \mathbf{R}$ be differentiable for each $n = 1, 2, \ldots$ with $|f_n'(x)| \leq 1$ for all n and all x. Assume
$$\lim_{n\to\infty} f_n(x) = g(x)$$
for all x. Prove that $g \colon \mathbf{R} \to \mathbf{R}$ is continuous.

3. Define the function ζ by
$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$
Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders on the interval $1 < x < \infty$.

4. Let
$$f(x) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt$$
for $x > 0$.
   (a) Show that $0 < f(x) < 1/x$.
   (b) Show that $f(x)$ is strictly decreasing for $x > 0$.

5. Prove that any bounded sequence of real numbers has a convergent subsequence.

6. A sequence of functions defined on a set A is said to be *equicontinuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, $y \in A$ and all positive integers n, if $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$. Prove that if $\{f_n\}$ is a uniformly convergent sequence of continuous functions on a compact set K, then $\{f_n\}$ is equicontinuous on K.