Math 787.03 Mock Exam 2, Summer, 2003

August 12, 2003

Four complete solutions are sufficient to pass. Please use only one side of every sheet, and use distinct sheets for distinct problems.

1. Find the smallest constant $C > 0$ such that
$$(a+b+c+d+e)^2 \leq C(a^2+b^2+c^2+d^2+e^2)$$
for every $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, and $e \geq 0$.

2. Assume that $f \in C^\infty([0,1])$ satisfies the following conditions:
   (a) f is not identically 0
   (b) $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$
   (c) for a sequence of real numbers a_n, the series $\sum_{n=1}^\infty a_n f^{(n)}(x)$ converges uniformly on $[0,1]$.

   Prove that $\lim_{n\to\infty} n! a_n = 0$.

3. Let $a > 0$ and f be continuously differentiable in $[0,a]$. Show that
$$|f(0)| \leq \frac{1}{a}\int_0^a |f(x)|\,dx + \int_0^a |f'(x)|\,dx.$$

4. Prove that if $\sum_{n=1}^\infty a_n$ is a convergent series of real numbers and the sequence of real numbers $\{b_n\}$ is monotonic and bounded, then the series $\sum_{n=1}^\infty a_n b_n$converges.

5. Suppose that f is a real valued function of one real variable such that $\lim_{x\to c} f(x)$ exists for all $c \in [a,b]$. Show that f is Riemann integrable on $[a,b]$.

6. Show that for any continuous function $f : [0,1] \to \mathbf{R}$ and any $\epsilon > 0$, there is a function of the form
$$g(x) = \sum_{k=0}^n C_k x^{4k}$$
for some natural number n, where $C_0, C_1, \ldots, C_n$ are rational numbers and $|g(x) - f(x)| < \epsilon$ for all $x \in [0,1]$.