Math 787.03 Mock Exam 3, Summer, 2003

August 27, 2003

1. Find the radius of convergence of the series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n^2}}{(n!)^n}$

Solution. (From the Autumn 93 qualifying exam.) First note that by comparing the sum with the integral we find

$$n(\ln(n) - 1) + 1 \leq \sum_{k=1}^{n} \ln k \leq (n+1)(\ln(n+1) - 1) + 1.$$

a. We have

$$n \ln|x| - (1+\frac{1}{n})(\ln(n+1)-1) - \frac{1}{n} \leq n \ln|x| - \frac{1}{n} \sum_{k=1}^{n} \ln k \leq n \ln|x| - \ln(n) + 1 - \frac{1}{n}.$$

If $|x| \leq 1$, then

$$\lim_{n\to\infty} \left(n \ln|x| - \frac{1}{n} \sum_{k=1}^{n} \ln k \right) = -\infty$$

and the sum converges. If $|x| > 1$, then the limit is infinity and the sum diverges. So the radius of convergence is 1.

b. Similarly we have

$$n \ln|x| - (n+1)(\ln(n+1)-1) - 1 \leq n \ln|x| - \sum_{k=1}^{n} \ln k \leq n \ln|x| - n(\ln(n)-1) - 1.$$

Then for all x,

$$\lim_{n\to\infty} \left(n \ln|x| - \sum_{k=1}^{n} \ln k \right) = -\infty,$$

so the radius of convergence is ∞.

2. Suppose $f : (0, \infty) \to \mathbf{R}$ is a continuous function such that

$$\lim_{n\to\infty} f(n^2 x) = a$$

for every $x \in (0, \infty)$. (Of course here n is an integer). Prove that $\lim_{x\to\infty} f(x) = a$.

Solution. (Stanford real analysis exam, fall 1998. Compare with Kaczor and Nowak, problem 1.7.22.)

Fix $\epsilon > 0$. For k a positive integer let

$$F_k = \{0\} \cup \bigcap_{n\geq k} \{x \in (0, \infty) : |f(n^2 x) - a| \leq \epsilon\}.$$

Then since f is continuous F_k is closed in $[0, \infty)$ (since we've added the point $\{0\}$). By the limit hypothesis,

$$[0, \infty) = \bigcup_{k=1}^{\infty} F_k.$$

Since $[0, \infty)$ is a complete metric space, the Baire Theorem says that there exists a k such that F_k has nonempty interior. Thus there is a $a > 0$, $\delta > 0$, and positive integer k such that

$$(a - \delta, a + \delta) \subset F_k.$$

Choose M large enough that both of the following inequalities hold:

$$M \geq ak^2 \qquad \text{and} \qquad \left(\frac{2}{M} + \frac{1}{M^2}\right) a < \delta.$$

Now let $x > M$. We will show that $|f(x) - a| \leq \epsilon$. Let $n = \left[\sqrt{\frac{x}{a}}\right]$. Then $n \geq k$ and

$$a \leq \frac{x}{n^2} \leq a \left(\frac{n+1}{n}\right)^2$$

so

$$0 \leq \frac{x}{n^2} - a \leq a \left(\frac{2}{n} + \frac{1}{n^2}\right) < \delta.$$

Then $\frac{x}{n^2} \in F_k$ and so $|f(x) - a| = |f(n^2 \frac{x}{n^2}) - a| \leq \epsilon$.

3. Suppose $f_n(x)$ is a sequence of non-decreasing functions on $[0, 1]$ (i.e., for all n, if $x_1 < x_2$ then $f_n(x_1) \leq f_n(x_2)$) which converge pointwise to a continuous function $g(x)$. Prove that the convergence is actually uniform on $[0, 1]$.

Solution. (Stanford real analysis exam, spring 2001.) Since g is continuous on the compact interval $[0, 1]$, it is uniformly continuous. Fix $\epsilon > 0$ and choose δ such that for all x, $y \in [0, 1]$,

$$|x - y| \leq \delta \implies |g(x) - g(y)| < \epsilon/2.$$

Let $n = [1/\delta]$. Let $t_k = k\delta$, $k = 0, \ldots, n$, and $t_{n+1} = 1$ (note $t_{n+1} - t_n \leq \delta$ also). Now choose M such that for all $m \geq M$, and all $k = 0, \ldots, n+1$,

$$|g(t_k) - f_m(t_k)| < \epsilon/2.$$

Now we will show that f_m converges uniformlly to g on $[0,1]$. Let $m \geq M$. For any $x \in [0,1]$, find k so that $x \in [t_k, t_{k+1}]$. Consider $|g(x) - f_m(x)|$.

If $0 < g(x) - f_m(x)$, then $-f_m(x) \leq -f_m(t_k)$ and

$$|g(x) - f_m(x)| \leq |g(x) - f_m(t_k)| \leq |g(x) - g(t_k)| + |g(t_k) - f_m(t_k)| \leq \epsilon$$

since $|x - t_k| \leq \delta$ and $m \geq M$.

If $0 < f_m(x) - g(x)$, then $f_m(x) \leq f_m(t_{k+1})$ and

$$|g(x) - f_m(x)| \leq |g(x) - f_m(t_{k+1}| \leq \epsilon$$

by the same reasoning as above. Since M is independent of x, this shows that f_m converges uniformly to g on $[0,1]$.

4. Suppose f and g are positive, Riemann integrable functions on $[0,1]$ such that $f(x)g(x) \geq 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\,dx \cdot \int_0^1 g(x)\,dx \geq 1.$$

Solution. (Stanford real analysis exam, Fall 1999.) Since $f(x)g(x) \geq 0$ and $f(x)g(x) \geq 1$ for all $x \in [0,1]$,then we also have $\sqrt{f(x)}\sqrt{g(x)} \geq 1$ for all $x \in [0,1]$. Since f and g are Riemann integrable then so is $\sqrt{fg}$ and applying the Cauchy-Schwarz inequality gives

$$1 = \int_0^1 1\,dt \leq \int_0^1 \sqrt{f(x)}\sqrt{g(x)}\,dx \leq \sqrt{\int_0^1 f(x)\,dx}\sqrt{\int_0^1 g(x)\,dx}.$$

Squaring both sides gives the result.

5. Let $\mathbf{N}$ be the set of all positive integers. Suppose $f : \mathbf{N} \to \mathbf{R}$ is a function such that

$$f(n+m) \leq f(n) + f(m)$$

for all $m, n \in \mathbf{N}$. Prove that $\lim_{n\to\infty} \frac{f(n)}{n}$ exists and is equal to $\inf_{n\in\mathbf{N}} \frac{f(n)}{n}$.

Solution. (Stanford real analysis exam, fall 1999.) Let $L = \inf_{n\in\mathbf{N}} \frac{f(n)}{n}$. First suppose $L > -\infty$. Fix $\epsilon > 0$. There exists an $n_0 \in \mathbf{N}$ such that

$$\frac{f(n_0)}{n_0} \leq L + \epsilon$$

(otherwise, all $\frac{f(n)}{n} > L + \epsilon$ so $L + \epsilon$ is a greater lower bound for the sequence $\frac{f(n)}{n}$ than L). Let $m > n_0$ and write $m = qn_0 + r$ where $q \in \mathbf{N}$ and $0 \le r < n_0$. Then $f(qn_0 + r) \le qf(n_0) + f(r)$ and

$$\begin{aligned}\frac{f(m)}{m} &\le \frac{qf(n_0)}{m} + \frac{f(r)}{m} \\ &= \frac{m-r}{m} \cdot \frac{f(n_0)}{n_0} + \frac{f(r)}{m}.\end{aligned}$$

Since r doesn't depend on m,

$$\limsup_{m\to\infty} \frac{f(m)}{m} \le \frac{f(n_0)}{n_0} \le L + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, conclude $\limsup_{m\to\infty} \frac{f(m)}{m} \le L$. On the other hand it is clear that $L \le \liminf_{m\to\infty} \frac{f(m)}{m}$. This shows that

$$\liminf_{m\to\infty} \frac{f(m)}{m} = \limsup_{m\to\infty} \frac{f(m)}{m} = L.$$

So $\lim_{m\to\infty} \frac{f(m)}{m}$ exists and equals L. If $L = -\infty$ then the same type of argument will show that $\lim_{m\to\infty} \frac{f(m)}{m} = -\infty$.

6. Let f be a differentiable function on $[0, 1]$ such that $\sup_{x\in[0,1]} |f'(x)| \le M$. Prove that for all positive integers n,

$$\left| \sum_{j=0}^{n-1} \frac{f(\frac{j}{n})}{n} - \int_0^1 f(x)\,dx \right| \le \frac{M}{2n}.$$

Solution. (See Berkeley problems, 1.5.10.) The Mean Value Theorem gives, for $j = 0, \ldots, n$, and $x \in [j/n, (j+1)/n]$,

$$|f(j/n) - f(x)| \le M(x - j/n).$$

Then

$$\begin{aligned}\left| \sum_{j=0}^{n-1} \frac{f(\frac{j}{n})}{n} - \int_0^1 f(x)\,dx \right| &\le \sum_{j=0}^{n} \int_{j/n}^{(j+1)/n} |f(j/n) - f(x)|\,dx \\ &\le \sum_{j=0}^{n-1} M \int_{j/n}^{(j+1)/n} (x - j/n)\,dx \\ &= M \sum_{j=0}^{n-1} \left(\frac{(j+1)^2}{2n^2} - \frac{j^2}{2n^2} - \frac{1}{n} \cdot \frac{j}{n} \right) \\ &= M \sum_{j=0}^{n-1} \frac{1}{2n^2} = \frac{M}{2n}.\end{aligned}$$