787.03, Su 2003 Problem Set 4

1. Let R be a non-constant rational function, $R(x) = \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0}$, a_m, $b_n \neq 0$. Prove that there exists $x_0 \in \mathbf{R}$ such that R is strictly monotone on (x_0, ∞).

   Solution. The quotient rule for derivatives shows that R' is again a rational function, i.e., $R' = P/Q$ for some polynomials P, Q. If R' is identically zero, then R is constant contradicting the assumption. So R' is a non-zero rational function. Then P and Q have a finite number of zeros (possibly none) so there is an x_o such that R' is a continuous (in fact smooth) non-zero function on (x_o, ∞). Then R' is either always positive or always negative on (x_o, ∞) so R is strictly monotone on (x_o, ∞).

2. Find

   (a) $\lim_{n\to\infty} \sin(\pi\sqrt{n^2+1})$
   Solution. Claim $\lim_{n\to\infty} \sin\left(\pi\sqrt{n^2+1}\right) = 0$ (note this is a sequential limit, i.e., n is a non-negative integer). Fix $\epsilon > 0$. The sine function is continuous and periodic on $\mathbf{R}$, hence uniformly continuous on $\mathbf{R}$. Choose $\delta > 0$ such that for all x, $y \in \mathbf{R}$, $|x-y| < \delta$ implies $|\sin(x) - \sin(y)| < \epsilon$. Note $\lim_{n\to\infty}\left(\sqrt{n^2+1} - n\right) = 0$. Choose $M > 0$ so that for all $n > M$, $|\pi\sqrt{n^2+1} - \pi n| < \delta$. Then

   $$|\sin(\pi\sqrt{n^2+1})| = |\sin(\pi\sqrt{n^2+1}) - \sin(\pi n)| < \epsilon.$$

   Since $\epsilon > 0$ was arbitrary, conclude $\lim_{n\to\infty} \sin(\pi\sqrt{n^2+1}) = 0$.

   (b) $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^2+n})$
   Solution. For this one we note that

   $$\sqrt{n^2+n} = n\sqrt{1+\frac{1}{n}} \sim n\left(1+\frac{1}{2n}\right).$$

   It's easy to show that $\lim_{n\to\infty}\left(\sqrt{n^2+n} - n - \frac{1}{2}\right) = 0$. Since

   $$\sin^2(\pi(n+\frac{1}{2})) = 1,$$

   we can show as above that $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^2+n}) = 1$.

3. Let f be a continuous function on $[0,1]$. Prove that

   $$\exp\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 \exp\left(f(x)\right)\,dx.$$

Solution. (This is a special case of Jensen's inequality. See Rudin's book, "Real and Complex Analyis," Theorem 3.3) For any n, we have by the arithmetic-geometric mean inequality,

$$\left(\prod_{k=1}^{n} \exp\left(f\left(\frac{k}{n}\right)\right)\right)^{1/n} = \exp\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)\right) \leq \frac{1}{n}\sum_{k=1}^{n} \exp\left(f\left(\frac{k}{n}\right)\right).$$

Note $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$. Letting $n \to \infty$ and using the continuity of exp gives the result.

4. Prove the theorem on term-by-term differentiation of power series: if $r > 0$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < r$, then $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges for $|x| < r$, and $f' = g$.

   Solution. See, e.g., Rudin, *Principles of Mathematical Analysis*, Theorem 8.1.

5. (i) Prove that for every $n = 1, 2, \ldots$, the series $\sum_{k=n+1}^{\infty} \left(\frac{1}{\sqrt{k-n}} - \frac{1}{\sqrt{k}}\right)$ converges.

   Solution. There are many ways to do this. Perhaps the easiest is to rewrite the series as $\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+n}}\right)$ and use the integral test (which we can do because $\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+n}} > 0$ and $f(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+n}}$ is decreasing, as can easily be seen by taking the deriviative):

$$\int_1^{\infty} \left[\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+n}}\right] dx$$
$$= \lim_{N\to\infty} 2\left(\sqrt{N} - \sqrt{N+n}\right) - 2 + 2\sqrt{n+1} = -2 + 2\sqrt{n+1}.$$

   The integral test says the sum converges, for every $n \geq 1$.

6. (ii) For each $n \in \mathbf{N}$, let S_n be the sum of the above series. Evaluate $\lim_{n\to\infty} \frac{S_n}{\sqrt{n}}$.

   Solution. Let $S_{N,n}$ be the N^{th} partial sum of the series corresponding to n. From the basic estimate in the integral test we have

$$\int_1^N f(x)\,dx + f(N) \leq S_{N,n} \leq f(1) + \int_1^N f(x)\,dx.$$

   Letting $N \to \infty$ gives

$$2\sqrt{n+1} - 2 \leq S_n \leq -1 - \frac{1}{\sqrt{1+n}} + 2\sqrt{1+n}.$$

   Dividing by $\sqrt{n}$ and letting $n \to \infty$ shows that $\frac{S_n}{\sqrt{n}} \to 2$.