787.03, Su 2003 Problem Set 6
Solutions

The problem set was to do the Autumn, 1994, Qualifying exam.

1. Prove the Cauchy inequality: Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be real numbers. Then
$$\left(\sum_{j=1}^{n} x_j y_j\right)^2 \leq \left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{j=1}^{n} y_j^2\right).$$

   Solution. See e.g. Rudin, Principles of Mathematatial Analysis, Theorem 1.35 (where it's called the Schwarz inequality).

2. Prove that if $\sum a_n$ is a convergent but not absolutely convergent series of real numbers and if c is any real number, then there exists a rearrangement $\sum b_n$ of the series $\sum a_n$ such that $\sum b_n = c$.

   Solution. See Rudin, loc. sit., Theorem 3.54.

3. Let $x_n = \frac{1}{n}$ if n is odd and $x_n = 1 - \frac{1}{n}$ if n is even. For $x \in (0, 1]$, let $f(x) = \sum_{x_n < x} \frac{1}{n^2}$. Describe f, in particular, determine all its discontinuities. Prove your assertions.

   Solution. For any $0 < x < 1/2$ then there is a positive integer k such that
$$\frac{1}{2k+1} < x \leq \frac{1}{2k-1} \text{ and } x \leq \frac{1}{2}.$$
   Then for all all x in the interval $(\frac{1}{2k+1}, \frac{1}{2k-1}]$ we have $f(x) = \sum_{l=k}^{\infty} \frac{1}{(2l+1)^2}$. Thus f is constant on the interval $(\frac{1}{2k+1}, \frac{1}{2k-1}]$, is continuous from the left at $x = \frac{1}{2k-1}$ and $f((\frac{1}{2k+1})^+) - f((\frac{1}{2k+1})^-) = \frac{1}{2k+1}$ so f has a jump discontinuity at $x = \frac{1}{2k+1}$, k any positive integer.

   Since $f(\frac{1}{2}^+) - f(\frac{1}{2}^-) = \frac{1}{2^2}$, f is continuous from the left at $\frac{1}{2}$ and discontinuous at $\frac{1}{2}$.

   Similarly f is continuous at all $x \in (1/2, 1)$ except at the points $x = 1 - \frac{1}{n}$ for $n = 4, 6, \ldots$. f is continuous at $x = 1$ since $f(1) = \sum_{n=2}^{\infty} \frac{1}{n^2} = \lim_{x \to 1^-} f(x)$.

4. Let the function f_0 be continuous on $[0, 1]$, let $f_n(x) = \int_0^x f_{n-1}(y)\, dy$ for $n = 1, 2, \ldots$, and let $M_n = \sup_{0 \leq x \leq 1} |f_n(x)|$.

   (a) Prove that $|f_n(x)| \leq M_0 \frac{x^n}{n!}$ for $x \in [0, 1]$, $n = 1, 2, \ldots$
   Solution. This is an easy induction argument.

(b) Deduce that, for any $A > 1$, $A^n M_n \to 0$ as $n \to \infty$.

Solution. We have, for all $x \in [0,1]$, $|f_n(x)| \leq M_0 \frac{x^n}{n!}$. Thus

$$0 \leq M_n = \sup_{x\in[0,1]} |f_n(x)| \leq \frac{M_0}{n!} \sup_{x\in[0,1]} x^n = \frac{M_0}{n!}.$$

Now for $A > 1$ we have

$$0 \leq A^n M_n \leq M_0 \frac{A^n}{n!}.$$

It remains only to show $\lim_{n\to\infty} \frac{A^n}{n!} = 0$. Let n_1 be the greatest integer less than A and for $n > n_1$ write $n = n_1 + m$ for some positive integer m. Let $c = \prod_{k=1}^{n_1} \frac{A}{k}$. Then for $n > n_1$ we have

$$0 \leq \frac{A^n}{n!} = c \cdot \prod_{k=1}^{n} \frac{A}{n_1 + k} \leq c \left(\frac{A}{n_1+1}\right)^m .$$

Since $\frac{A}{n_1+1} < 1$, the right hand side of the above goes to zero as $n \to \infty$.

5. Prove or disprove.

(a) If f is continuously differentiable on $[0,1]$ then there exists a sequence of polynomials, $\{p_n\}$, such that $p_n \to f$ uniformly on $[0,1]$ and $p_n' \to f'$ uniformly on $[0,1]$.

Solution. Proof: Since f' is continuous on $[0,1]$, we can choose a sequence of polynomials $\{q_n\}$ such that $q_n \to f'$ uniformly on $[0,1]$. We claim that the polynomials $p_n(x)$ defined by

$$p_n(x) = \int_0^x q_n(t)\,dt + f(0)$$

converge uniformly to f. To see this fix $\epsilon > 0$. Since the q_n converge uniformly to f', we can choose N such that for all $n \geq N$,

$$\sup_{t\in[0,1]} |q_n(t) - f'(t)| < \epsilon.$$

Then for all $n > N$ and all $x \in [0,1]$,

$$\begin{aligned}
|f(x) - p_n(x)| &= \left| f(x) - f(0) - \int_0^x q_n(t)\,dt \right| \\
&\left| \int_0^x f'(t)\,dt - \int_0^x q_n(t)\,dt \right| \\
&\leq \int_0^x |f'(t) - p_n(t)|\,dt \\
&\leq x \sup_{t\in[0,1]} |f'(t) - p_n(t)|\,.
\end{aligned}$$

Since $0 \le x \le 1$,

$$|f(x) - p_n(x)| \le \sup_{t\in[0,1]} |f'(t) - p_n(t)| \le \epsilon.$$

This shows that p_n converges uniformly to f on $[0,1]$ also.

(b) If f is infinitely differentiable on $[0,1]$ then there exists a sequence of polynomials, $\{g_n\}$, such that for every k, $k = 0, 1, 2, \dots$, $g_n^{(k)} \to f^{(k)}$ as $n \to \infty$, uniformly on $[0,1]$.

Solution 1. (Thanks to Hongyuan Wang). The polynomials constructed in the proof of the Weierstrass theorem (Rudin, Principles of Mathematical Analysis) are

$$P_n(x) = \int_0^1 f(x+t)Q_n(t)\,dt$$

where $Q_n(t) = c_n(1-x^2)^n$ and the c_n is chosen so that

$$\int_{-1}^1 Q(t)\,dt = 1.$$

It is shown in Rudin that P_n converges uniformly to f on $[0,1]$. By differentiating under the integral sign (which can be easily justified if $f \in C^\infty([0,1])$), we can use the same proof to see that $p_n^{(k)}$ converges uniformly to $f^{(k)}$ as $n \to \infty$. (The details should be included in your solution.)

Solution 2. (Thanks to Jacqueline Kirby.) For $g \in C^0([0,1])$ let $\|g\|_0$ denote the C^0 norm, i.e., $\|g\|_0 = \sup_{[0,1]} |g(x)|$.

Step 1. For each $k = 0, 1, \dots$ there exists a sequence of polynomials $\{P_{n,k}\}_{n=1}^\infty$ such that

i. $P_{n,k}^{(k)}$ converges uniformly to $f^{(k)}$ on $[0,1]$ as $n \to \infty$.

ii. For each $j = 0, 1, \dots k$, and all $n \in \mathbf{N}$, $\|P_{n,k}^{(j)} - f^{(j)}\|_0 \le \|P_{n,k}^{(k)} - f^{(k)}\|_0$.

Proof of step 1. Using the Weierstrass theorem choose a sequence of polynomials P_n such that P_n converges to $f^{(k)}$ uniformly on $[0,1]$. Let $P_{n,1}(x) = \int_0^x P_n(t)\,dt + f^{(k-1)}(0)$. Then $P_{n,1}$ is a polynomial, $P_{n,1}' = P_n$ converges uniformly to $f^{(k)}$ on $[0,1]$ and

$$\|P_{n,1} - f^{(k-1)}\|_0 =$$
$$\sup_{[0,1]} \left| \int_0^x P_n(t)\,dt + f^{(k-1)}(0) - \int_0^x f^{(k)}(t)\,dt - f^{(k-1)}(0) \right| \le \|P_n - f^{(k)}\|_0.$$

Having constructed polynomials $P_{n,1}$, $P_{n,2}$, $\ldots$, $P_{n,l}$ such that $P_{n,l}^{(l)}$ converges uniformly to $f^{(k)}$ on $[0,1]$ and

$$\|P_{n,l} - f^{(k-l)}\|_0 \leq \|P'_{n,l} - f^{(k-l+1)}\|_0 \leq \cdots \leq \|P_{n,l}^{(l)} - f^{(k)}\|,$$

let $P_{n,l+1}(x) = \int_0^x P_{n,l}(t)\,dt + f^{k-l-1}(0)$. Then $P_{n,l+1}$ is a polynomial, $P'_{n,l+1} = P_{n,l}$, so $P_{n,l+1}^{(l+1)} = P_{n,l}^{(l)}$ converges uniformly to $f^{(k)}$ by the induction hypothesis and

$$\begin{aligned}
&\|P_{n,l+1} - f^{(k-l-1)}\|_0 \\
&= \sup_{[0,1]} \left| \int_0^x P_{n,l}(t)\,dt + f^{(k-l-1)}(0) - \int_0^x f^{(k-l)}(t)\,dt - f^{(k-l-1)}(0) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \|P_{n,l} - f^{(k-l)}\|_0.
\end{aligned}$$

This completes the proof of step 1.

Step 2. For each $l = 1, 2, \ldots$, choose an n_l such that $\|P_{n_l,l}^{(l)} - f^{(l)}\|_0 \leq \frac{1}{l}$ (which we can do by step 1, part i above). Now consider the sequence of polynomials $Q_l = P_{n_l,l}$. We claim that for each $j = 0, 1, \ldots$, $Q_l^{(j)}$ converges uniformly to $f^{(j)}$ as $l \to \infty$. Given j and $\epsilon > 0$, choose L such that $\frac{1}{L} < \epsilon$ and $L > j$. Then for any $l > L$,

$$\|Q_l^{(j)} - f^{(j)}\|_0 = \|P_{n_l,l}^{(j)} - f^{(j)}\|_0 \leq \|P_{n_l,l}^{(l)} - f^{(l)}\|_0 \text{ (by step 1, part ii)} \leq \frac{1}{l}$$

This shows that $Q_l^{(j)}$ converges uniformly to $f^{(j)}$ as $l \to \infty$.

6. Let f and g be non-negative functions on $[0, \infty)$, both locally Riemann integrabale. Prove that if $\liminf_{x\to\infty} f(x) \neq 0$ then

$$\int_0^\infty |f(x)\cos x + g(x)\sin x|\; dx = \infty.$$

Solution. (A function is locally Riemann integrable on a set A if f is bounded on A and Rieman integrable on every compact interval $[a, b]$ contained in A). Since $f \geq 0$ and $\liminf_{x\to\infty} f(x) \neq 0$, there is a $\delta > 0$ and an integer M such that for all $x \geq 2\pi M$, we have $0 < \delta \leq f(x)$. For any integer $N > M$ let

$$A_N = \cup_{k=M}^N (2\pi k, \frac{\pi}{2} + 2\pi k).$$

Note cos and sin are both positive on A_N, and $0 < \delta \leq f(x)$ for $x \in A_N$

(and g is non-negative). Then

$$\begin{aligned}\int_0^\infty |f(x)\cos x + g(x)\sin x|\,dx &\geq \int_{A_N} |f(x)\cos x + g(x)\sin x|\,dx \\ &= \int_{A_N} f(x)\cos x + g(x)\sin x\,dx \\ &\geq \delta \int_{A_N} \cos x\,dx \\ &= \delta \sum_{k=M}^{N} 1 = \delta(N - M + 1).\end{aligned}$$

Since N is arbitrary, this shows the integral diverges.