Solutions to Homework 7

Due Wednesday, August 4, 2004.

Chapter 4.1) 3, 4, 9, 20, 27, 30. Chapter 4.2) 4, 9, 10, 11, 12.

Chapter 4.1.

Solution to problem 3. The sum has the form

$$a_1 - a_2 + a_3 - \cdots$$

with $a_k = 1/k$. Since the a_k are positive and decreasing, the series converges by the alternating series test. Let L be the sum of the series. The proof of the alternating series test shows that

$$S_{2k-1} \leq L \leq S_{2k} \text{ and } |S_{2k} - S_{2k-1}| = a_k.$$

Since the sum of the series is between 0.1 and 1, to guarantee n digit accuracy we must have $a_k < 0.5 \times 10^{-n}$ (so we can be sure whether to round the n^{th} digit up or down), i.e., $2 \times 10^n < k$. For 10 digit accuracy: $2 \times 10^{10} < k$. For 20 digit accuracy: $2 \times 10^{20} < k$. For 100 digit accuracy: $2 \times 10^{100} < k$. (The point is the series converges very slowly).

Solution to problem 4. It is possible to have different A's for each ϵ (for example the Cauchy criteria does this), but if a series does not converge then it must diverge: if you have a different value of A for each ϵ, then the sequence of A's is a Cauchy sequence which converge to some A which works for all ϵ. I.e., let ϵ_N be a sequence tending to 0^+ and A_N be a sequence such that

$$|A_N - E_n| < \epsilon_N \text{ for all } n \geq N.$$

Then if $N < M$ we have for all $n \geq M$,

$$|A_M - A_N| \leq |A_M - E_n| + |A_N - E_n| \leq \epsilon_N + \epsilon_M.$$

Since both ϵ_M and ϵ_N go to zero we can make this less than any fixed ϵ by requiring N to be sufficiently large. Thus the A_N are a Cauchy sequence and converge to some A. Then the series converges to this A: given $\epsilon > 0$ chose N such that $|A - A_N| < \epsilon/2$ and $\epsilon_N < \epsilon/2$. Then for all $n \geq N$,

$$|A - E_n| \leq |A - A_N| + |A_N - E_n| \leq \epsilon/2 + \epsilon/2.$$

Solution to problem 9. The ratio test gives

$$\left|\frac{B_{2k+1}}{B_{2k}}\right| \cdot \frac{(2k-1)(2k)10^{2k-1}}{(2k+1)(2k+2)10^{2k+1}}$$

The factors not involving the B_2k approach 10^{-2} as $k \to \infty$. Then after doing some algebra the factors involving B_{2k} become

$$\left|\frac{B_{2k+1}}{B_{2k}}\right| = (2k+2)^2(1+1/k)^{2k}\frac{1}{2\pi}e^{-2}\sqrt{1+\frac{1}{k}}$$

which goes to infinity as $k \to \infty$. So the series does not converge.

Solution to problem 20. Choose N such that for all $n \geq N$, $|a_{n+1}/a_n| \leq \alpha$. (We may assume $\alpha > 0$, since if $\alpha = 0$ then $a_{N+1} = 0$ and the ratio $|a_{N+2}/a_{N+1}|$ is not defined.) We will prove by induction that for all $n \geq N$, $|a_n| \leq |a_N|\alpha^{n-N}$. We have clearly that $|a_{N+1}| \leq \alpha|a_N|$. Fix an $n > N+1$ and assume that $|a_n| \leq |a_N|\alpha^{n-N}$. Then

$$|a_{n+1}| \leq \alpha|a_n| \leq \alpha|a_N|\alpha^{n-N} = |a_N|\alpha^{n+1-N},$$

which proves the result by induction. As an immediate consequence we have

$$\sqrt[n]{|a_n|} \leq \alpha(|a_N|\alpha^{-N})^{1/n}.$$

Now

$$\lim_{n\to\infty}(|a_N|\alpha^{-N})^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(|a_N|\alpha^{-N})} = 1.$$

So given $\epsilon > 0$ we can choose N_1 such that for all $n \geq N_1$,

$$|(|a_N|\alpha^{-N})^{1/n} - 1| < \epsilon/\alpha$$

and in particular,

$$(|a_N|\alpha^{-N})^{1/n} < 1 + \epsilon/\alpha.$$

Then for all $n \geq M = \max(N, N_1)$ we have

$$\sqrt[n]{|a_n|} \leq \alpha(|a_N|\alpha^{-N})^{1/n}. < \alpha(1+\epsilon/\alpha) = \alpha + \epsilon.$$

To show that it does *not* necessarily imply $\sqrt[n]{|a_n|} \leq \alpha$, consider the (divergent) series $2+2+2+\cdots$. Then $|a_{n+1}/a_n| = 1$, but $\sqrt[n]{|a_n|} = 2^{1/n} > 1$. (But of course $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$.)

Solution to problem 27.

$$S_{10} \cong 1.87089362621604811409110898257 5$$
$$S_{100} \cong 1.872521265312390568584675164909$$
$$S_{300} \cong 1.872521265312390568584675164910$$

We have

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{k}{2k-1}\right)^k} = \frac{1}{2}$$

so the series converges by the root test. (Note one would expect the series to converge also by comparison with a geometric series, since for any $\alpha \in (1/2, 1)$, all but finitely many of the terms are less than α.)

Solution to problem 30.

$$S_{10} \cong 4511.87$$
$$S_{100} \cong 1.7001 \times 10^{42}$$
$$S_{300} \cong 7.0826 \times 10^{128}$$

One approach to prove divergence would be to use the limit ratio test:

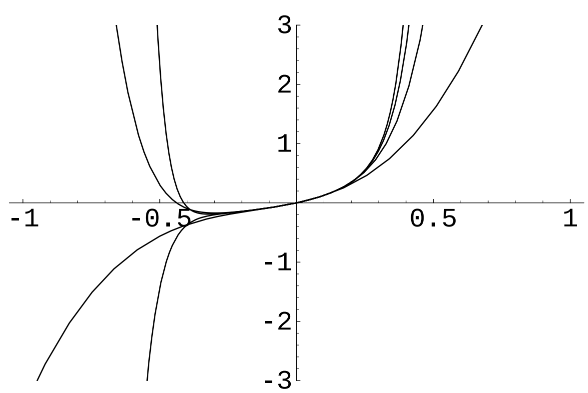
$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$$
$$= \lim_{k\to\infty} \frac{1}{k+1} \cdot (k+1) \cdot \left(\frac{k+1}{k}\right)^k$$
$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e.$$

Since $e > 1$ the series diverges.

Chapter 4.2.

Solution to problem 4.

Here is a graph of $S_n(x)$ for $n = 3$, 6, 9, and 12:



The graphs seem to approximate each other very well for $|x|$ less than about .3 but then they seem to diverge from each other when $|x|$ is larger than about .3. An easy computation shows the radius of convergence is $1/e \cong 0.37$. So one expects these graphs to approximate the function

$$f(x) = \sum_{k=1}^{\infty} \frac{k^k}{k!} x^k$$

well on the interval $|x| < 1/e$.

Solution to problem 9.

By the ratio test we find the radius of convergence is 1.

Solution to problem 10. Let

$$a_k = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{3 \cdot 5 \cdot 7 \cdots (2k+1)}$$

One can use Gauss's test from section 4.3. Since this is section 4.2, I'll try to write a proof without using it. I'll need a series of lemmas:

Lemma 1. For all k, $a_k \geq \frac{2}{2k+1}$.

Proof. For $k = 1$, $a_1 = \frac{2}{3} \geq \frac{2}{3}$. Suppose $a_k \geq \frac{2}{2k+1}$. Then

$$a_{k+1} = a_k \cdot \frac{2k+2}{2k+3} \geq \frac{2}{2k+1} \cdot \frac{2k+2}{2k+3} \geq \frac{2}{2k+3}$$

since $\frac{2k+2}{2k+1} > 1$.

Lemma 2. For all k, $a_k \geq \frac{1}{2k}$.

Proof. Use Lemma 1 and the easily verified fact that

$$\frac{2}{2k+1} = \frac{1}{k+1/2} \geq \frac{1}{2k}.$$

Corollary. The series diverges at $x = 1$ by comparison with the harmonic series.

Lemma 3. For all k, $a_k \geq a_{k+1} \geq 0$.

Proof. Note $a_{k+1} = a_k \cdot \frac{2k+2}{2k+3}$ and note $\frac{2k+2}{2k+3} \leq 1$. Clearly all the a_k's are greater than or equal to 0.

Lemma 4. $\lim_{k \to 0} a_k = 0$.

Proof. Note the a_k can be written as

$$a_k = \frac{2^k k!}{\frac{(2k+1)!}{2^k k!}} = \frac{2^{2k}(k!)^2}{(2k+1)!}.$$

Stirling's formula says that

$$k! = \left(\frac{k}{e}\right)^k \sqrt{2\pi k}\; e^{E(k)}$$

where $\lim_{k\to\infty} E(k) = 0$ (equation (A.77), page 301). Plugging this into the above and simplifying a bit gives

$$a_k = \left(\frac{2k}{2k+1}\right)^{2k} \frac{e}{\sqrt{2\pi(2k+1)}} \cdot \frac{k}{2k+1} \cdot e^{2E(k)-E(2k+1)}.$$

As $k \to \infty$, the first factor approaches the limit e^{-1} and the last factor approaches 1. Then it's easy to see that a_k goes to zero as $k \to \infty$.

Corollary. The series converges at $x = -1$.

Proof. The series when $x = -1$ is

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

which converges by Lemmas 3 and 4 and the alternating series test.

Solution to problem 11. Using the hint $1 \cdot 3 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$ and Stirling's formula (equation (A.77), page 301) gives

$$\begin{aligned}
1 \cdot 3 \cdots (2n-1) &= \frac{(2n)!}{2^n n!} \\
&= \frac{1}{2^n} \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi 2n}\; e^{E(2n)}}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\; e^{E(n)}} \\
&= \frac{1}{2^n} \cdot 2^{2n} \cdot \left(\frac{n^2}{e^2} \cdot \frac{e}{n}\right)^n \sqrt{2}\; e^{E(2n)-E(n)} \\
&= 2^{n+\frac{1}{2}} \left(\frac{n}{e}\right)^n e^{E(2n)-E(n)} \\
&= 2^{n+\frac{1}{2}} n^n e^{-n+F(n)}
\end{aligned}$$

where $F(n) = E(2n) - E(n)$ goes to zero as $n \to \infty$ (since $E(n)$ and $E(2n)$ do).

If we ignore the $F(n)$, then the ratio of the two sides of the equation (4.11) gets close to 1, but the absolute value of the difference between them gets large.

If we look at the ratio of both sides, they are approximately 0.9958444146, 0.9979191384, and 0.9995834222 respectively when $n = 10$, 20, and 100 (the ratios are very close to one).

If we look at the absolute value of the difference between both sides, they are approximately 2.7321359×10^6, 6.669117×10^{20}, and 2.778188×10^{183} when $n = 10$, 20, and 100 respectively.

So without the error terms it is only in the first sense that the two are good approximations of each other.

Solution to problem 12. Using

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n \cdot n!}$$

and Stirling's Formula gives

$$\lim_{n \to \infty} \sqrt[n]{\frac{n^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}} = \frac{e}{2},$$

so the radius of convergence is $\frac{2}{e}$.