Math 650 Summer 2004 Test 2 Solutions. Friday, August 6, 2004.

Name: ______________________________

1. Use the Mean Value Theorem to prove that if f is continuous on $[a, b]$, differentiable on (a, b) and $f'(x) > 0$ for all x in (a, b), then f is strictly monotonically increasing on $[a, b]$. (In other words, prove that if $a \leq x_1 < x_2 \leq b$, then $f(x_1) < f(x_2)$.)

   Solution. Let $a \leq x_1 < x_2 \leq b$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$ (since $[x_1, x_2] \subseteq [a, b]$ and $(x_1, x_2) \subseteq (a, b)$). So there is a $c \in (x_1, x_2) \subseteq (a, b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

   (the last inequality by the hypothesis on f'). Since $x_2 - x_1 > 0$, we have $f(x_2) > f(x_1)$.

2. Use the *definition* of continuity to write a proof that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   is continuous at 0.

   Solution. Fix $\epsilon > 0$ and let $\delta = \min(1, \epsilon)$. Then $\delta^2 \leq \epsilon$. If $x = 0$ we certainly have $|f(x) - 0| < \epsilon$. If $0 < |x - 0| < \delta$ then

$$|f(x) - 0| \leq |x^2 \sin(1/x)| \leq |x^2| < \delta^2 \leq \epsilon.$$

3. Compare the Cauchy and Lagrange form of the remainder for the Taylor series for $f(x) = \log(1 + x)$ when $x = 1$, expanded around $a = 0$. Which gives a tighter bound on the error (when each remainder is estimated to be as large as possible)?

   Solution. (Note the problem should have said $\ln(1 + x)$, not $\log(1 + x)$.) We compute $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$. The Lagrange remainder is

$$L_n(0, 1) = \frac{(-1)^{n-1}}{n} \cdot (1 + c)^{-n}$$

   for some $c \in (0, 1)$. Then

$$|L_n(0, 1)| \leq \frac{1}{n} \cdot \frac{1}{(1 + c)^n} \leq \frac{1}{n}.$$

   The Cauchy remainder is

$$C_n(0, 1) = (-1)^{n-1}(1 + c)^{-n}(1 - c)^{n-1}$$

for some $c \in (0,1)$. The function $g(c) = (1+c)^{-n}(1-c)^{n-1}$, with n a positive integer, is monotonically decreasing on $(0,1)$ (easiest to check by logarithmic differentiation) with maximum $g(0) = 1$. Thus

$$|C_n(0,1)| \leq 1.$$

The Lagrange remainder gives a tighter estimate when $n > 1$.

4. Prove that the series with partial sums

$$S_n = \sum_{k=1}^{n} \frac{2^k}{\sqrt{k}}$$

diverges.

Solution. Since $\lim_{k\to\infty} \frac{e^{k \ln 2}}{\sqrt{k}} = \infty$ the terms certainly don't go to zero as $k \to \infty$. (Or you could use the root or ratio tests.)

(a) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n!} x^n$$

Solution. Writing the series as $\sum_{n=1}^{\infty} a_n x^n$ we have after a short computation

$$\frac{a_{n+1}}{a_n} = \frac{4n+2}{n+1}.$$

The radius of convergence is $\frac{1}{4}$.

(b) Determine the convergence of the series above when $|x|$ is equal to the radius of convergence.

Solution. When $|x| = \frac{1}{4}$ we have

$$\left|\frac{a_{n+1}}{a_n} x\right| = \left|\frac{n+\frac{1}{2}}{n+1}\right|.$$

Thus, in the notation of Theorem 4.7, $B_{t-1} = B_0 = \frac{1}{2}$ and $b_{t-1} = b_0 = 1$. We are in the situation of parts 3 and 4 of the Theorem: $b_{t-1} > B_{t-1} \geq b_{t-1} - 1$. Thus the series is not absolutely convergent and converges if and only if it is alternating when $|x| = \frac{1}{4}$. The series is alternating when $x = -\frac{1}{4}$ and the terms in the series are all positive when $x = +\frac{1}{4}$. The series diverges when $x = \frac{1}{4}$ and converges when $x = -\frac{1}{4}$. (Note you'll come to the same conclusion even if you didn't cancel out the common factor of $n+1$ in part a.)