

Orbits of the geodesic flow and chains on the boundary of a Grauert tube

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Abstract. A C^ω Riemannian manifold (X, g) determines an integrable complex structure on a tubular neighborhood $T^{*\epsilon}X$ of X in T^*X , and a CR-structure on the boundary $\partial T^{*\epsilon}X$. There are two natural families of curves on $\partial T^{*\epsilon}X$: the orbits of the geodesic flow, and a CR-invariant family called chains; it is natural to ask whether they are related. We show that if orbits of the geodesic flow are chains on $\partial T^{*\epsilon}X$ for all ϵ sufficiently small, then (X, g) is Einstein. As a partial converse, we show that if (X, g) is harmonic, then orbits of the geodesic flow are chains. To prove this we study the Fefferman metric associated with $\partial T^{*\epsilon}X$.

0. Introduction.

Let (X, g) be a compact, connected, C^ω Riemannian manifold. For sufficiently small ϵ this data determines in a canonical way an integrable complex structure on the tube

$$T^{*\epsilon}X := \{\xi \in T^*X : |\xi|_g < \epsilon\}$$

(see [15], [22]) such that the level sets of the length function are strictly pseudoconvex domains in a Stein manifold. We will explore the relationship between CR invariants of the boundary of the tube, M_ϵ , and the metric g . In principle the infinite order jet of the complex structure at the zero section is computable from g , but it is not obvious how to relate this data to boundary CR invariants. A further complication is fact that the M_ϵ are not CR equivalent for different values of ϵ [17]. So one first needs to understand properties which persist as $\epsilon \rightarrow 0^+$.

The reason the CR invariants of M_ϵ are interesting is that a strictly pseudoconvex domain in a Stein manifold is determined by its CR structure at the boundary: if Ω_1, Ω_2 are two such domains, then they are biholomorphic if and only if their boundaries are CR diffeomorphic*. Thus the equivalence problem for such domains was in principle solved by Chern and Moser [9], who constructed a complete set of invariants for nondegenerate CR manifolds M of hypersurface type. In the process

* The proof of the “if” part is the same as for domains in \mathbf{C}^n [23] using the extension theorems of [16]; the “only if” part is Fefferman’s theorem [12].

they identified an interesting family of CR invariant curves called chains. For each $v \in TM$ transverse to the maximal complex subspace, there is a unique chain with tangent vector v . They showed that along a chain, M can be represented in a normal form which osculates M to maximal order by the holomorphic image of a quadric. It is known that chains carry a lot of information about the CR structure. For example, a chain-preserving diffeomorphism between two non-degenerate CR manifolds is either a CR or conjugate CR diffeomorphism [8].

On the other hand, the boundaries M_ϵ carry a natural family of curves determined by the metric, namely, the orbits of the geodesic flow. Since the identity component of the holomorphic automorphism group of $T^{*\epsilon}X$ is the lift of the isometries of X , [5], these orbits are at least invariant under the identity component of the automorphism group. As a first step in relating Riemannian invariants of X to CR invariants of M_ϵ , we give a partial answer to the question, “When are the orbits of the geodesic flow chains (up to reparameterization)?” (there are of course many other chains which are *not* initially tangent to the geodesic flow). We find that if the orbits of the geodesic flow are chains for all sufficiently small ϵ , then (X, g) must be Einstein. In the converse direction we show that if X is a harmonic manifold, then the orbits of the geodesic flow are chains.

Chains were defined in [9, §5d] as solutions of a system of ordinary differential equations involving the Chern-Moser invariants on the Chern structure bundle. Extending the seminal work of Fefferman [13] on domains in \mathbf{C}^n , Burns et al. [6] and Webster [25] showed that if M is strictly pseudoconvex, then its chains can be described as the projections of (non-vertical) null geodesics for a certain conformal class of indefinite metrics on a circle bundle over M . This conformal class, called the Fefferman metric, is invariant under CR diffeomorphisms of M . In fact, all the CR invariants can be reconstructed from conformal invariants of the Fefferman metric [6]. Since it is difficult to work directly with the Chern structure bundle, our approach will be via the Fefferman metric.

A pseudohermitian structure on M is a choice of contact form θ annihilating the maximal complex subspace. Associated with a pseudohermitian structure is a fiber metric on the canonical bundle K_M of M , and the Fefferman manifold can be identified with the unit circle bundle in K_M . J. Lee [20] gave a description of the Fefferman metric in terms of intrinsically defined forms and normalizations on K_M . We use Lee’s approach to the Fefferman metric to obtain our main result, theorem 3.3, which computes the Lee representative of the Fefferman metric in terms of the Kähler geometry of $T^{*\epsilon}X$. This allows us to relate the CR invariants of M_ϵ to the infinite order jet of the complex structure at the zero section, and hence to the Riemann metric g .

The organization of this paper is as follows. In §1 we give background material, establish some notations and conventions, and prove a “well-known” but important lemma. We remark that an analog of this lemma is true for other functions of ϕ , and that many of our computations can be carried out in other interesting situations. For

example, on a negative holomorphic hermitian line bundle $\log|\cdot|$ satisfies the homogeneous Monge-Ampère equation, and one could obtain results similar to theorem 3.3 for the boundary of the unit disk bundle. In §2 we describe the Lee construction of the Fefferman metric and use it to give a criteria for the orbits of the geodesic flow to be chains (lemma 2.3). In §3 we analyze the effect of the homogeneous Monge-Ampère condition on the Lee representative of the Fefferman metric. We find that the volume normalization used to construct the Fefferman metric has a simple form (corollary 3.2). The main result is theorem 3.3, which gives a description of the Fefferman metric in terms of the Kähler geometry of the tubes. This leads to a more explicit condition for the orbits of the geodesic flow to be chains (corollary 3.6). In §4 we use a result of R. Aguilar [1] show that if X is a harmonic manifold, then the orbits of the geodesic flow are chains. The Damek-Ricci spaces give non-symmetric (non-compact) examples of this phenomenon. We also compute some of the Webster invariants of M_ϵ , and remark that the M_ϵ associated with the Damek-Ricci spaces give new examples of pseudo-Einstein CR manifolds.

In §5 we prove that a necessary condition for the orbits of the geodesic flow to be chains for all ϵ sufficiently small is that X must be Einstein. The relationship between the Ricci tensor $\tilde{\rho}$ of $T^{*\epsilon}X$ and the Ricci tensor Ric_X of X is given by proposition 5.2, which shows that $\iota_X^* \tilde{\rho} = \frac{2}{3} \text{Ric}_X$. To prove this we use another result of R. Aguilar [1], which relates $\tilde{\rho}$ to the Ricci tensor of a pseudo-Riemannian metric associated with the square of the distance function on a neighborhood of the diagonal in $X \times X$. We obtain a criteria for the orbits of the geodesic flow to be chains in terms of the scalar and Ricci curvature of $T^{*\epsilon}X$ (equation 5.3), and derive the necessary condition by expanding this in a power series in local coordinates. If $\dim X = 2$ or 3 , then the orbits of the geodesic flow are chains if and only if X has constant sectional curvature (corollary 5.4).

1. Background material.

Geometric preliminaries.

In this paper (X, g) will be a connected, real-analytic Riemann manifold of dimension $n + 1$ such that $T^{*\epsilon}X$ admits an adapted complex structure for some $\epsilon > 0$ (see below; this is always true if X is compact). Our convention for wedge products is the “alternating” one, i.e., $\alpha \wedge \beta = (\alpha \otimes \beta - \beta \otimes \alpha)/2$ for 1-forms α, β . We will use the usual summation conventions (repeated indices are summed).

Let ϕ be a smooth, positive, strictly plurisubharmonic function on an $(n + 1)$ -dimensional complex manifold Ω . Let h be the Kähler metric associated with the Kähler form $2\sqrt{-1} \partial\bar{\partial}\phi$, i.e.,

$$h(V, W) = 2\sqrt{-1} \partial\bar{\partial}\phi(V, J\bar{W}) \quad V, W, \in \mathbf{CT}\Omega.$$

In local holomorphic coordinates, $h = 2\phi_{i\bar{j}}dz^i \cdot d\bar{z}^j$ (the subscripts denoting partial derivatives). We denote the Ricci form of h by ρ and the Ricci tensor by $\tilde{\rho}$. In local coordinates, $\rho = -2\sqrt{-1}(\log \det \phi_{k\bar{l}})_{i\bar{j}}dz^i \wedge d\bar{z}^j$ and $\tilde{\rho} = -2(\log \det \phi_{k\bar{l}})_{i\bar{j}}dz^i \cdot d\bar{z}^j$. See [3], 2.99 or [18], IX.5 eq. 26. We use the wedge product convention of [18] and relate the Ricci form to the Ricci tensor as in [3], 2.26.

A homogeneous Monge-Ampère type problem.

Let $\iota_X: X \rightarrow \Omega$ be an embedding of X as a totally real submanifold of a complex manifold Ω . It is well-known that such an embedding exists if X is connected and real analytic, and the germ of the embedding is unique. There is a unique antiholomorphic involution σ on a neighborhood of X whose fixed point set is X . Consider the following ‘‘homogeneous Monge-Ampère type’’ problem: Find a smooth, non-negative, strictly plurisubharmonic function φ on a neighborhood of X such that $\varphi^{-1}(0) = X$, $\sigma^*\varphi = \varphi$, $\iota_X^*h = g$ where h is the Kähler metric associated with φ , and $(\partial\bar{\partial}\sqrt{\varphi})^{n+1} = 0$. The main result of [15] is that if (X, g) is compact, connected and real analytic, then there is a unique solution to this problem, and the solution is real analytic. Given such a solution, there is a unique diffeomorphism ψ from a neighborhood of X in T^*X to a neighborhood of X in Ω such that $\psi^*\text{Im}\bar{\partial}\varphi$ is the canonical one-form on T^*X . It can be shown that $\psi^*\varphi$ is the length squared function. The identification ψ gives $T^{*\epsilon}X$ an integrable complex structure, canonically determined by the metric. Using the metric to identify T^*X with TX one obtains the adapted complex structure discovered independently and simultaneously by Lempert and Szöke [22]. If X is not compact, then the adapted complex structure exists locally but may not extend to a tube of uniform radius. If X is homogeneous or covers a compact quotient, then the adapted complex structure does extend to some $T^\epsilon X$.

Pseudohermitian manifolds.

A pseudohermitian manifold is a CR manifold M with a choice of a smooth one-form θ annihilating the maximal complex subspace of TM . See J. Lee’s paper [20] for more details, the notations and conventions of which we will follow closely. We will study the pseudohermitian manifolds $M_\epsilon = \partial T^{*\epsilon}X$, with the pseudohermitian structure $\theta = \iota_\epsilon^*(-\text{Im}\bar{\partial}\phi)$, where ι_ϵ is the inclusion and ϕ is the g -length squared function. Our perspective will be to consider M_ϵ simply as the ϵ^2 -level set of a solution ϕ of the homogeneous Monge-Ampère type problem above.

Define a vector field T on M_ϵ by $T \lrcorner d\theta = 0$, $\theta(T) = 1$. T is a constant multiple of the infinitesimal generator of the geodesic flow, so its orbits (considered as unparameterized curves) are the orbits of the geodesic flow. Since $\theta(T) = 1$, T is transverse to the maximal complex subspace of TM_ϵ . At each point $p \in M_\epsilon$, there is a unique chain passing through p with tangent vector $T(p)$. In this paper we will investigate whether this chain is the orbit of T through p (up to reparameterization).

Equivalent formulations of the homogeneous Monge-Ampère condition.

The metric identification of $\mathbf{CT}^*\Omega$ with $\mathbf{CT}\Omega$ induces an inner product on $\mathbf{CT}^*\Omega$. Define a vector field Z of type $(1,0)$ on Ω by

$$Z \lrcorner \partial \bar{\partial} \phi = |\partial \phi|^{-2} \bar{\partial} \phi \quad \text{where } |\partial \phi|^2 = |\bar{\partial} \phi|^2 = h(\partial \phi, \partial \phi).$$

Note that $Z = |\bar{\partial} \phi|^{-2} \sharp \bar{\partial} \phi$, and $\bar{\partial} \phi(\bar{Z}) = \partial \phi(Z) = 1$. We will need the following “well-known” lemma.

Lemma 1.1. *The following are equivalent:*

1. $(\partial \bar{\partial} \sqrt{\phi})^{n+1} = 0$
2. $|\partial \phi| = \sqrt{2\bar{\phi}}$
3. $[Z, \bar{Z}] = (2\phi)^{-1}(Z - \bar{Z})$
4. *If Ξ is defined by $\Xi \lrcorner \sqrt{-1} \partial \bar{\partial} \phi = -\text{Im } \bar{\partial} \phi$, then $\Xi \phi = 2\phi$.*

Proof. A computation shows that 1) is equivalent to

$$2\phi (\partial \bar{\partial} \phi)^{n+1} = (n+1) \partial \phi \wedge \bar{\partial} \phi \wedge (\partial \bar{\partial} \phi)^n. \quad (1.1)$$

Contracting the identity $0 = \partial \phi \wedge (\partial \bar{\partial} \phi)^{n+1}$ with Z shows that

$$(\partial \bar{\partial} \phi)^{n+1} = |\partial \phi|^{-2} (n+1) \partial \phi \wedge \bar{\partial} \phi \wedge (\partial \bar{\partial} \phi)^n.$$

Therefore 1) is equivalent to 2). Note that $L_Z \partial \phi = -|\partial \phi|^{-2} \bar{\partial} \phi$ and

$$L_Z \partial \bar{\partial} \phi = d|\partial \phi|^{-2} \wedge \bar{\partial} \phi + |\partial \phi|^{-2} \partial \bar{\partial} \phi.$$

Using these identities and the identity $[Z, \bar{Z}] \lrcorner \partial \bar{\partial} \phi = L_Z (\bar{Z} \lrcorner \partial \bar{\partial} \phi) - \bar{Z} \lrcorner L_Z \partial \bar{\partial} \phi$ we can compute

$$[Z, \bar{Z}] \lrcorner \partial \bar{\partial} \phi = d|\partial \phi|^{-2} - (Z \lrcorner \partial \phi)^{-2} \partial \phi + \bar{Z} \lrcorner \partial \phi)^{-2} \bar{\partial} \phi + |\partial \phi|^{-4} d\phi.$$

Assuming 2), the right hand side reduces to $(2\phi)^{-2} d\phi = (2\phi)^{-1} (Z - \bar{Z}) \lrcorner \partial \bar{\partial} \phi$, which gives 3). Assuming 3), the left hand side becomes $(2\phi)^{-2} d\phi$, and contracting this equation with $Z + \bar{Z}$ gives 2). 4) is equivalent to 2) because $\Xi = |\partial \phi|^2 \Re(Z)$ and $\Re(Z) \phi = 1$. //

2. Chains on CR manifolds and the Fefferman metric.

Let M be a pseudohermitian manifold and let K^* be the canonical bundle of M minus the zero section. Let $C := K^*/\mathbf{R}_+$ be the quotient by the natural \mathbf{R}_+ action. The pseudohermitian structure θ determines a “volume normalization” map

$$\iota_\theta: C \rightarrow K^*$$

by associating with an equivalence class in C the unique representative $\zeta_o \in K^*$ satisfying

$$\sqrt{-1} n^2 n! \theta \wedge (T]\zeta_o) \wedge (T]\bar{\zeta}_o) = \theta \wedge d\theta^n \quad (2.1)$$

where T is the vector field defined by $T]d\theta = 0$, $\theta(T) = 1$ (see [20], §3). An $(n+1)$ -form on K^* satisfying (2.1) is said to be volume normalized. Since K is a bundle of $(n+1)$ -forms, there is a tautologous form ξ on K . The volume normalization allows this tautologous form to be pulled back to C . Let $\zeta := \iota_\theta^* \xi$ and let $\pi: C \rightarrow M$ be the natural projection. Let L_θ be the Levi form: for $V, W \in \ker \theta$, $L_\theta(V, W) = d\theta(V, J\bar{W})$, and $L_\theta(T, V) = 0$ for $V \in \mathbf{CTM}$. The following theorem, due to J. Lee, gives an intrinsic description of the Fefferman metric.

Theorem 2.1 (Lee [20]).

1. *There is a unique n -form η on C such that $\zeta = \pi^* \theta \wedge \eta$ and $V]\eta = 0$ for any lift V of T .*
2. *There is a unique real 1-form σ on C satisfying*

$$d\zeta = \sqrt{-1} (n+2) \sigma \wedge \zeta \quad (2.2)$$

$$\sigma \wedge d\eta \wedge \bar{\eta} = (\text{Tr } d\sigma) \sqrt{-1} \sigma \wedge \pi^* \theta \wedge \eta \wedge \bar{\eta}. \quad (2.3)$$

3. *The conformal class of the Fefferman metric is represented by*

$$g_\theta = \pi^* L_\theta + 2\pi^* \theta \cdot \sigma.$$

Note that $d\sigma$ is the pullback of a globally defined form on M . This can be seen by following the construction of σ in [20]. In particular, $\text{Tr } d\sigma$ makes sense*. We remark that the fibers of C are totally geodesic, and the natural S^1 action is isometric. For our purposes the following definition of chains will be convenient.

* If ω is a two-form on M and Z_α , $\alpha = 1, \dots, n$, is a unitary basis for $T^{(1,0)}M$ (with respect to the Levi form), then $\text{Tr } \omega = -\sqrt{-1} \sum_1^n \omega(Z_\alpha, \bar{Z}_\alpha)$.

Definition 2.2. *A chain on M is the projection of a non-vertical null geodesic of g_θ .*

Changing the pseudohermitian structure to $\tilde{\theta} = e^{2f}\theta$ changes the Lee representative of the Fefferman metric to $g_{\tilde{\theta}} = e^{2f}g_\theta$ ([20], theorem 3.8). Null geodesics are conformally invariant, so this definition is independent of the choice of pseudohermitian structure. When the Levi form is positive definite, the equivalence of chains and non-vertical null geodesics was established in [6] and [25]. If the Levi form has negative eigenvalues, then there are non-vertical null geodesics which do not project to chains (see [19]).

Since $d\sigma$ is the pullback of a form on M , there is a unique 2-form ϖ on M such that $d\sigma = \pi^*\varpi$. We will express the condition that the orbits of T are chains as an equation on M .

Lemma 2.3. *The orbits of T are chains if and only if $T]\varpi = 0$.*

Proof. There is a unique null lift V of T to C . Since S^1 acts transitively by isometries on the fibers of C , the orbits of T are chains if and only if V is autoparallel. For any vector field W on C we have, using the standard formula for the Levi-Cevita connection ∇ of g_θ ,

$$g_\theta(W, \nabla_V V) = V(g_\theta(W, V)) - g_\theta([V, W], V).$$

Since V is null, $\sigma(V) = 0$ and $\sigma(W) = g_\theta(V, W)$. Then

$$\begin{aligned} g_\theta(W, \nabla_V V) &= V(\sigma(W)) - W(\sigma(V)) - \sigma([V, W]) \\ &= (V]d\sigma)(W) = \pi^*(T]\varpi)(W). \end{aligned}$$

Therefore $\nabla_V V = 0$ if and only if $T]\varpi = 0$. //

3. Chains and the homogeneous Monge-Ampère equation.

In this section we will assume only that M^{2n+1} is a hypersurface in a complex manifold Ω of the form $M = \{\phi = \epsilon^2\}$, where ϕ is a strictly plurisubharmonic function on a neighborhood of M with $d\phi \neq 0$ at M and satisfying the homogeneous Monge-Ampère type equation

$$\left(\partial\bar{\partial}\sqrt{\phi}\right)^{n+1} = 0. \tag{3.1}$$

We remark that if M is real analytic and the canonical bundle of M is trivial, then such a defining function can be found on a neighborhood of M by solving a Cauchy-Kowalewski problem (see [21], chapter 3.2). We will equip M with the

pseudohermitian structure $\theta := \iota^*(-\text{Im } \bar{\partial}\phi)$, where $\iota: M \rightarrow \Omega$ is the inclusion, and compute the corresponding representative g_θ of the Fefferman metric. The main result is theorem 3.3, which shows that under these assumptions the 1-form σ in theorem 2.1 can be expressed in terms of the Hermitian connection induced by $h = 2\phi_{i\bar{j}}dz^i \cdot d\bar{z}^j$ on the canonical bundle of Ω , K_Ω . As an application we find a necessary and sufficient condition for the orbits of T to be chains in terms of the Ricci curvature of h (corollary 3.6).

We will first show that the volume normalization map has an especially simple form in this situation. Define an inner product on K_Ω by

$$\sqrt{-1}^{n^2+n}(n+1)!\nu \wedge \bar{\mu} = (\nu, \mu) (\partial\bar{\partial}\phi)^{n+1}. \quad (3.2)$$

In local coordinates, $|fdz^1 \wedge \cdots \wedge dz^{n+1}| = |f| (\det \phi_{\alpha\bar{\beta}})^{-1/2}$.

Proposition 3.1. *If ν is a non-vanishing local section of K_Ω , then $\iota^*(|\partial\phi| |\nu|^{-1} \nu)$ is volume normalized.*

Proof. (This does not use the homogeneous Monge-Ampère condition.) Let Z be as in section 1. Then $\partial\phi(Z) = 1$ and $d\iota(T) = \sqrt{-1}(Z - \bar{Z}) \circ \iota$. Put $\mu = \nu$ in (3.2), contract with Z and \bar{Z} , and wedge with $\partial\phi$ to get

$$\sqrt{-1}^{n^2+n} n! (-1)^n \partial\phi \wedge (Z]\nu) \wedge (\bar{Z}]\bar{\nu}) = |\nu|^2 |\partial\phi|^{-2} \partial\phi \wedge (\partial\bar{\partial}\phi)^n.$$

Pulling back to M and using $\iota^*(\partial\phi) = \sqrt{-1}\theta$, $\iota^*(\partial\bar{\partial}\phi) = -\sqrt{-1}d\theta$ gives

$$\sqrt{-1}^{n^2} n! \theta \wedge (T]\iota^*\nu) \wedge (T]\iota^*\bar{\nu}) = \iota^*(|\nu|^2 |\partial\phi|^{-2}) \theta \wedge d\theta^n,$$

which shows that $\iota^*(|\partial\phi| |\nu|^{-1} \nu)$ satisfies the volume normalization (2.1). //

Corollary 3.2. *If ϕ satisfies (3.1), then $\xi_o := \sqrt{2} \epsilon \iota^*(|\nu|^{-1} \nu)$ is volume normalized.*

Proof. This follows from part 2 of lemma 1.1. //

We can now analyze the effect of the homogeneous Monge-Ampère type condition (3.1) on the representative of the Fefferman metric given by theorem 2.1. Let ν be a *closed* non-vanishing local section of K_Ω and let ξ_o be as in corollary 3.2. Define a fiber coordinate γ on C by writing $\zeta = e^{\sqrt{-1}\gamma} \pi^* \xi_o$ (with ζ as in section 2).

Theorem 3.3. *The one-form σ in part 2 of theorem 2.1 is given by*

$$\sigma = \frac{1}{n+2} \left(d\gamma + (\iota \circ \pi)^* \text{Im } \partial \log |\nu|^2 \right) + \pi^*(f\theta)$$

where

$$f = \frac{1}{2(n+1)} \left(\frac{n}{2\epsilon^2} - \frac{1}{n+2} \text{Tr } i^* \rho / 2 \right)$$

and ρ is the Ricci form of the Kähler metric associated with ϕ .

Proof.

Step 1. *An equation for f .*

Since ν is closed and ξ_o is an $(n+1, 0)$ -form, we have

$$d\xi_o = \sqrt{-1} i^* (\text{Im } \partial \log |\nu|^2) \wedge \xi_o. \quad (3.3)$$

Thus

$$\sigma := \frac{1}{n+2} (d\gamma + (i \circ \pi)^* \text{Im } \partial \log |\nu|^2)$$

is real, satisfies equation (2.2), and is determined up to a multiple of $\pi^* \theta$ (see [20], proof of proposition 3.4). Put $\sigma' = \sigma + f\pi^* \theta$. Then σ' also satisfies (2.2), and, using the transformation rules in [20], loc. cit., σ' satisfies (2.3) if and only if f satisfies

$$\sigma \wedge d\eta \wedge \bar{\eta} = \sqrt{-1} (\text{Tr } d\sigma + (2n+2)f) \sigma \wedge \pi^* \theta \wedge \eta \wedge \bar{\eta}. \quad (3.4)$$

Here η is as in part 1 of theorem 2.1. Writing $\zeta = e^{\sqrt{-1}\gamma} \pi^* \xi_o$ and contracting the equation in part 2 of theorem 2.1 with any lift of T we find $\eta = e^{\sqrt{-1}\gamma} \pi^* (T \rfloor \xi_o)$. For notational convenience we will put $\eta_o := T \rfloor \xi_o$ and

$$\sigma_o := \frac{1}{n+2} i^* (\text{Im } \partial \log |\nu|^2), \quad \text{so that} \quad \sigma = \frac{1}{n+2} d\gamma + \pi^* \sigma_o. \quad (3.5)$$

If we compute $d\eta$, expand the left hand side of equation (3.4), observe that a $(2n+2)$ -form on M is identically zero and that $\sigma_o \wedge \eta_o \wedge \bar{\eta}_o = \sigma_o(T) \theta \wedge \eta_o \wedge \bar{\eta}_o$ (since in terms of an admissible coframe $\{\theta, \theta^i, \bar{\theta}^i\}$ (see [20]), $\eta = \alpha \theta^1 \wedge \cdots \wedge \theta^n$) and finally contract with $\partial/\partial\gamma$, we obtain the following equation for f :

$$d\eta_o \wedge \bar{\eta}_o = \sqrt{-1} (\text{Tr } d\sigma_o + (2n+2)f + (n+2)\sigma_o(T)) \theta \wedge \eta_o \wedge \bar{\eta}_o. \quad (3.6)$$

Step 2. *Relate tangential derivatives of the CR structure to the transverse derivative of the complex structure.*

Note T is i -related to $\sqrt{-1}(Z - \bar{Z})$, where Z is as in section 1. Let $\xi_1 := |\nu|^{-1} \nu$ and $\eta_1 := Z \rfloor \xi_1$, so that $\xi_o = \sqrt{2} \epsilon i^* \xi_1$ and $\eta_o = \sqrt{-1} (\sqrt{2} \epsilon) i^* \eta_1$. We will show that

$$d\eta_o \wedge \bar{\eta}_o = 2\epsilon^2 i^* ((L_Z \xi_1) \wedge \bar{\eta}_1 - \bar{Z} (\log |\nu|^{-1}) \xi_1 \wedge \bar{\eta}_1). \quad (3.7)$$

Lemma 3.4. $d\eta_o \wedge \bar{\eta}_o = L_T \xi_o \wedge \bar{\eta}_o$.

Proof. Using the identity $L_T = d \circ (T\rfloor) + (T\rfloor) \circ d$ gives $L_T \xi_o \wedge \bar{\eta}_o = (d\eta_o + T\rfloor d\xi_o) \wedge \bar{\eta}_o$. Since $d\xi_o = \sqrt{-1}(n+2)\sigma_o \wedge \xi_o$ (by equation (3.3)) and $\xi_o = \theta \wedge \eta_o$ (contract the identity $\theta \wedge \xi_o = 0$ with T), we have

$$\begin{aligned} (T\rfloor d\xi_o) \wedge \bar{\eta}_o &= \sqrt{-1}(n+2)T\rfloor(\sigma_o \wedge \theta \wedge \eta_o) \wedge \bar{\eta}_o \\ &= \sqrt{-1}(n+2)(\sigma_o(T)\theta \wedge \eta_o \wedge \bar{\eta}_o - \sigma_o \wedge \eta_o \wedge \bar{\eta}_o) = 0. \end{aligned}$$

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Since T is ν -related to $\sqrt{-1}(Z - \bar{Z})$, we have $L_T \xi_o = \sqrt{-1}(\sqrt{2}\epsilon)\iota^*(L_{Z-\bar{Z}}\xi_1)$. A computation using the definition of ξ_1 and the fact that ν is closed gives $L_{\bar{Z}}\xi_1 = \bar{Z}(\log|\nu|^{-1})\xi_1$. Therefore

$$L_T \xi_o = \sqrt{-1}(\sqrt{2}\epsilon)\iota^*(L_Z \xi_1 - \bar{Z}(\log|\nu|^{-1})\xi_1)$$

and equation (3.7) follows from lemma 3.4.

Step 3. *Computing the transverse derivative of the complex structure.*

We will show that

$$(L_Z \xi_1) \wedge \bar{\eta}_1 \equiv (n(2\phi)^{-1} - Z \log|\nu|^{-1}) \xi_1 \wedge \bar{\eta}_1 \pmod{d\phi}. \quad (3.8)$$

First contract the identity $\sqrt{-1}^{n^2+n}(n+1)!\xi_1 \wedge \bar{\xi}_1 = (\partial\bar{\partial}\phi)^{n+1}$ (from equation (3.2)) with \bar{Z} to obtain

$$(-1)^{n+1}\sqrt{-1}^{n^2+n}(n+1)!\xi_1 \wedge \bar{\eta}_1 = \bar{Z}\rfloor(\partial\bar{\partial}\phi)^{n+1}. \quad (3.9)$$

We will Lie differentiate both sides of equation (3.9) with respect to Z to get an expression for $(L_Z \xi_1) \wedge \bar{\eta}_1$. To compute the derivative of the left hand side we use

Lemma 3.5.

$$L_Z(\xi_1 \wedge \bar{\eta}_1) = (L_Z \xi_1) \wedge \bar{\eta}_1 + (-(2\phi)^{-1} + Z \log|\nu|^{-1}) \xi_1 \wedge \bar{\eta}_1.$$

Proof. Use the Leibniz rule for Lie derivatives and $L_Z \bar{\xi}_1 = Z(\log|\nu|^{-1})\bar{\xi}_1$:

$$\begin{aligned} L_Z \bar{\eta}_1 &= L_Z(\bar{Z}\rfloor \bar{\xi}_1) = [Z, \bar{Z}]\rfloor \bar{\xi}_1 + \bar{Z}\rfloor L_Z \bar{\xi}_1 \\ &= (2\phi)^{-1}(Z - \bar{Z})\rfloor \bar{\xi}_1 + \bar{Z}\rfloor (Z \log|\nu|^{-1})\bar{\xi}_1 \\ &= (-(2\phi)^{-1} + Z \log|\nu|^{-1})\bar{\eta}_1 \end{aligned}$$

(the third equality uses part 3 of lemma 1.1). //

Next we compute the Lie derivative of the right hand side of equation (3.9):

$$L_Z (\bar{Z}] (\partial\bar{\partial}\phi)^{n+1}) = [Z, \bar{Z}] (\partial\bar{\partial}\phi)^{n+1} + \bar{Z}] L_Z (\partial\bar{\partial}\phi)^{n+1}.$$

By part 3 of lemma 1.1, $[Z, \bar{Z}] (\partial\bar{\partial}\phi) \equiv 0 \pmod{d\phi}$. We compute, using part 2 of lemma 1.1, $L_Z \partial\bar{\partial}\phi = -2(2\phi)^{-2} \partial\phi \wedge \bar{\partial}\phi + (2\phi)^{-1} \partial\bar{\partial}\phi$. Therefore

$$L_Z (\bar{Z}] (\partial\bar{\partial}\phi)^{n+1}) \equiv (n+1)(2\phi)^{-1} \bar{Z}] (-2(2\phi)^{-1} \partial\phi \wedge \bar{\partial}\phi \wedge (\partial\bar{\partial}\phi)^n + (\partial\bar{\partial}\phi)^{n+1}) \pmod{d\phi}.$$

Using equation (1.1) we obtain

$$\begin{aligned} L_Z (\bar{Z}] (\partial\bar{\partial}\phi)^{n+1}) &\equiv (n-1)(2\phi)^{-1} \bar{Z}] (\partial\bar{\partial}\phi)^{n+1} \\ &\equiv (n-1)(2\phi)^{-1} (-1)^{n+1} \sqrt{-1}^{n^2+n} (n+1)! \xi_1 \wedge \bar{\eta}_1 \pmod{d\phi} \end{aligned}$$

(the last equivalence is by equation (3.9)). Equating the Lie derivatives of each side of equation (3.9) gives equation (3.8).

Step 4. *The equation for f revisited.*

Using the definition (3.5) of σ_o and the fact that T is ι -related to $\sqrt{-1}(Z - \bar{Z})$, we find $(n+2)\sigma_o(T) = ((Z + \bar{Z}) \log |\nu|) \circ \iota$. Since $\iota^*(\xi_1 \wedge \bar{\eta}_1) = (2\epsilon^2)^{-1} \sqrt{-1} \xi_o \wedge \bar{\eta}_o$ (see step 2), we have by equations (3.7) and (3.8)

$$d\eta_o \wedge \bar{\eta}_o = \sqrt{-1} (n(2\epsilon^2)^{-1} + (n+2)\sigma_o(T)) \xi_o \wedge \bar{\eta}_o. \quad (3.10)$$

Substituting this in the left hand side of equation (3.6) and using $\xi_o = \theta \wedge \eta_o$ gives

$$f = \frac{1}{2n+2} \left(\frac{n}{2\epsilon^2} - \text{Tr } d\sigma_o \right)$$

Now $(n+2)d\sigma_o = \iota^*(\sqrt{-1} \partial\bar{\partial} \log |\nu|^2) = \iota^* \rho / 2$, where ρ is the Ricci form of the Kähler metric associated with ϕ . (See [3], 2.99 or [18], IX.5 eq. 26; here ν is a section of K_Ω .) With this choice of f , equation (3.6) and hence Lee's equations (2.2)–(2.3) are satisfied. //

As a corollary we find a necessary and sufficient condition for the orbits of T to be chains in term of the Ricci form ρ of h . For a function u , we denote by $d_b u$ the projection of du onto the annihilator of T , i.e., $d_b u = du - (Tu)\theta$.

Corollary 3.6. *The orbits of T are chains if and only if*

$$T] \iota^* \rho = \frac{-1}{2(n+1)} d_b (\text{Tr } \iota^* \rho). \quad (3.11)$$

Proof. By theorem 3.3, $d\sigma = \pi^*(d\sigma_o + df \wedge \theta + fd\theta)$. Since $(n+2)d\sigma_o = \iota^* \rho$, we have, in the notation of lemma 2.3,

$$\varpi = \frac{1}{n+2} \iota^* \rho + df \wedge \theta + fd\theta.$$

Contracting with T , using the expression for f in theorem 3.3 and applying lemma 2.3 proves the corollary. //

4. Applications.

Webster-Ricci and scalar curvature.

We will use the results of section 3 to express the Webster-Ricci and scalar curvatures* of M in terms of the Ricci curvature of h . We recall that Webster [26] associates to a pseudohermitian structure θ and an admissible coframe $\{\theta^\alpha\}$ a uniquely determined set of locally defined 1-forms, $\omega_\alpha^\beta, \tau^\beta$, on M satisfying

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta \\ \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} &= dh_{\alpha\bar{\beta}} \\ \tau_\alpha \wedge \theta^\alpha &= 0. \end{aligned}$$

We have used the Levi matrix $h_{\alpha\bar{\beta}}$ to raise and lower indices: $\tau_\alpha = h_{\alpha\bar{\beta}} \tau^{\bar{\beta}}$ and $\omega_{\alpha\bar{\beta}} = \omega_\alpha^\gamma h_{\gamma\bar{\beta}}$, etc. The forms ω_α^β can be interpreted geometrically as a metric connection with torsion on $T^{(1,0)}M$. The curvature of this Webster connection is

$$\Pi_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha.$$

Webster showed that this can be written

$$\Pi_\beta^\alpha = R_\beta^\alpha{}_{\rho\bar{\sigma}} \theta^\rho \wedge \theta^{\bar{\sigma}} + W_\beta^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar{\rho}} \theta^{\bar{\rho}} \wedge \theta + \sqrt{-1} \theta_\beta \wedge \tau^\alpha - \sqrt{-1} \tau_\beta \wedge \theta^\alpha.$$

Let $R_{\alpha\bar{\beta}} = R_\gamma^\gamma{}_{\alpha\bar{\beta}}$. The Webster-Ricci tensor \tilde{r} is the hermitian form on $T^{(1,0)}M$ given by

$$\tilde{r}(X, \bar{Y}) = R_{\alpha\bar{\beta}} X^\alpha Y^{\bar{\beta}} = 2\sqrt{-1} d\omega_\alpha^\alpha(X, J\bar{Y}), \quad X, Y \in T^{(1,0)}M.$$

The Webster scalar curvature is $R = h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}} = \text{Tr} \sqrt{-1} d\omega_\alpha^\alpha$.

Proposition 4.1. *Let M be given the pseudohermitian structure $\theta = i^*(-\text{Im} \bar{\partial}\phi)$ where ϕ satisfies equation (3.1). Then there is an admissible coframe $\{\theta^\alpha\}$ such that*

$$\begin{aligned} \sqrt{-1} \omega_\alpha^\alpha &= i^*(\text{Im} \partial \log |\nu|^2) + n(2\epsilon^2)^{-1} \theta \\ \sqrt{-1} \Pi_\alpha^\alpha &= i^* \rho / 2 + n(2\epsilon^2)^{-1} d\theta \\ \tilde{r} &= (i^* \tilde{\rho})|_{T^{(1,0)}M} + n(\epsilon^2)^{-1} L_\theta \\ R &= \text{Tr} i^* \rho / 2 + n^2(2\epsilon^2)^{-1} \end{aligned}$$

where ρ is the Ricci form and $\tilde{\rho}$ is the Ricci tensor of the Kähler metric associated with ϕ : $\tilde{\rho}(X, \bar{Y}) = \rho(X, J\bar{Y})$.

Proof. As in section 3, let ν be a closed local section of K_Ω and let $\xi_o = \sqrt{2} \epsilon i^*(|\nu|^{-1} \nu)$ be the corresponding volume normalized section of K_M .

* Note that in principle one can extract all the psuedoconformal invariants from the conformal class of the Fefferman metric (not just the psuedohermitian invariants). See [6].

Lemma 4.2. *There is a unitary admissible local coframe $\{\theta^\alpha\}$ such that $\xi_o = \theta \wedge \theta^1 \wedge \dots \wedge \theta^n$.*

Proof. Let $\{\tilde{\theta}^\alpha\}$ be any unitary admissible coframe. Then $\theta \wedge \tilde{\theta}^1 \wedge \dots \wedge \tilde{\theta}^n$ is volume normalized. Writing $\xi_o = h\theta \wedge \tilde{\theta}^1 \wedge \dots \wedge \tilde{\theta}^n$, the volume normalization implies $|h| = 1$. Restricting to a smaller open set and choosing a branch of the logarithm, we can put $\theta^\alpha := h^{\frac{1}{n}} \tilde{\theta}^\alpha$. //

Using this admissible coframe, we conclude from theorem 5.1 of [20] that

$$\sigma = \frac{1}{n+2} \left(d\gamma + \pi^* \left(\sqrt{-1} \omega_\alpha^\alpha - \frac{1}{2(n+1)} R\theta \right) \right).$$

Equating this with the expression for σ in theorem 3.3 and rearranging gives

$$2(n+1) (\sqrt{-1} \omega_\alpha^\alpha - (i^* \text{Im } \partial \log |\nu|^2)) = (R + n(n+2)(2\epsilon^2)^{-1} - \text{Tr } i^* \rho / 2) \theta. \quad (4.1)$$

In particular $\sqrt{-1} \omega_\alpha^\alpha - (i^* \text{Im } \partial \log |\nu|^2) \equiv 0 \pmod{\theta}$. But we can compute the θ -component of the left hand side of (4.1) as follows. We have $\eta_o = \theta^1 \wedge \dots \wedge \theta^n$ (see proof of theorem 3.3, step 1), so

$$d\eta_o = \sum_{\beta=1}^n (-1)^{\beta+1} \theta^1 \wedge \dots \wedge d\theta^\beta \wedge \dots \wedge \theta^n.$$

Using Webster's equations and $\tau^\beta \equiv 0 \pmod{\theta^\gamma}$ we find

$$d\eta_o = -\omega_\beta^\beta \wedge \eta_o \pmod{\theta^\gamma}.$$

Then

$$d\eta_o \wedge \bar{\eta}_o = -\omega_\beta^\beta \wedge \eta_o \wedge \bar{\eta}_o = -\omega_\beta^\beta(T) \theta \wedge \eta_o \wedge \bar{\eta}_o.$$

Comparing this with the right hand side of (3.10) and using (3.5) gives

$$\sqrt{-1} \omega_\beta^\beta(T) = n(2\epsilon^2)^{-1} + i^*(\text{Im } \partial \log |\nu|^2)(T).$$

From equation (4.1) we conclude

$$\sqrt{-1} \omega_\beta^\beta = i^*(\text{Im } \partial \log |\nu|^2) + n(2\epsilon^2)^{-1} \theta.$$

The rest of the theorem follows easily from the definitions. //

Remark. Combining this with corollary 3.6, we see that the orbits of T are chains if and only if $-(n+1)T \lrcorner i^* \rho = d_b R$.

Harmonic manifolds.

We will now assume that X is a harmonic manifold. Roughly speaking this means that there is a function $F: [0, \epsilon_o) \rightarrow \mathbf{R}$ such that for all $v \in TX$,

$$\det \left(\text{Exp}_{\pi(v)} \right) (v) = F(|v|).$$

Harmonic manifolds admit solutions to Laplace's equation $\Delta f = 0$ depending only on the geodesic distance $d(m, \cdot)$, and their harmonic functions satisfy the mean value property. See Besse [4] for more details. It is known that a compact harmonic manifold with finite fundamental group is a two-point homogeneous space [24], but there are examples of non-symmetric harmonic manifolds [10].

Let $\epsilon' > 0$ be small enough that the adapted complex structure exists* on $T^{*\epsilon'} X$. We recall that the g -length squared function ϕ solves the homogeneous Monge-Ampère type problem described in section 1. As above let $0 < \epsilon < \epsilon'$ and $M = \partial T^{*\epsilon} X$. We equip M with the pseudohermitian structure $\theta := \iota^*(-\text{Im} \bar{\partial} \phi)$ and let T be the corresponding transverse vector field.

Theorem 4.3. *If X is a harmonic manifold, then the orbits of T are chains on M .*

Proof. R. Aguilar has shown that if X is harmonic, then the Ricci form of the Kähler metric associated with ϕ has the form $\rho = \sqrt{-1} \partial \bar{\partial} F(\phi)$ for some (real analytic) real function F of one variable (see [1]). It follows easily that $\iota^* \rho$ is a constant multiple of $d\theta$, and that both sides of (3.11) are identically zero. //

Remark. We conclude from proposition 4.1 and the proof of theorem 4.3 that M is pseudo-Einstein, i.e., the Webster-Ricci tensor is a constant multiple of the Levi form. Thus we have many examples of compact strictly pseudoconvex CR manifolds (without transverse CR symmetry) which are globally pseudo-Einstein (cf. [11]).

5. A necessary condition for the orbits of T to be chains.

As above, let $\epsilon' > 0$ be small enough that the adapted complex structure exists on $\Omega = T^{*\epsilon'} X$ and let $M_\epsilon = \partial T^{*\epsilon} X$, $0 < \epsilon < \epsilon'$, with the pseudohermitian structure $\theta = \iota_\epsilon^*(-\text{Im} \bar{\partial} \phi)$, where $\iota_\epsilon: M_\epsilon \rightarrow \Omega$ is the inclusion map. In this section we will prove the following necessary condition for the orbits of the geodesic flow to be chains on M_ϵ for all ϵ sufficiently small.

Theorem 5.1. *If there is an $\epsilon_o > 0$ such that the orbits of the geodesic flow are chains on M_ϵ for all $0 < \epsilon \leq \epsilon_o$, then (X, g) is Einstein.*

To prove this we will first relate the Ricci tensor $\tilde{\rho}$ of the Kähler metric h associated with ϕ to the Ricci tensor Ric_X of (X, g) . Then we will expand equation (3.11) in powers of $\text{Im} z^i$, where z^i are local holomorphic coordinates such that X is given locally by $\text{Im} z^1 = \dots = \text{Im} z^{n+1} = 0$. Equating the leading terms on each side of (3.11) will show that (X, g) must be Einstein. Let $\iota_X: X \rightarrow \Omega$ be the inclusion map.

* Such an ϵ' exists for the non-compact, non-symmetric Damek-Ricci harmonic spaces (because they are homogeneous).

Proposition 5.2. $\iota_X^* \tilde{\rho} = \frac{2}{3} \text{Ric}_X$.

Proof. We recall the analytic continuation and restriction procedure of [15] and Aguilar's result [1] relating $\tilde{\rho}$ to the Ricci tensor of a certain pseudo-Riemannian metric on a neighborhood of the diagonal Δ in $X \times X$. Note that the map $j(z) = (z, \sigma(z))$, where σ is the antiholomorphic involution fixing X , embeds Ω as a totally real submanifold of $\Omega \times \Omega$. Let $C_\Delta^\omega(X \times X, \otimes^k \mathbf{CT}^*(X \times X))$ denote the set of germs of complex valued k -covariant tensors given in local (real analytic) coordinates by a convergent power series in some neighborhood of Δ . Similarly let $C_X^\omega(\Omega, \otimes^k \mathbf{CT}^* \Omega)$ denote those on Ω converging in a neighborhood of $\iota_X(X)$. There is a bijection

$$\mathcal{A}: C_\Delta^\omega(X \times X, \otimes^k \mathbf{CT}^*(X \times X)) \rightarrow C_X^\omega(\Omega, \otimes^k \mathbf{CT}^* \Omega)$$

given* by analytic continuation to $\Omega \times \Omega$ followed by restriction to $j(\Omega) \subset \Omega \times \Omega$. With our conventions, $\phi = \mathcal{A}(-d^2/4)$ where d is the Riemannian distance function. (See [15]. The constant factor can be checked by looking at \mathbf{R}^n with the usual metric.) Put $\psi = -d^2/4$ and consider the pseudo-Riemannian metric on a neighborhood \mathcal{X} of the diagonal in $X \times X$ given in local product coordinates (u_+, u_-) by

$$g_{\mathcal{X}} = 2 \frac{\partial^2 \psi}{\partial u_+^i \partial u_-^j} du_+^i \cdot du_-^j.$$

Let $\text{Ric}_{\mathcal{X}}$ denote the Ricci tensor of $g_{\mathcal{X}}$. R. Aguilar [1] showed that

$$\text{Ric}_{\mathcal{X}} = -2 \partial_{u_+^i} \partial_{u_-^j} \log \det \frac{\partial^2 \psi}{\partial u_+^k \partial u_-^l} du_+^i \cdot du_-^j. \quad (5.1)$$

From the familiar expression for $\tilde{\rho}$ in holomorphic coordinates, it follows that $\mathcal{A}(\text{Ric}_{\mathcal{X}}) = \tilde{\rho}$. Let ι_Δ denote the diagonal embedding of X in $X \times X$. Then $\iota_X^* \circ \mathcal{A} = \iota_\Delta^*$ (it suffices to check this for functions, and that is easy to see in local coordinates), and so $\iota_X^* \tilde{\rho} = \iota_\Delta^* \text{Ric}_{\mathcal{X}}$. To complete the proof of proposition 5.2 we will show that $\iota_\Delta^* \text{Ric}_{\mathcal{X}} = \frac{2}{3} \text{Ric}_X$.

Fix $x_o \in X$. Let e_i be an orthonormal frame at x_o and u^i the corresponding geodesic coordinates on X centered at x_o . Let (u_+^i, u_-^j) be the induced product coordinates on $X \times X$ near $\iota_\Delta(x_o)$. For (u_+, u_-) sufficiently close to the diagonal, let $\theta(u_+, u_-) := \left| \det \left(T \exp_{u_+} \right) \left(\exp_{u_+}^{-1}(u_-) \right) \right|$. By [27], 6.37,

$$\theta(u_+, u_-) = 2^{-n-1} \sqrt{\det g_{ij}(u_+)} \sqrt{\det g_{ij}(u_-)} \left/ \left| \det \frac{\partial^2 \psi}{\partial u_+^i \partial u_-^j} \right| \right.$$

* In terms of (real analytic) local product coordinates (u_+^i, u_-^j) on $X \times X$ and their analytic continuation to holomorphic coordinates (z^i, w^j) on $\Omega \times \Omega$, the map \mathcal{A} amounts to replacing u_+^i , resp. u_-^j , in the power series expansion by z^i , resp. \bar{z}^j .

(here g_{ij} is computed with respect to the coordinates u^i). By (5.1), we have

$$\iota_{\Delta}^* \text{Ric}_{\mathcal{X}} = 2\iota_{\Delta}^* \left(\partial_{u_+^i} \partial_{u_-^j} \log \theta \right) du^i \cdot dw^j. \quad (5.2)$$

To compute the derivatives we express $\log \theta$ in geodesic coordinates centered at u_+ . Extend the e_i to a local orthonormal frame field near x_o and let $w^i(u_+, u_-)$ be defined by $\exp_{u_+} w^i e_i = u_-$. The Riemannian volume element ω on X is

$$\omega = \theta(u_+, u_-) dw^1 \dots dw^{n+1}$$

(here we consider w^i as geodesic coordinates centered at u_+). The power series expansion of θ is ([14], corollary 9.9)

$$\theta = 1 - \frac{1}{6} R_{kl}(u_+) w^k w^l + O(w^3)$$

where R_{kl} are the components of Ric_X in the w^i coordinates (again considered as geodesic coordinates centered at u_+). Differentiating this gives

$$\iota_{\Delta}^* \left(\partial_{u_+^i} \partial_{u_-^j} \log \theta \right) (x_o) = -\frac{1}{3} R_{kl}(x_o) \frac{\partial w^k}{\partial u_+^i} \frac{\partial w^l}{\partial u_-^j} \Big|_{\iota_{\Delta}(x_o)}.$$

Differentiating the relation $\exp_{u_+} w^i e_i = u_-$ and evaluating at $u_+ = u_- = x_o$ gives $\partial_{u_-^j} w^l = \delta_j^l$ and $\partial_{u_+^i} w^k = -\delta_i^k$. Inserting this into (5.2) and noting that $w^i(x_o, \cdot) = u^i$ completes the proof of proposition 5.2. //

Example. Let $X = S^{n+1}$ with the metric g of constant sectional curvature 1. We have $\theta(u_+, u_-) = (\sin d/d)^n$ ([2], p. 57), so

$$\text{Ric}_{\mathcal{X}} = 2\partial_{u_+^i} \partial_{u_-^j} \log \left(\frac{\sin 2\sqrt{-\psi}}{2\sqrt{-\psi}} \right)^n du_+^i \cdot du_-^j.$$

Since $\mathcal{A}(\psi) = \phi$,

$$\tilde{\rho} = \mathcal{A}(\text{Ric}_{\mathcal{X}}) = 2\partial_{z^i} \partial_{\bar{z}^j} \log \left(\frac{\sinh 2\sqrt{\phi}}{2\sqrt{\phi}} \right)^n dz^i \cdot d\bar{z}^j.$$

Let $F(\phi) = (\sinh 2\sqrt{\phi}/2\sqrt{\phi})^n$. Recall $h = 2\phi_{i\bar{j}} dz^i \cdot d\bar{z}^j$ satisfies $\iota_{S^{n+1}}^* h = g$ (see section 1). Then

$$\iota_{S^{n+1}}^* \tilde{\rho} = \frac{F'(0)}{F(0)} g = \frac{2n}{3} g = \frac{2}{3} \text{Ric}_{S^{n+1}}.$$

Remark. We recall that since X is totally geodesic in (Ω, h) , it is a minimal submanifold. Using a result of Chen, Leung, and Nagano ([7], p. 51) and proposition 5.2, we conclude that if $\text{Ric}_X \leq 0$, then X is a stable minimal submanifold of (Ω, h) . If $\text{Ric}_X > 0$ and $H^1(X, \mathbf{R}) \neq 0$, then X is unstable.

We continue with the proof of theorem 5.1 by relating $\text{Tr} \iota_{\epsilon}^* \rho$ to invariants of the Kähler metric h .

Lemma 5.3. $\text{Tr } \iota_\epsilon^* \rho = \iota_\epsilon^*(s/2 - 2\phi\tilde{\rho}(Z, \bar{Z}))$ where s is the scalar curvature of h .

Proof. Choose a local unitary frame for $T^{(1,0)}\Omega$ near a point $p \in M_\epsilon$ of the form $Z_1, \dots, Z_{n+1} = \sqrt{2}\bar{\phi}Z$ with $\partial\phi(Z_\alpha) = 0$ for $\alpha = 1, \dots, n$. Then $Z \circ \iota_\epsilon$ is a local unitary frame for $T^{(1,0)}M_\epsilon$, so $\text{Tr } \iota_\epsilon^* \rho = -\sqrt{-1} \sum_1^n \rho(Z_\alpha, \bar{Z}_\alpha) \circ \iota_\epsilon = \sum_1^n \tilde{\rho}(Z_\alpha, \bar{Z}_\alpha) \circ \iota_\epsilon$. On the other hand, $(Z_i + \bar{Z}_i)/2, J(Z_i + \bar{Z}_i)/2$ form an orthonormal frame for $T\Omega$ near p , so $s = 2 \sum_1^{n+1} \tilde{\rho}(Z_i, \bar{Z}_i)$. //

We observe that the orbits of the geodesic flow are chains on M_ϵ , $0 < \epsilon \leq \epsilon'$, if and only if for all $W \in C^\infty(\Omega, T^{(1,0)}\Omega \cap \ker \partial\phi)$,

$$2\phi Z \lrcorner \tilde{\rho}(\bar{W}) = \frac{1}{2(n+1)} \bar{W}(\phi s - \tilde{\rho}(2\phi Z, 2\phi \bar{Z})). \quad (5.3)$$

Indeed this follows from equation (3.11) and lemma 5.3 after multiplying both sides of (3.11) by $2\epsilon^2$, writing $d\iota_\epsilon(T) = \sqrt{-1}(Z - \bar{Z})$, and decomposing into parts of type (1, 0) and (0, 1).

Now suppose (5.3) holds. Choose local holomorphic coordinates $z^i = x^i + \sqrt{-1}y^i$ such that $\iota_X(X)$ is given locally by $y^1 = \dots = y^{n+1} = 0$. Let $W_i = 2\phi(\partial_{z^i} - \phi_i Z)$, $i = 1, \dots, n+1$. Since $Z\phi = 1$, $W_i \in \ker \partial\phi$. Since $2\phi Z = \phi^k \partial_{z^k}$, $2\phi Z$ and W extend real analytically across $y = 0$. We will put $W = W_i$ in (5.3) and compute the first term in the expansion of each side in powers of y . Let $|y|^2 = g_{rs}y^r y^s$. A computation gives

$$\phi = |y|^2 + O(y^4), \quad \phi_i = -\sqrt{-1}g_{ir}y^r + O(y^2), \quad \phi^i = 2\sqrt{-1}y^i + O(y^2),$$

and

$$2\phi Z = 2\sqrt{-1}y^k \partial_{z^k} + O(y^2), \quad W_i = 2(|y|^2 \delta_i^k - g_{ir}y^r y^k) \partial_{z^k} + O(y^3).$$

By proposition 5.2, $\tilde{\rho} = (2/3)R_{mn}dz^m \cdot d\bar{z}^n + O(y)$ where R_{mn} is the Ricci tensor of (X, g) (in the coordinates given by the restriction of the x^i to X). Since $\bar{W}_i \phi = 0$, $\bar{W}_i(\phi s) = O(y^4)$. The expansion of (5.3) is

$$\begin{aligned} & \frac{8\sqrt{-1}}{3} (y^l R_{il} |y|^2 - g_{ir}y^r R_{kl}y^k y^l) + O(y^4) \\ &= \frac{-1}{2(n+1)} \cdot \frac{8\sqrt{-1}}{3} (y^l R_{il} |y|^2 - g_{ir}y^r R_{kl}y^k y^l) + O(y^4). \end{aligned}$$

If (5.3) holds, we must have $y^l R_{il} |y|^2 = g_{ir}y^r R_{kl}y^k y^l$, for all y and all $i = 1, \dots, n+1$. This says that for all $v, w \in TX$, $g(v, v)\text{Ric}_X(v, w) = g(v, w)\text{Ric}_X(v, v)$. If e_i is a local orthonormal frame for TX , then Ric_X must have the form $\text{Ric}_X = \sum_1^{n+1} r_i (e^i)^2$. Putting $v = \sum_1^{n+1} e_k$ and $w = e_i$ shows that all the r_i are equal, and so X must be Einstein. //

Corollary 5.4. *Suppose $\dim X = 2$ or 3 . Then the orbits of the geodesic flow are chains on M_ϵ for all ϵ sufficiently small if and only if X has constant sectional curvature.*

Proof. Since the constant curvature metrics are harmonic, the “if” part follows from theorem 4.3. The “only if” part follows from the fact that the Einstein metrics in dimension 2 or 3 are exactly those of constant sectional curvature. //

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