An inversion formula for the Segal-Bargmann transform on a symmetric space of non-compact type.

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Abstract

We prove analogs of the heat kernel transform inversion formulae of Golse, Leichtnam and the author [18, Theorems 0.3, 0.4] in the setting of a Riemannian symmetric space of Helgason's non-compact type.

Keywords: Segal-Bargmann transform, Heat kernel, non-compact symmetric space, Fourier-Bros-Iagolnitzer transform.

In [18, Theorems 0.3, 0.4] F. Golse, E. Leichtnam and the author proved inversion formulae for a heat kernel transform on a compact real analytic Riemannian manifold, X. This transform consists of applying the solution operator to the heat equation to suitable initial data and analytically continuing the result to an appropriate complexification of X. This process is known variously as the Fourier-Bros-Iagolnitzer transform [18], the Segal-Bargmann transform [8], [21], or simply the heat kernel transform [17]. [16]. In this paper we will prove analogs of the inversion formulae of [18] in the non-compact setting of a Riemannian symmetric space, G/K, of Helgason's non-compact type.

Passing to the non-compact setting introduces several essential difficulties. In the first place the techniques used in the compact case rely heavily on the discreet Fourier series of eigenfunctions of the Laplacian. In the non-compact case we must use instead the Helgason Fourier transform to express L^2 functions as continuous superpositions of eigenfunctions of the Laplacian (which are not in L^2). Second it is not clear what the complexification of G/K should be. In the compact symmetric space case it is convenient to use the group theoretic complexification, $G_{\mathbb{C}}/K_{\mathbb{C}}$, because the heat kernel and eigenfunctions of the Laplacian extend to entire functions on $G_{\mathbb{C}}/K_{\mathbb{C}}$ (see [7] and [21]). In the non-compact case the group theoretic complexification is not as useful because the presence of negative sectional curvature forces singularities in the analytic continuation of these Riemannian objects. When

G/K is of "complex type" Hall and Mitchell [9] show that the heat kernel pulls back to a meromorphic function on the complex tangent space to the identity coset with controllable singularities. In this way they are able to develop inversion and isometry formulae for the heat kernel transform in the complex case. When G/K is not of complex type the singularities in the analytically continued heat kernel are more difficult to understand. Our approach will be instead to use the "complex crown," Ξ, of G/K as the complexification of G/K (see Section 1). The attractive feature of the complex crown is that it is the largest G-invariant open set in $G_{\mathbb{C}}/K_{\mathbb{C}}$ to which the Iwasawa decomposition, and hence the eigenfunctions of G-invariant differential operators, can be analytically continued. This complexification was used by Krötz, Ólafsson and Stanton [16] to give a description of the image of the heat kernel transform.

The main result of this paper is that the inversion formulae for the heat kernel transform given in [18] on a compact manifold may be adapted to a Riemannian symmetric space of Helgason's non-compact type. Recall Ξ inherits the cotangent fibration $\pi\colon \Xi \to G/K$ and that, via the left action of G, each fiber $\pi^{-1}(gK)$ can be equipped with a Riemannian metric isometric to a normal coordinate neighborhood of the identity coset in a dual Riemannian symmetric space of the compact type (see Section 1). Let v^d_{gK}, ∇^d be the volume form, respectively gradient of this metric. Let $k(o,\cdot,t)$ be the heat kernel of the dual symmetric space with the first argument held fixed at the identity coset and let $k(gK,\cdot,t)$ be the corresponding function on $\pi^{-1}(gK)$, which we will denote by $k_t(\cdot)$. Let $\mathcal{U}_{gK}$ be a connected, relatively compact open subset of $\pi^{-1}(gK)$ containing gK with piecewise smooth boundary. Let f be a function on G/K with suitable regularity (see Theorem 1). We will show that for all $(gK,t) \in G/K \times (0,\infty)$,

$$f(gK) = \int_{z\in\mathcal{U}_{gK}} k_t(z)(e^{-t\Delta}f)(z)\, v^d_{gK}(z) + \int_0^t \int_{\partial\mathcal{U}_{gK}} \iota(V_s)\, v^d_{gK}\, ds \tag{1}$$

where V_s is the vector field

$$V_s(z) = k_s(z)\nabla^d(e^{-s\Delta}f)(z) - (e^{-s\Delta}f)(z)\nabla^d k_s(z).$$

This gives a prescription for recovering f from the analytic continuation of $e^{-t\Delta}$ to $\mathcal{U}_{gK}$ (with t fixed) together with the values of $e^{-s\Delta}f$ and its first derivatives, $0 < s < t$, on the boundary of $\mathcal{U}_{gK}$. Letting $t \to \infty$ we obtain our second inversion formula: under the same hypotheses,

$$f(gK) = \int_{s=0}^{\infty} \int_{\partial\mathcal{U}_{gK}} \iota(V_s) v^d_{gK}\, ds. \tag{2}$$

Thus one can recover f from the knowledge of $e^{-s\Delta}f$ and its first derivatives on $\partial\mathcal{U}_{gK}\times(0,\infty)$. The main tool in the proof of these results is the Helgason Fourier transform.

If we take flat Euclidean space $\mathbb{R}^n$ for both G/K and the dual space and let $\partial\mathcal{U}_{gK}$ be a sphere of radius r, then (2) reduces to a result of G. Lebeau (see e.g. [13, Equation (9.6.8)]):

$$f(x) = 2^{-1}(2\pi)^{-n} \int_0^{\infty} r^n \lambda^{n-1} \int_{|\omega|=1} e^{-\lambda r^2/2}\, (1+ < \omega, -i\nabla/r\lambda >)\, T_\lambda f(x - ir\omega)\, d\omega\, d\lambda$$

where $T_\lambda f = \lambda^{-n/2}(2\pi)^{n/2}e^{-\Delta/2\lambda}f$. As an application of our formula we give estimates on the growth of $e^{-t\Delta}f$ on $\partial\mathcal{U}_{gK}$ as $t \to 0^+$ which imply that f is a real analytic function on a neighborhood of gK in G/K (see Section 4).

1 The Inversion Formulae

We recall that a connected real analytic Riemannian manifold X is said to be a Riemannian globally symmetric space if each point $p \in X$ is an isolated fixed point of an involutive isometry, s_p, of X. Let G be the identity component of the isometry group of X and let K be the subgroup of G which fixes a (fixed) point $o \in X$. Then G acts transitively and X is real analytically diffeomorphic to G/K under the map $gK \to g \cdot o$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Conjugation by s_o gives an involutive automorphism of G and induces an involution of $\mathfrak{g}$ whose $+1$ eigenspace is $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces. The symmetric space G/K is said to be of the non-compact type if $\mathfrak{g}$ is a non-compact semisimple Lie algebra (see the definition in [10], page 130) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. In the sequel we will assume that G/K is of non-compact type.

Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\Omega \subset \mathfrak{a}$ be the convex, open subset of $\mathfrak{a}$ defined by

$$\Omega = \{X \in \mathfrak{a}\colon |\alpha(X)| < \frac{\pi}{2} \text{ for all } \alpha \in \Delta.\}$$

The complex crown, Ξ, of G/K is

$$\Xi = G\exp(\sqrt{-1}\,\Omega) \cdot o.$$

Let $\mathrm{Ad}(K)\Omega = \{\mathrm{Ad}(k)X\colon k \in K, X \in \Omega\}$ and let $G \times_K \mathrm{Ad}(K)\Omega$ be $G \times \mathrm{Ad}(K)\Omega$ modulo the equivalence relation $(g, V) \sim (gk, \mathrm{Ad}(k^{-1})V)$, $k \in K$. The map

$$\Phi\colon G \times_K \mathrm{Ad}(K)\Omega \to \Xi,$$

sending an equivalence class $[g, \mathrm{Ad}(k)H]$ to $gk\exp(iH) \cdot o$, is a real analytic G-equivariant diffeomorphism (see [1]). The map

$$a\colon TG/K \to G \times_K \mathfrak{p},$$

sending $d\tau(g)d\pi_{G/K}(V)$ to $[g, V]$, where τ is the action of G on G/K and $\pi_{G/K}\colon G \to G/K$ is the coset projection, is also a real analytic diffeomorphism. Thus the map

$$(\Phi \circ a)^{-1}\colon \Xi \to (\Phi \circ a)^{-1}(\Xi) \subset TG/K$$

is a real analytic diffeomorphism onto its image. In this way the domain Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ inherits the cotangent fibration (which we will continue to denote by π). Similarly the domain $(\Phi\circ a)^{-1}(\Xi)$ is the largest connected open subset in TG/K which admits the adapted canonical complex structure of [6] and [19] (see [3]).

The map $(\Phi \circ a)^{-1}\big|_{\pi^{-1}(eK)}$ identifies $\pi^{-1}(eK)$ with a normal coordinate neighborhood in a dual symmetric space of compact type. Since the isotropy subgroup K acts by isometries, the left action of G equips each fiber $\pi^{-1}(gK)$ with the Riemannian structure of a neighborhood of the origin in a dual symmetric space (see [18, Section 8]; compare with [21, Section 2] in the compact case). Let ∇^d denote the gradient with respect to the dual metric on $\pi^{-1}(gK)$ and let v^d_{gK} denote the volume form of the dual metric on on $\pi^{-1}(gK)$. For a vector field V on $\pi^{-1}(gK)$ let $\iota(V)v^d_{gK}$ denote the $n-1$ form on $\pi^{-1}(gK)$ obtained by contraction with V.

We recall Helgason's Fourier transform of a function on G/K is the function on $(\mathfrak{a}^*)_{\mathbb{C}}\times K/M$ given by

$$\hat{f}(\lambda, kM) = \int_{G/K} f(x)e^{(-i\lambda+\rho)(A(x,kM))}\,dx$$

defined for all (λ, kM) for which the integral exists. The Fourier transform is a bijection from the space of smooth, compactly supported functions on G/K onto a space of entire functions of exponential type in λ enjoying a certain Weyl invariance property (see [11], Chapter III, Theorem 5.1). For $f \in C_c^\infty(G/K)$, the Fourier transform is inverted by

$$f(x) = w^{-1}\int_{\mathfrak{a}^*}\int_{K/M} e^{(i\lambda+\rho)(A(x,kM))}\hat{f}(\lambda, kM)\,d\mu(\lambda, kM)$$

where $d\mu$ is the Plancherel measure,

$$d\mu(\lambda, kM) = |\boldsymbol{c}(\lambda)|^{-2}\,d\lambda\,d(kM)$$

and $\boldsymbol{c}$ is Harish-Chandra's $\boldsymbol{c}$-function (see [11], Chapter III, Theorem 1.3). One has Parseval's formula

$$\int_{G/K} f_1(x)\overline{f_2(x)}\,dx = w^{-1}\int_{\mathfrak{a}^*}\int_{K/M}\hat{f}_1(\lambda, kM)\overline{\hat{f}_2}(\lambda, kM)\,d\mu$$

which implies that the Fourier transform and the inversion formula extend to $L^2(G/K)$ (in the L^2 sense).

Our goal is to prove the following inversion formulae.

Theorem 1. *Suppose f is a square-integrable continuous function on G/K and $(|\lambda|^2+|\rho|^2)^p\,\hat{f} \in L^2(\mathfrak{a}^*\times K/M, d\mu)$ where $p > (n-1)/2 + n/4$. Then:*

1. *For all $(gK, t) \in G/K \times (0,\infty)$,*

$$f(gK) = \int_{z\in\mathcal{U}_{gK}} k_t(z)e^{-t\Delta}f(z)\,v^d_{gK}(z) + \int_0^t\int_{\partial\mathcal{U}_{gK}}\iota(V_s)\,v^d_{gK}\,ds.$$

where V_s is the vector field

$$V_s(z) = k_s(z)\nabla^d(e^{-s\Delta}f)(z) - (e^{-s\Delta}f)(z)\nabla^d k_s(z).$$

2. For all $gK \in G/K$,

$$f(gK) = \int_0^\infty \int_{\partial \mathcal{U}_{gK}} \iota(V_s) v_{gK}^d \, ds.$$

Remark 1. By the Plancherel Theorem and the self-adjointness of Δ^k it suffices to assume that $f \in C^{2p}(G/K) \cap L^2(G/K)$ and $\Delta^p f \in L^2(G/K)$.

Remark 2. The second integral on the right hand side of part 1 should be interpreted as an improper integral,

$$\lim_{\delta \to 0^+} \int_\delta^t \int_{\partial \mathcal{U}_{gK}} \iota(V_s) \, v_{gK}^d \, ds.$$

The integral on the right hand side of part 2 should be interpreted as

$$\lim_{\delta \to 0^+} \int_\delta^T \int_{\partial \mathcal{U}_{gK}} \iota(V_s) \, v_{gK}^d \, ds + \lim_{t \to \infty} \int_T^t \int_{\partial \mathcal{U}_{gK}} \iota(V_s) \, v_{gK}^d \, ds.$$

2 The Basic Computation

Let f be an L^2 function on G/K. Then $e^{-t\Delta} f$ extends to a holomorphic function on Ξ (see Proposition 2). We will regard $\mathcal{U}_{gK} \subset \pi^{-1}(gK)$ as a normal coordinate neighborhood of the point gK in a Riemannian manifold isometric to a symmetric space of compact type dual to G/K. So $\mathcal{U}_{gK}$ comes equipped with a Riemannian metric g^d, a Riemannian volume n-form v_{gK}^d, Laplacian Δ^d, and a heat kernel $k(w, z, s)$. One has $0 = \partial_s \left(e^{-s\Delta} f\right) + \Delta \left(e^{-s\Delta} f\right)$ on G/K. By analytic continuation, $e^{-s\Delta} f$ satisfies a "backwards" heat equation on $\pi^{-1}(gK)$:

$$0 = \partial_s \left(e^{-s\Delta} f\right) - \Delta^d \left(e^{-s\Delta} f\right) \tag{3}$$

(see [18, Section 6]). Let $k_s(z)$ denote $k(gK, z, s)$. Multiplying Equation (3) by $k_s(z)$ and integrating over $(\delta, t) \times \mathcal{U}_{gK}$ gives, for $0 < \delta < t$,

$$0 = \int_\delta^t \int_{\mathcal{U}_{gK}} k_s(z) \partial_s e^{-s\Delta} f(z) \, v_{gK}^d(z) \, ds - \int_\delta^t \int_{\mathcal{U}_{gK}} k_s(z) \Delta^d e^{-s\Delta} f(z) \, v_{gK}^d(z) \, ds. \tag{4}$$

As in [18, Section 7] and [21], we make the following computation. In the first integral in Equation (4) we interchange the order of integration (this is permissible because the integrand is smooth on $[\delta, t] \times \overline{\mathcal{U}_{gK}}$) and integrate by parts in s to find

$$\begin{aligned}\int_\delta^t \int_{\mathcal{U}_{gK}} k_s(z) \partial_s e^{-s\Delta} f(z) \, v_{gK}^d(z) \, ds &= \int_{\mathcal{U}_{gK}} k_t^d(z) e^{-t\Delta} f(z) \, v_{gK}^d(z) \\ &- \int_{\mathcal{U}_{gK}} k_\delta^d(z) e^{-\delta\Delta} f(z) \, v_{gK}^d(z) - \int_{\mathcal{U}_{gK}} \int_\delta^t e^{-s\Delta} f(z) \partial_s k_s(z) \, ds \, v_{gK}^d(z).\end{aligned} \tag{5}$$

In the second integral in Equation (4) we apply Green's Theorem in the form

$$\int_{\mathcal{U}_{gK}} k_s(z) \Delta^d e^{-s\Delta} f(z) \, v_{gK}^d(z) = \int_{\mathcal{U}_{gK}} e^{-s\Delta} f(z) \Delta^d k_s(z) \, v_{gK}^d(z) - \int_{\partial \mathcal{U}_{gK}} \iota(V_s) \, v_{gK}^d \tag{6}$$

where V_s is the vector field

$$V_s(z) = k_s(z)\nabla^d(e^{-s\Delta}f)(z) - (e^{-s\Delta}f)(z)\nabla^d k_s(z)$$

and ∇^d is the gradient with respect to the dual metric g^d. Inserting Equations (5) and (6) into (4) and using the fact that k_s satisfies the *forward* heat equation $0 = \partial_s k_s + \Delta^d k_s$ on $\pi^{-1}(gK)$ gives

$$\int_{\mathcal{U}_{gK}} k_\delta(z)(e^{-\delta\Delta}f)(z)v^d_{gK}(z) = \int_{\mathcal{U}_{gK}} k_t(z)(e^{-t\Delta}f)(z)v^d_{gK}(z) + \int_\delta^t \int_{\partial\mathcal{U}_{gK}} \iota(V_s)v^d_{gK}\,ds. \qquad (7)$$

3 The Proof of Theorem 1

Only two of the three integrals in Equation (7) depend on δ. To prove Theorem 1.1 it suffices to show that

$$\lim_{\delta\to 0^+}\int_{\mathcal{U}_{gK}} k_\delta(z)(e^{-\delta\Delta}f)(z)v^d_{gK}(z) = f(gK). \qquad (8)$$

It then follows automatically that the second integral on the right hand side of (7) converges (as an improper integral). Note both factors in the integrand on the left side of (8) become singular as δ goes to zero: $k_\delta(z)$ approaches a delta function centered at gK and $\left(e^{-\delta\Delta}f\right)(z)$ will not necessarily converge (unless $z \in G/K$). In this section we will prove Theorem 1.1 by proving (8) (see Proposition 3).

The following is the analog of Boutet de Monvel's estimate [2] for the growth of the eigenfunctions of the Laplacian on a compact Riemannian manifold in the complex domain (see also [17, Equation (6.5)], and [15, Lemma 2.1.2]). Let a be the analytic continuation of the middle Iwasawa projection from $G = NAK$ to $G\exp(i\Omega)K_{\mathbb{C}}$. For all $H \in \Omega$ we have $a(G\exp(iH)) \subset A\exp(i\,\mathrm{conv}(\mathcal{W}H)) \subset A\exp(i\Omega)$ where $\mathrm{conv}(\mathcal{W}H)$ is the convex hull of the orbit $\mathcal{W}H$ of H under the action of the Weyl group [5, Theorem A]. The middle Iwasawa projection descends to a holomorphic map from Ξ to $A_{\mathbb{C}}$.

Proposition 1. *Let Q be a relatively compact subset of Ξ and let $z \in Q$. Then for all $\lambda \in \mathfrak{a}^*$*

$$|a(z)^{(i\lambda+\rho)}| \le C_Q e^{|\lambda||H|}$$

where $C_Q = \max_{z\in Q} e^{\rho(\mathrm{Re}\log(a(z)))}$ and H is the unique element of $\mathfrak{a}$ such that

$$z = g\exp(iH)K_{\mathbb{C}} \in G\exp(i\Omega)K_{\mathbb{C}}.$$

Remark 3. If we identify Ξ with an open subset of T^*G/K then $|H|$ is the length of the cotangent vector corresponding to z. If $z \in \mathcal{U}_{gK}$ and we identify $\mathcal{U}_{gK}$ with an open subset of a Riemannian manifold isometric to the dual symmetric space, then $|H|$ is the Riemannian distance from z to gK.

Proof. The exponential map is a diffeomorphism from $\mathfrak{a}+i\Omega$ onto $A\exp(i\Omega)$ and by definition

$$a(z)^{i\lambda+\rho} = e^{(i\lambda+\rho)(\log a(z))}.$$

Then for $z \in Q$ we have

$$\begin{aligned}|a(z)^{i\lambda+\rho}| &= |e^{(i\lambda+\rho)(\operatorname{Re}\log a(z)+i\operatorname{Im}\log a(z))}| \\ &= e^{\rho(\operatorname{Re}\log a(z))}e^{-\lambda(\operatorname{Im}\log a(z))} \\ &\leq C_Q e^{-\lambda(\operatorname{Im}\log a(z))}\end{aligned}$$

where $C_Q = \sup_{z\in Q} e^{\rho(\operatorname{Re}\log a(z))}$ is finite since Q is a relatively compact subset of Ξ. Now $-\lambda(\operatorname{Im}\log a(z)) \leq |\lambda||\operatorname{Im}\log a(z)|$. To prove the proposition we will show that

$$|\operatorname{Im}\log a(z)| \leq |H|.$$

By the Gindikin-Krötz Complex Convexity Theorem [5, Theorem A] we may write

$$a(z) = \exp(B)\exp(iY)$$

with $B \in \mathfrak{a}$ and $Y \in \operatorname{conv}(\mathcal{W}H) \subset \mathfrak{a}$. Since $[B, iY] = 0$ we have $a(z) = \exp(B+iY)$. Then $\log a(z) = B + iY$ and so $\operatorname{Im}\log a(z) = Y \in \operatorname{conv}(\mathcal{W}H)$. Since $|\mathcal{W}H| = |H|$, $\operatorname{conv}(\mathcal{W}H)$ is contained in the ball of radius $|H|$ in $\mathfrak{a}$ and so $|\operatorname{Im}\log a(z)| \leq |H|$. □

Recall the spectral resolution of the heat operator is, for $gK \in G/K$ and $t > 0$,

$$\left(e^{-t\Delta}f\right)(gK) = w^{-1}\int_{\mathfrak{a}^*\times K/M} e^{-t(|\lambda|^2+|\rho|^2)}\hat{f}(\lambda, kM)a(k^{-1}gK)^{i\lambda+\rho}\,d\mu \tag{9}$$

where w is the order of the Weyl group and $d\mu$ is the Plancherel measure on $\mathfrak{a}^* \times K/M$ (see [17, Equation (6.8)]). Note that the right hand side of (9) is the inverse Fourier transform of $e^{-t(|\lambda|^2+|\rho|^2)}\hat{f}$. We will prove the following well-known results about solving the heat equation via the spectral resolution (9) to prepare for the analogous results we will need in the complex domain. One can also solve the heat equation (in L^p) by convolution with the heat kernel. This is not as useful for our purposes as (9) because it would require information about the analytic continuation of the heat kernel to the complex domain which is not as readily available.

Lemma 1. *Let f be a square-integrable continuous function on G/K. Let $I(f)(t, gK)$ denote the integral on the right hand side of (9).*

1. *Fix a compact set Q contained in the real locus G/K and $t_o > 0$. Let P be a polynomial and h a continuous function on G/K. Then there is an $H \in L^1(\mathfrak{a}^* \times K/M, d\mu)$ such that for all $(gK, t) \in Q \times (t_o, \infty)$,*

$$|e^{-t(|\lambda|^2+|\rho|^2)}\hat{f}(\lambda, kM)a(k^{-1}gK)^{i\lambda+\rho}P(|\lambda|)h(gK)| \leq H(\lambda, kM).$$

2. $(\Delta + \partial_t) I(f)(t, gK) = 0$ *on* $(0, \infty) \times G/K$.

3. $L^2 - \lim_{t\to 0^+} I(f)(t, \cdot) = f$.

4. Let l *be greater than* $n/4$. *If* $f \in L^2(G/K)$ *and* $(|\lambda|^2 + |\rho|^2)^l \hat{f} \in L^2(\mathfrak{a}^* \times K/M, d\mu)$, *then for all* $gK \in G/K$, $\lim_{t\to 0^+} I(f)(t, gK) = f(gK)$ *pointwise.*

Proof. Note that $|a(k^{-1}gK)^{i\lambda+\rho}| = |a(k^{-1}gK)^{\rho}|$. Then $C_Q := \sup_{gK\in Q, k\in K} |a(k^{-1}gK)^{i\lambda+\rho}|$ is independent of λ and finite since a is continuous. If $P(|\lambda|)$ is a polynomial and h is a continuous function on G/K with $\sup_Q |h| = C_h$ then the absolute value in question is bounded by $C_Q C_h P(|\lambda|) e^{-t_o(\lambda^2+\rho^2)} |\hat{f}(\lambda, kM)|$. This is in $L^1(\mathfrak{a}^* \times K/M, d\mu)$ due to the exponential decay in λ and the fact that $|\boldsymbol{c}(\lambda)|^{-2}$ is bounded by a polynomial in $|\lambda|$. This proves part 1. Part 2 follows from the fact that differentiating under the integral sign in $I(f)$ can be justified by the result of part 1 and the Dominated Convergence Theorem together with the fact that $\Delta \left(a(k^{-1}gK)^{i\lambda+\rho}\right) = (|\lambda|^2 + |\rho|^2) a(k^{-1}gK)^{i\lambda+\rho}$. To prove part 3 note that by the Plancherel Theorem,

$$\|I(f)(t, \cdot) - f\|^2_{L^2(G/K)} = \int_{\mathfrak{a}^* \times K/M} \left(e^{-t(|\lambda|^2+|\rho|^2)} - 1\right)^2 |\hat{f}(\lambda, kM)|^2 \, d\mu. \tag{10}$$

Since the integrand in (10) is bounded by the $d\mu$-integrable function $|\hat{f}|^2$ we can use the Dominated Convergence Theorem to exchange limit and integral and conclude the limit as $t \to 0^+$ in (10) is 0. This proves part 3. To prove part 4 we start with a lemma.

Lemma 2. *Let* l *be greater than* $n/4$. *Suppose* f *is a square-integrable continuous function on* G/K *and* $(|\lambda|^2 + |\rho|^2)^l \hat{f} \in L^2(\mathfrak{a}^* \times K/M, d\mu)$. *Then* $\hat{f}$ *is in* $L^1(\mathfrak{a}^* \times K/M, d\mu)$ *and* f *is equal to the inverse Fourier transform of* $\hat{f}$ *at all points of* G/K.

Proof. Note

$$\int_{\mathfrak{a}^* \times K/M} |\hat{f}(\lambda, kM)| \, d\mu \le \|(|\lambda|^2 + |\rho|^2)^l \hat{f}\|_{L^2(\mathfrak{a}^* \times K/M, d\mu)} \|(|\lambda|^2 + |\rho|^2)^{-l}\|_{L^2(\mathfrak{a}^* \times K/M, d\mu)}.$$

Since $d\mu \le C(|\lambda|^2 + |\rho|^2)^{d/2} \, d\lambda \, d(kM)$ where $d = \dim(N)$ (see (15) below), K/M is compact, and $4l > n = d + \dim(\mathfrak{a}^*)$, we conclude the L^2-norm of $(|\lambda|^2 + |\rho|^2)^{-l}$ is finite. This together with the hypothesis shows $\hat{f} \in L^1(\mathfrak{a}^* \times K/M, d\mu)$. Since $a(k^{-1}gK)^{i\lambda+\rho}$ is bounded by a constant on any compact subset of G/K (see (12) with $H = 0$), Lebesgue's Dominated Convergence Theorem shows that the inverse Fourier transform of $\hat{f}$ is continuous. Since $f \in L^2(G/K)$, f is equal to the inverse Fourier transform of $\hat{f}$ in $L^2(G/K)$ and so pointwise almost everywhere. Since both are continuous, they are equal pointwise everywhere. □

To prove part 4 we use Lemma 2 to observe that for all $gK \in G/K$,

$$|I(f)(t, gK) - f(gK)| \le \int_{\mathfrak{a}^* \times K/M} |(e^{-t(|\lambda|^2+|\rho|^2)} - 1) \hat{f}(\lambda, kM) a(k^{-1}gK)^{i\lambda+\rho}| \, d\mu.$$

If Q is any compact subset of G/K, then for $(t, \lambda, gK, kM) \in [0, \infty) \times \mathfrak{a}^* \times Q \times K/M$ the integrand is dominated by $C_Q|\hat{f}(\lambda, kM)|$ (by (12) with $H = 0$). This is in $L^1(\mathfrak{a}^* \times K/M, d\mu)$ by the proof of Lemma 2. The result of part 4 now follows from the Dominated Convergence Theorem. $\square$

The following proposition is the analog of Proposition 2.4.1 in [18].

Proposition 2. *Let $f \in L^2(G/K)$ and $t > 0$.*

1. *The integral*

$$w^{-1} \int_{\mathfrak{a}^* \times K/M} e^{-t(|\lambda|^2+|\rho|^2)} \left| \hat{f}(\lambda, kM) a(k^{-1}z)^{i\lambda+\rho} \right| |\mathbf{c}(\lambda)|^{-2} \, d\lambda \, d(kM) \tag{11}$$

converges and is a bounded function of z in any fixed compact subset of Ξ. The right hand side of (9) can be analytically continued to Ξ by replacing gK by z in the integral (9).

2. *Let p be a non-negative integer, let $l > n/4$ and suppose $(|\lambda|^2 + |\rho|^2)^{p+l} \hat{f} \in L^2(\mathfrak{a}^* \times K/M, d\mu)$. If $Q \subset \Xi$ is compact, then for all $z \in Q \backslash (G/K)$ with $z = g \exp(iH) K_{\mathbb{C}}$, $g \in G$, $H \in \Omega \backslash \{0\}$, and all $t > 0$,*

$$|\left(e^{-t\Delta} f\right)(z)| \leq C_{Q,p}(H) C_{f,p+l} e^{-t|\rho|^2} t^p e^{|H|^2/4t}$$

where $H \to C_{Q,p}(H)$ is a positive locally bounded function of H on $\Omega \backslash \{0\}$, $C_{Q,0}(H) = 1$, and $C_{f,p+l}$ is the $L^2(\mathfrak{a}^* \times K/M, d\mu)$ norm of $(|\lambda|^2 + |\rho|^2)^{p+l} \hat{f}$.

Remark 4. Lemma 2 shows that the hypotheses in Proposition 2.2 imply that $\hat{f} \in L^1(\mathfrak{a}^* \times K/M, d\mu)$.

Proof of 1. Let Q be a compact subset of Ξ and $z \in Q$. Since Ξ is K-invariant the K-orbit $K \cdot Q$ is also a compact subset of Ξ. Using Proposition 1 we have

$$\sup_{k \in K, z \in Q} |a(k^{-1}z)^{i\lambda+\rho}| \leq C_{K \cdot Q} e^{|\lambda||H|} \tag{12}$$

where again we have written $k^{-1}z = k^{-1}g \exp(iH) K_{\mathbb{C}} \in G \exp(i\Omega) K_{\mathbb{C}}$. Thus

$$\int_{\mathfrak{a}^* \times K/M} e^{-t(|\lambda|^2+|\rho|^2)} \left| \hat{f}(\lambda, kM) a(k^{-1}z)^{i\lambda+\rho} \right| d\mu$$
$$\leq C_{K \cdot Q} e^{-t|\rho|^2} \int_{\mathfrak{a}^* \times K/M} e^{-t|\lambda|^2 + |\lambda||H|} \left| \hat{f}(\lambda, kM) \right| d\mu. \tag{13}$$

By the Schwarz inequality this is less than or equal to

$$C_{K \cdot Q} e^{-t|\rho|^2} \|e^{-\frac{t}{2}|\lambda|^2} \hat{f}\|_{L^2(\mathfrak{a}^* \times K/M, d\mu)} \|e^{-\frac{t}{2}|\lambda|^2} e^{|\lambda||H|}\|_{L^2(\mathfrak{a}^* \times K/M, d\mu)}.$$

We note by the Plancherel Theorem that

$$\|e^{-\frac{t}{2}|\lambda|^2}\hat{f}\|_{L^2(\mathfrak{a}^*\times K/M,d\mu)} \le \|f\|_{L^2(G/K)}.$$

Since H must lie in the bounded subset Ω of $\mathfrak{a}$ and $|\boldsymbol{c}(\lambda)|^{-2}$ can be bounded above by a polynomial in $|\lambda|$, there is a constant C_t such that

$$\|e^{-\frac{t}{2}|\lambda|^2}e^{|\lambda||H|}\|_{L^2(\mathfrak{a}^*\times K/M,d\mu)} \le C_t.$$

Thus

$$\int_{\mathfrak{a}^*\times K/M} e^{-t(|\lambda|^2+|\rho|^2)}\left|\hat{f}(\lambda,kM)a(k^{-1}z)^{i\lambda+\rho}\right| |\boldsymbol{c}(\lambda)|^{-2}\,d\lambda\,d(kM) \le C_{K\cdot Q}C_t e^{-t|\rho|^2}\|f\|_{L^2(G/K)}.$$

This shows the integral (11) converges and is bounded on compact subsets of Ξ. The analyticity of the integral obtained from (9) by formally replacing $gK \in G/K$ by $z \in \Xi$ now follows from Morera's and Fubini's Theorem. □

Remark 5. The proof shows that part one of Proposition 2 is true if we only assume that for all $t > 0$,

$$e^{-\frac{t}{2}|\lambda|^2}\hat{f} \in L^2(\mathfrak{a}^*\times K/M, d\mu).$$

Proof of 2. Using (13) we see that $|\left(e^{-t\Delta}f\right)(z)|$ is bounded above by the L^2-inner product

$$C_{K\cdot Q}e^{-t|\rho|^2}\langle\left(|\lambda|^2+|\rho|^2\right)^{-p-l}e^{-t|\lambda|^2+|\lambda||H|},\left(|\lambda|^2+|\rho|^2\right)^{p+l}|\hat{f}|\rangle_{L^2(\mathfrak{a}^*\times K/M,d\mu)}.$$

Using the Schwarz inequality we obtain

$$|\left(e^{-t\Delta}f\right)(z)| \le C_{K\cdot Q}C_{f,p+l}e^{-t|\rho|^2}\|\left(|\lambda|^2+|\rho|^2\right)^{-p-l}e^{-t|\lambda|^2+|\lambda||H|}\|_{L^2(\mathfrak{a}^*\times K/M,d\mu)}$$

where C_{p+l} is the $L^2(\mathfrak{a}^*\times K/M)$ norm of $(|\lambda|^2+|\rho|^2)^{p+l}\hat{f}$. To prove part 2 we must show that there is a positive locally bounded function $C_p(H)$ on $\Omega\backslash\{0\}$ such that

$$\|\left(|\lambda|^2+|\rho|^2\right)^{-p-l}e^{-t|\lambda|^2+|\lambda||H|}\|_{L^2(\mathfrak{a}^*\times K/M,d\mu)} \le C_p(H)t^p e^{|H|^2/4t}. \tag{14}$$

We have $|\boldsymbol{c}(\lambda)|^{-2} \le (C_1 + C_2|\lambda|^{\frac{d}{2}})^2$ where $d = \dim(N)$ [12, Chapter IV, Proposition 7.2] Since $|\rho|$ is a positive constant one can show by a limiting argument that

$$|\boldsymbol{c}(\lambda)|^{-2} \le C(|\rho|^2+|\lambda|^2)^{\frac{d}{2}}. \tag{15}$$

(Here and in the following we will use C to denote various constants independent of H, λ and t.) Using the Iwasawa decomposition, $\mathfrak{g} = \mathfrak{k}+\mathfrak{a}+\mathfrak{n}$, we note that $n = \dim\mathfrak{a}+d$. Then

$$\|\left(|\lambda|^2+|\rho|^2\right)^{-p-l}e^{-t|\lambda|^2+|\lambda||H|}\|^2_{L^2(\mathfrak{a}^*\times K/M,d\mu)} \le C\left(\int_{|\lambda|<1}+\int_{|\lambda|\ge 1}\right)\left(|\lambda|^2+|\rho|^2\right)^{-2p-2l+\frac{d}{2}}e^{-2t|\lambda|^2+2|\lambda||H|}\,d\lambda. \tag{16}$$

Since H is restricted to lie in the bounded subset $\Omega\backslash\{0\}$ of $\mathfrak{a}$, the integral over $|\lambda| < 1$ is bounded above by a constant independent of H and t. So the integral over $|\lambda| < 1$ satisfies an estimate like (14). To estimate the integral over $|\lambda| \geq 1$ we note that $(|\lambda|^2 + |\rho|^2)^{-2l+\frac{d}{2}}$ is integrable over $\mathfrak{a}$ since it is $O(|\lambda|^{-4l+d})$ and $-4l + d < -\dim(\mathfrak{a})$. Thus

$$\begin{aligned}\int_{|\lambda|\geq 1} \left(|\lambda|^2 + |\rho|^2\right)^{-2p-2l+\frac{d}{2}} e^{-2t|\lambda|^2+2|\lambda||H|}\, d\lambda &\leq C \sup_{|\lambda|\geq 1} \left(|\lambda|^2 + |\rho|^2\right)^{-2p} e^{-2t|\lambda|^2+2|\lambda||H|} \\ &\leq C \left(\sup_{|\lambda|\geq 1} |\lambda|^{-2p} e^{-t|\lambda|^2+|\lambda||H|}\right)^2 .\end{aligned}$$

To complete the proof of part 2 we will reprove in the present context Lemma 2.5 of [18].

Lemma 3 (Lemma 2.5 of [18])**.** *For all non-negative integers p and all $t > 0$,*

$$\sup_{|\lambda|\geq 1} |\lambda|^{-2p} e^{-t|\lambda|^2+|\lambda||H|} \leq t^p e^{|H|^2/4t} C_p(H) < \infty$$

where $H \to C_p(H)$ is a positive, locally bounded function on $\Omega\backslash\{0\}$ and $C_0(H) = 1$.

Proof. Completing the square and putting $\sqrt{t}|\lambda| = v$ gives

$$\sup_{\lambda\geq 1} |\lambda|^{-2p} e^{-t|\lambda|^2+|\lambda||H|} \leq t^p e^{|H|^2/4t} \sup_{\substack{v>0 \\ \lambda\geq 1}} v^{-2p} e^{-(|\lambda||H|/2-v^2)^2/v^2} .$$

For $p = 0$ the conclusion is obvious so we will assume $p \geq 1$. If $0 < v \leq \sqrt{|H|}/2$ then, since $|\lambda| \geq 1$, we have $\left(|\lambda||H|/2 - v^2\right)^2 \geq |H|^2/16$ and so

$$v^{-2p} e^{-(|\lambda||H|/2-v^2)^2/v^2} \leq v^{-2p} e^{-|H|^2/16v^2} \leq A_p(H) < \infty$$

where $A_p(H) = (16/|H|^2)^p\, p^p e^{-p}$ is positive and locally bounded on $\Omega\backslash\{0\}$. If $v > \sqrt{|H|}/2$ then

$$v^{-2p} e^{-(|\lambda||H|/2-v^2)^2/v^2} < v^{-2p} \leq B_p(H) < \infty$$

where $B_p(H) = (4/|H|)^p$. Then we can take $C_p(H) = \max(A_p(H), B_p(H))$. □

This shows that the integral in (16) satisfies the estimate in (14). □

Proposition 3. *Let $p > (n-1)/2 + n/4$. Suppose f is a square-integrable continuous function on G/K and $(|\lambda|^2 + |\rho|^2)^p \hat{f} \in L^2(\mathfrak{a}^* \times K/M, d\mu)$. Then*

$$\lim_{\delta\to 0^+} \int_{\mathcal{U}_{gK}} k_\delta^d(z)(e^{-\delta\Delta}) f(z)\, v_{gK}^d(z) = f(gK). \tag{17}$$

Proof. We write (17) in terms of the spectral decomposition and reverse the order of integration (which is legitimate for all $\delta > 0$ since the factor $e^{-\delta(|\lambda|^2+|\rho|^2)}$ guarantees convergence; note $\mathcal{U}_{gK}$ is a relatively compact subset of Ξ so $a(k^{-1}z)^{i\lambda+\rho}$ is a smooth function of k, z, and λ):

$$\lim_{\delta\to 0^+}\int_{\mathcal{U}_{gK}} k^d_\delta(z)(e^{-\delta\Delta})f(z)\, v^d_{gK}(z)$$
$$= \lim_{\delta\to 0^+} w^{-1}\int_{\mathfrak{a}^*\times K/M}\hat{f}(\lambda,kM)e^{-\delta(|\lambda|^2+|\rho|^2)}\int_{\mathcal{U}_{gK}} k^d_\delta(z)a(k^{-1}z)^{i\lambda+\rho}\, v^d_{gK}(z)\, d\mu. \quad (18)$$

The properties of the dual heat kernel k^d_δ imply that

$$\lim_{\delta\to 0^+} e^{-\delta(|\lambda|^2+|\rho|^2)}\int_{\mathcal{U}_{gK}} k^d_\delta(z)a(k^{-1}z)^{i\lambda+\rho}\, v^d_{gK}(z) = a(k^{-1}gK).$$

If we can interchange the limit as $\delta \to 0$ with the integral over $\mathfrak{a}^* \times K/M$ in (18) then

$$\lim_{\delta\to 0^+}\int_{\mathcal{U}_{gK}} k^d_\delta(z)(e^{-\delta\Delta})f(z)v(z) = w^{-1}\int_{\mathfrak{a}^*\times K/M}\hat{f}(\lambda,kM)a(k^{-1}gK)^{i\lambda+\rho}\, d\mu.$$

By Lemma 2 this is equal to $f(gK)$ for all $gK \in G/K$. To prove the Proposition we need to justify the interchange of limit and integral in (18) by showing that the integrand (of the integration over $\mathfrak{a}^*\times K/M$) on the right hand side of (18) can be dominated by a $d\mu$-integrable function independent of δ. Since $\mathcal{U}_{gK}$ is a relatively compact subset of a normal coordinate neighborhood in the dual compact symmetric space we can use the small time, off-diagonal estimate of Kannai [14] which implies that there is a $T > 0$ such that for $(\delta, z) \in (0, T)\times\mathcal{U}_{gK}$,

$$0 < k^d_\delta(z) \le C(4\pi\delta)^{-n/2}e^{-d^2(gK,z)/4\delta}. \quad (19)$$

Here $d(gK, z)$ is the distance function in the dual compact symmetric space from the base point gK to z. This is equal to the length of the cotangent vector corresponding to z. Using the estimate (19), Proposition 1 and Remark 3 we estimate the integrand of (18) (for $0 < \delta < T$) as

$$\left|\hat{f}(\lambda,kM)e^{-\delta(|\lambda|^2+|\rho|^2)}\int_{\mathcal{U}_{gK}} k^d_\delta(z)a(k^{-1}z)^{i\lambda+\rho}\, v^d_{gK}(z)\right|$$
$$\le C|\hat{f}(\lambda,kM)|e^{-\delta(|\lambda|^2+|\rho|^2)}(4\pi\delta)^{-n/2}\int_{\mathcal{U}_{gK}} e^{-d^2(gK,z)/4\delta+|\lambda|d(gK,z)}\, v^d_{gK}(z) \quad (20)$$

(again we use C to denote various constants). Since $\mathcal{U}_{gK}$ is a relatively compact subset of a normal coordinate neighborhood in the dual compact symmetric space we can estimate the Riemannian volume v^d_{gK} and its associated measure by a constant multiple of Lebesgue measure in exponential coordinates centered at gK. Then the integral in (20) can be estimated by a Gaussian integral over $\mathbb{R}^n$:

$$\int_{\mathcal{U}_{gK}} e^{-d^2(gK,z)/4\delta+|\lambda|d(gK,z)}\, v^d_{gK}(z) \le C\int_{\mathbb{R}^n} e^{-|v|^2/4\delta+|\lambda||v|}\, dv. \quad (21)$$

The integral on the right hand side of (21) can be estimated by changing to polar coordinates and completing the square:

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-|v|^2/4\delta+|\lambda||v|}\,dv &\le C\int_{-\infty}^{\infty} e^{-r^2/4\delta+|\lambda|r}|r|^{n-1}\,dr \\ &= Ce^{\delta|\lambda|^2}\int_{-\infty}^{\infty} e^{-(r/2\sqrt{\delta}-|\lambda|\sqrt{\delta})^2}|r|^{n-1}\,dr \\ &\le Ce^{\delta|\lambda|^2}(2\sqrt{\delta})^n\int_{-\infty}^{\infty} e^{-w^2}(|w|+|\lambda|\sqrt{\delta})^{n-1}\,dr \\ &= Ce^{\delta|\lambda|^2}(2\sqrt{\delta})^n\sum_{j=0}^{n-1} c_j\left(|\lambda|\sqrt{\delta}\right)^j \qquad (22)\end{aligned}$$

where $c_j = \binom{n-1}{j}\int_{-\infty}^{\infty} e^{-w^2}|w|^{n-1-j}\,dw > 0$. Then (22), (21) and (20) shows that the integrand of (18) is bounded by

$$C|\hat{f}(\lambda,kM)|\sum_{j=1}^{n-1} c_j e^{-\delta|\rho|^2}\sqrt{\delta}^j|\lambda|^j$$

Since the factors $e^{-\delta|\rho|^2}\sqrt{\delta}^j$ are bounded on $(0,\infty)$ by a constant, the integrand of (18) can be estimated by a constant times $|\hat{f}(\lambda,kM)|\,(|\lambda|^2+|\rho|^2)^{(n-1)/2}$. Using Lemma 2 the hypothesis of Proposition 3 implies that $|\hat{f}(\lambda,kM)|\,(|\lambda|^2+|\rho|^2)^{(n-1)/2}$ is in $L^1(\mathfrak{a}^*\times K/M, d\mu)$. Then we may apply the Dominated Convergence Theorem to interchange limit and integral in (18). □

This completes the proof of Theorem 1.1. To prove Theorem 1.2 it suffices to show that

$$\lim_{t\to\infty}\int_{z\in\mathcal{U}_{gK}} k_t^d(z)\left(e^{-t\Delta}f\right)(z)\,v_{gK}^d(z) = 0. \qquad (23)$$

There is a positive constant C such that for all $t>0$ and all $z\in\mathcal{U}_{gK}$, $0<k_t^d(z)\le C$ (see e.g., [4, Theorem 5.5.6]). By Proposition 2.2 we have for all $t>0$ and all $z\in\mathcal{U}_{gK}$,

$$|\left(e^{-t\Delta}f\right)(z)| \le Ce^{-t|\rho|}e^{|H|^2/4t}.$$

Since $|\rho|>0$ and $\mathcal{U}_{gK}$ is relatively compact, (23) follows immediately. This proves Theorem 1.2.

4 Analytic Regularity

Following [20, Definition 3.2.3] we say that f is microlocally exponentially small at a point $(x_o,-\xi_o)\in T^*G/K$ if there is a neighborhood $\mathcal{Z}$ in Ξ of the point z_o corresponding to

$(x_o, -\xi_o)$ under the cotangent identification and positive constants C, δ such that for all z in $\mathcal{Z}$ and all $0 < t \le 1$ we have

$$|e^{-t\Delta} f(z)| \le C e^{-d(z_o,x_o)^2/4t} e^{-\delta/t} \qquad (24)$$

where $d(z_o, x_o)$ is the distance from z_o to x_o with respect to the dual symmetric space metric on the fiber over x_o (see Remark 3).

Theorem 2. *Suppose f satisfies the hypotheses of Theorem 1 and is microlocally exponentially small at all points ξ in a cosphere fiber over g_oK,*

$$S^{*r}_{g_oK} G/K := \{(g_oK, \xi) \colon |\xi| = r\},$$

where r is small enough that $S^{*r}_{g_oK}G/K$ can be identified with a subset of Ξ. Then there is a neighborhood $\mathcal{V}$ of g_oK in G/K and a real analytic function on $\mathcal{V}$ which is equal to f on $\mathcal{V}$.

Proof. We apply the inversion formula with $\mathcal{U}_{gK}$ the subset of Ξ corresponding to the set

$$\left\{\xi \in T^*_{gK}G/K \colon |\xi| < r\right\}.$$

It will be convenient to pull back the integrals in the inversion formula, Theorem 1, part 1, to the identity coset, o, by the left action of g, $\tau(g)$. By the construction of the metric on $\mathcal{U}_{gK}$ this map is an isometry from $\mathcal{U}_o$ equipped with the dual symmetric space metric to $\mathcal{U}_{gK}$. This gives

$$\begin{aligned} f(gK) &= \int_{\mathcal{U}_o} \tau(g)^* \left(k_t(e^{-t\Delta} f)\, v^d_{gK}\right) + \int_0^t \int_{\partial\mathcal{U}_o} \tau(g)^* \left(\iota(V_s)\, v^d_{gK}\right) ds. \\ &= \int_{\mathcal{U}_o} k_t(z) \tau(g)^* (e^{-t\Delta} f)(z)\, v^d_o(z) \\ &\qquad + \int_0^t \int_{\partial\mathcal{U}_o} \iota\left(k_s \nabla^d \left(\tau(g)^* e^{-s\Delta} f\right) - \tau(g)^*(e^{-s\Delta} f) \nabla^d k_s\right) v^d_o\, ds. \end{aligned} \qquad (25)$$

We claim that the first integral on the right hand side of the second equality in (25) is a real analytic function of gK on a neighborhood of g_oK independent of any estimates on $e^{-t\Delta}f$ (since t is fixed). There is a neighborhood $\mathcal{L}$ of g_oK in $G_{\mathbb{C}}$ such that for each $z \in \mathcal{U}_o$ the map $g \mapsto e^{-t\Delta} f(gz)$, thought of as a real analytic function on $\mathcal{L} \cap G$, can be analytically continued to a holomorphic function on $\mathcal{L}$ which is uniformly bounded for $(g, z) \in \mathcal{L} \times \overline{\mathcal{U}_o}$. Since the integrand is uniformly bounded, depends holomorphically on $g \in G_{\mathbb{C}}$ and $\overline{\mathcal{U}_o}$ is compact, we can apply Cauchy's estimates and the Dominated Convergence Theorem to justify differentiating inside the integral and conclude the integral is a holomorphic function on $\mathcal{L}$. The integral is right K-invariant (since K acts by isometries on $\mathcal{U}_o$) so its restriction to $G \cap \mathcal{L}$ descends to a real-analytic function on a neighborhood of g_oK in G/K. This proves the claim.

It remains to show that the second integral on the right hand side of (25) is real analytic on a neighborhood of g_oK. We can choose a neighborhood $\mathcal{W}$ of g_o in $G_{\mathbb{C}}$ such that for each $(z,s) \in \partial\mathcal{U}_o \times (0,1]$ the maps

$$G \ni g \mapsto e^{-s\Delta} f(gz), \qquad G \ni g \mapsto \nabla^d(\tau(g)^* e^{-s\Delta} f)(z)$$

can be analytically continued to holomorphic functions on $\mathcal{W}$. This shows that the integrand in the second integral in (25) extends to a holomorphic function of g on $\mathcal{W}$. Using the hypothesis and Cauchy's estimates we can find (after shrinking $\mathcal{W}$) positive constants C, δ such that for all $(g,z,s) \in \mathcal{W} \times \partial\mathcal{U}_o \times (0,1]$,

$$\max\left(|e^{-s\Delta} f(gz)|, |\nabla^d\left(\tau(g)^* e^{-s\Delta} f\right)|\right) \leq C e^{d^2(z,o)/4s} e^{-\delta/s}.$$

(Here we have used $d(z,o) = r$ for all $z \in \partial\mathcal{U}_o$.) We also have Gaussian estimates for k_s and $\nabla^d k_s$ ([14, Remark 3.1]): there is a constant C such that for all $(z,s) \in \partial\mathcal{U}_o \times (0,1]$,

$$\max\left(k_s(z), \nabla^d k_s(z)\right) \leq C(4\pi s)^{-n/2-1} e^{-d^2(z,o)/4s}.$$

Multiplying these estimates together shows that in any local coordinate system (U, u_i) on $\partial\mathcal{U}_o$ there is a positive constant C' such that for all $(g,z,s) \in \mathcal{W} \times U \times (0,1]$,

$$|\iota\left(k_s \nabla^d\left(\tau(g)^* e^{-s\Delta} f\right) - \tau(g)^*(e^{-s\Delta} f)\nabla^d k_s\right) v_o^d| \leq C' s^{-n/2-1} e^{-\delta/s} du^1 \cdots du^{n-1}.$$

Since $\partial\mathcal{U}_o$ is compact we can again apply Cauchy's estimates and the Dominated Convergence Theorem to differentiate under the integral and prove that the second integral on the right hand side of (25) extends to a holomorphic function of g on $\mathcal{W}$. As above the K-invariance of the integral implies that its restriction to $\mathcal{W} \cap G$ descends to a real analytic function on a neighborhood of g_oK in G/K. □

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