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The Segal–Bargmann Transform on a Symmetric Space of Compact Type

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We study the Segal–Bargmann transform on a symmetric space X of compact type, mapping $L^2(X)$ into holomorphic functions on the complexification $X_{\mathbf{C}}$. We invert this transform by integrating against a "dual" heat kernel measure in the fibers of a natural fibration of $X_{\mathbf{C}}$ over X. We prove that the Segal–Bargmann transform is an isometry from $L^2(X)$ onto the space of holomorphic functions on $X_{\mathbf{C}}$ which are square integrable with respect to a natural measure. These results extend those of B. Hall in the compact group case.

1. INTRODUCTION

It is well known that a solution of the heat equation on $\mathbf{R}^n$ can be analytically continued to an entire function on $\mathbf{C}^n$. The classical Segal–Bargmann transform, in the form we will be interested in, is the map

$$f \in L^2(\mathbf{R}^n, dx) \to e^{-t\Delta} f \in \mathcal{O}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, (2\pi t)^{-n/2} e^{-|\mathrm{Im}\, z|^2/2t}\, dz),$$

where $\mathcal{O}(\mathbf{C}^n)$ denotes the holomorphic functions on $\mathbf{C}^n$. This is an isometric isomorphism of Hilbert spaces, identifying $L^2(\mathbf{R}^n)$ with a natural L^2 space of entire functions on $\mathbf{C}^n$. B. Hall has obtained an interesting generalization of the Segal–Bargmann transform by replacing $\mathbf{R}^n$ with a compact Lie group H equipped with a bi-invariant metric [5]. Hall proved that solutions of the heat equations on H can be analytically continued to entire functions on the complexified Lie group $H_{\mathbf{C}}$, and that the Segal–Bargmann transform

$$f \in L^2(H, dh) \to e^{-t\Delta} f \in \mathcal{O}(H_{\mathbf{C}}) \cap L^2(H_{\mathbf{C}}, dh\, \sigma_{t/2}(p)\, dp)$$

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is an isometric isomorphism (we consider the "K-averaged" version of the transform). Here $P=\exp\sqrt{-1}\,\mathfrak{h}$, $H_{\mathbf{C}}$ is identified with $H\times P$, dh is Riemannian measure, and $\sigma_t(p)\,dp$ is the heat kernel measure for the heat operator $\partial_t+\Delta_P$, $\Delta_P\geqslant 0$, on P identified with the symmetric space $H_{\mathbf{C}}/H$ by the quotient map. In a subsequent paper [6], Hall proved the following "fiberwise" inversion formula for the Segal–Bargmann transform: for $f\in C^\infty(H)$,

$$f(h)=\int_P e^{-t\Delta}f(hp^2)\,\sigma_{t/4}(p)\,dp, \tag{1}$$

and used this to give an elegant proof of its isometricity (our conventions differ slightly from Hall's; see Section 5).

Independently, Golse, Leichtnam, and the author studied a Segal–Bargmann type transform called the Fourier–Bros–Iagolnitzer (F.B.I.) transform [3]. This transform can be defined on any compact real analytic Riemannian manifold (X, g). We showed that solutions of the heat equation can be analytically continued to a small tube $X^R_{\mathbf{C}}$ about X in its complexification. The radius R of this tube, measured by an exhaustion function canonically determined by g, does not depend on the initial heat data. The F.B.I. transform is the map

$$f\in C^\infty(X)\to e^{-t\Delta}f\in \mathcal{O}(X^R_{\mathbf{C}}).$$

The F.B.I. transform is injective, and we proved an inversion formula for it similar to Hall's. Our inversion formula involves integrating over the fibers $Y_R(x)$, $x\in X$, of the fibration of $X^R_{\mathbf{C}}$ over X obtained by identifying $X^R_{\mathbf{C}}$ with a neighborhood of the zero section in TX (see [4, 9]). We constructed a "pseudo-heat kernel" form $K(\cdot, t)\,\mu^{Y_R}$ in each fiber $Y_R(x)$ for R sufficiently small and gave the following integral prescription for recovering $f\in C^\infty(X)$ from its F.B.I. transform:

$$f(x)=\int_{Y_R(x)} e^{-t\Delta}f(\cdot)\,K(\cdot, t)\,\mu^{Y_R} + \int_0^t\int_{\partial Y_R(x)} [e^{-s\Delta}f(\cdot)\,i^{Y_R}_{\mathrm{grad}^{Y_R} K(\cdot, s)}\mu^{Y_R} - K(\cdot, s)\,i^{Y_R}_{\mathrm{grad}^{Y_R} e^{-s\Delta}f(\cdot)}\mu^{Y_R}]\,ds. \tag{2}$$

See [3, Theorem 0.3] for details. Our purpose here is to generalize Hall's inversion formula for the Segal–Bargmann transform by assuming that a complexification of X admits an unbounded canonical exhaustion function and letting $R\to\infty$ in (2). This assumption imposes severe restrictions on the geometry of (X, g); for example, the sectional curvatures must be nonnegative [9]. The list of known examples whose complexifications admit an unbounded canonical exhaustion is quite short: locally symmetric spaces of nonnegative curvature, normal homogeneous spaces of compact Lie groups and their discrete quotients, and certain surfaces of revolution (see [11, 13, 12], respectively). (Added in proof. Recently R. Aguilar has found many more examples.)

We will assume that X is a connected symmetric space U/K of Helgason's compact type. For simplicity we will assume that U/K is simply connected; the main results of this paper hold after passing to quotients by a freely acting finite group of isometries. It follows from the results of Hall [5, Sect. 11] that solutions of the heat equation on X can be analytically continued to entire functions on $X_{\mathbf{C}}$. We study the Segal–Bargmann transform in this setting. Under these assumptions each (maximal) fiber $Y(x)$ can be identified with a dual symmetric space of the noncompact type. To prove our inversion formula we will use the corresponding fiber heat kernel measure in (2) and let $R \to \infty$. Let $K_{g_Y}(\,\cdot, t)$ denote the heat kernal in the fiber $Y(x)$ evaluated at the base point $x \in X$, let v_{g_Y} be the Riemannian measure on $Y(x)$, and let $Y_R(x)$ be the distance ball of radius R in $Y(x)$. Use the Riemannian measure on X to form $L^2(X)$. Our main results are the following generalizations of Hall's fiberwise inversion formulae for compact Lie groups to the symmetric spaces of compact type.

THEOREM 1. *For all $f \in L^2(X)$, $t > 0$,*

$$f(x) = L^2 - \lim_{R \to \infty} \int_{Y_R(x)} e^{-t\Delta} f(\,\cdot\,)\, K_{g_Y}(\,\cdot, t)\, v_{g_Y}.$$

THEOREM 2. *For all $f \in C^\infty(X)$, $t > 0$, $x \in X$,*

$$f(x) = \int_{Y(x)} e^{-t\Delta} f(\,\cdot\,)\, K_{g_Y}(\,\cdot, t)\, v_{g_Y}$$

and the integral is absolutely convergent.

In Section 4 we consider the measure ω_t on $X_{\mathbf{C}}$ induced by the Riemannian measure on X and the heat kernel measure in the fibers at time $2t$, rescaled by a factor of 2 in the fibers. Let $\mathscr{H}_t$ denote the Hilbert space of holomorphic functions in $L^2(X_{\mathbf{C}}, \omega_t)$.

THEOREM 3. *The Segal–Bargmann transform is an isometric isomorphism from $L^2(X)$ onto $\mathscr{H}_t$.*

The organisation of this paper is as follows. In Section 2 we provide some background material and prove Theorem 2 for joint eigenfunctions of the U-invariant differential operators. Our strategy is to prove an analog of (2) for any $R>0$, then estimate the growth of the joint eigenfunctions in the complex domain and pass to the limit as $R\to\infty$. Section 3 contains the proofs of Theorems 1 and 2, and in Section 4 we define the measure ω_t and prove Theorem 3. In Section 5 we show that our inversion formula reduces to (1) when H is a compact, connected semisimple Lie group identified with a symmetric space of compact type.

2. INVERSION OF THE SEGAL–BARGMANN TRANSFORM ON JOINT EIGENFUNCTIONS

We assume that $X=U/K$ is a simply connected Riemannian (globally) symmetric space of Helgason's compact type. The Riemannian symmetric pair (U, K) is associated with an orthogonal symmetric Lie algebra $(\mathfrak{u}, \theta)$ of compact type (see [7, pp. 230, 244]). In particular $\mathfrak{u}$ is semisimple and U, K are compact. By the proof of [7, Proposition 4.2, Chap. V] we may assume that U is simply connected and K is connected. The Riemannian symmetric metric g determines an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}_*$, where $\mathfrak{p}_*$ is the -1 eigenspace of θ in the eigenspace decomposition $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}_*$. We assume that U acts effectively on U/K; then $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{u}$. Using [7, Proof of Proposition 5.2, Chap. VIII] and the remarks preceding Proposition 5.1, we may assume that the inner product on $\mathfrak{p}_*$ is the restriction of an $\mathrm{Ad}(U)$-invariant inner product on $\mathfrak{u}$ (we thank the referee for pointing this out). We will normalize the metric so that the total volume is one.

Let $U_{\mathbf{C}}$ be the Lie group complexification of U and let $K_{\mathbf{C}}$ be the connected subgroup of $U_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. Then $K_{\mathbf{C}}$ is a closed subgroup and the natural map, $gK\to gK_{\mathbf{C}}$, embeds $X=U/K$ as a closed, totally real submanifold of $X_{\mathbf{C}}=U_{\mathbf{C}}/K_{\mathbf{C}}$ [3, Proposition 8.1]. We will identify U/K with its image in $U_{\mathbf{C}}/K_{\mathbf{C}}$. Let Δ be the (nonnegative) Laplacian on $(U/K, g)$. By [5, Lemma 13] solutions of the heat equation $(\partial_t+\Delta)\,u=0$ can be analytically continued to entire functions on $U_{\mathbf{C}}/K_{\mathbf{C}}$.

A representation δ of U on a vector space V is said to be spherical if there is a nonzero vector in V fixed by $\delta(K)$. Let $\hat{U}_K$ denote the set of equivalence classes of irreducible unitary finite dimensional spherical representations of U. For each $\delta\in\hat{U}_K$ let V_δ be a representation space for δ with inner product $\langle , \rangle$. Since U/K is a symmetric space, the subspace of K-fixed vectors is spanned by a single unit vector $\mathbf{e}$. There is an orthogonal Hilbert space decomposition of $L^2(U/K)$ into irreducible subspaces under the U action,

$$L^2(U/K)=\bigoplus_{\delta\in\hat{U}_K} C_\delta(U/K), \tag{3}$$

where the $C_\delta(U/K)$ are equivalent irreducible representations of type δ. Explicitly, $C_\delta(U/K)$ consists of the functions $uK \to \langle \mathbf{v}, \delta(u)\,\mathbf{e} \rangle$, $\mathbf{v} \in V_\delta$. $C_\delta(U/K)$ is a joint eigenspace for the U-invariant differential operators $\mathbf{D}(U/K)$, i.e., for each $\delta \in \hat{U}_K$, there is a homorphism c_δ: $\mathbf{D}(U/K) \to \mathbf{C}$ such that for all $f \in C_\delta(U/K)$, $Df = c_\delta(D)\,f$. See [8, Theorem 4.3, Chap. V]. Since U acts on U/K by isometries, Δ is a U-invariant differential operator and the elements of $C_\delta(U/K)$ are eigenfunctions for Δ. Since solutions of the heat equation can be analytically continued to $U_{\mathbf{C}}/K_{\mathbf{C}}$, it follows that each $f \in C_\delta(U/K)$ can be analytically continued to $U_{\mathbf{C}}/K_{\mathbf{C}}$ (see also the proof of Lemma 1).

We consider the fibration π: $U_{\mathbf{C}}/K_{\mathbf{C}} \to U/K$ obtained by identifying $U_{\mathbf{C}}/K_{\mathbf{C}}$ with TU/K as in [11, 12]. This identification is given explicitly by the map

$$TU/K \ni d\tau(u)\, dp_e(V) \to u \exp \sqrt{-1}\; VK_{\mathbf{C}} \in U_{\mathbf{C}}/K_{\mathbf{C}}, \tag{4}$$

where $V \in \mathfrak{p}_*$, τ is the action of U on U/K, and p: $U \to U/K$ is the coset projection (see [3, Proposition 8.2 and Remark 8.4]). We will denote by Y, or $Y(x)$ if necessary, the fiber $\pi^{-1}(x)$ in $U_{\mathbf{C}}/K_{\mathbf{C}}$ over $x \in U/K$. An essential ingredient in our inversion formula for the Segal–Bargmann transform is a symmetric space metric on each fiber Y, which we now define.

Let G be the connected subgroup of $U_{\mathbf{C}}$ with Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \sqrt{-1}\,\mathfrak{p}_*$. Then K (which we may assume is connected) is a closed subgroup of G. The metric g on U/K determines in an obvious way a positive definite $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$. The homogeous manifold G/K with the corresponding G-invariant metric is a symmetric space of the noncompact type dual to U/K. The map (4) and the fact that G/K is diffeomorphic with $\mathfrak{p}$ show that the natural inclusion

$$gK \in G/K \to gK_{\mathbf{C}} \in U_{\mathbf{C}}/K_{\mathbf{C}}$$

is a diffeomorphism from G/K onto the fiber $Y(o)$, where o is the identity coset. We equip $Y(o)$ with the symmetric space metric given by this identification and use the transitive U-action among the fibers to give each $Y(x)$ a symmetric space metric; since K acts on $Y(o)$ by isometries this is well-defined. We denote this metric by g_Y and its Riemannian measure by v_{g_Y}.

Let ϕ: $U_{\mathbf{C}}/K_{\mathbf{C}} \to [0, \infty)$ be the function corresponding to the Riemannian length-squared function on TU/K under the identification (4), and let

$$Y_R(x) := \{\zeta \in Y(x) : \phi(\zeta) \leqslant R^2\}.$$

Let $K_{g_Y}(\,\cdot, t)$ denote the heat kernel for (Y, g_Y) evaluated at the base point x.

PROPOSITION 1. *For all $f_\delta \in C_\delta(U/K)$, $t>0$, $R>0$, and $x \in U/K$,*

$$f_\delta(x) = \int_{Y_R(x)} e^{-t\Delta} f_\delta(\,\cdot\,)\, K_{g_Y}(\,\cdot, t)\, v_{g_Y}$$
$$+ \int_0^t \int_{\partial Y_R(x)} [K_{g_Y}(\,\cdot, s)(\hat{n} e^{-s\Delta} f_\delta) - (\hat{n} K_{g_Y}(\,\cdot, s))\, e^{-s\Delta} f_\delta]\, v_{\partial Y_R}\, ds,$$

where $\hat{n}$ is the outward unit normal vector field on ∂Y_R and $v_{\partial Y_R}$ is the Riemannian measure of the induced metric on ∂Y_R.

Proof. This can be proved in the same way as [3, Theorem 0.3] using the estimate (6) below. The starting point is the following relationship between the Laplacians Δ on U/K and Δ_{g_Y} on Y.

PROPOSITION 2 [3]. *For every holomorphic function F on $U_{\mathbf{C}}/K_{\mathbf{C}}$, there is a neighborhood W of U/K in $U_{\mathbf{C}}/K_{\mathbf{C}}$ such that $\Delta(F|_{U/K})$ and $-\Delta_{g_Y}(F|_{Y \cap W})$ have the same analytic continuation to W.*

Proof. We use the notation of [3, Definition 1.13]. By [3, Theorem 8.5.1], $g^Y = -g_Y$. By [3, Theorem 1.19.ii], $\Delta_{g_Y} = -\Delta^Y$. The proposition now follows from [3, Theorem 1.16]. ∎

By [5, Lemma 13], $e^{-t\Delta} f_\delta$, $\Delta e^{-t\Delta} f_\delta$, and $\partial_t e^{-t\Delta} f_\delta$ can be analytically continued to $U_{\mathbf{C}}/K_{\mathbf{C}}$. Analytically continuing the equation $(\Delta + \partial_t)\, e^{-t\Delta} f_\delta = 0$ and restricting to Y_R, we obtain

$$(-\Delta_{g_Y} + \partial_t)(e^{-t\Delta} f_\delta|_{Y_R}) = 0$$

for all t, $R > 0$. As in the proof of [3, Theorem 0.3], we multiply by K_{g_Y} and integrate: for $0 < \varepsilon < t$,

$$\int_\varepsilon^t \int_{Y_R(x)} K_{g_Y}(\,\cdot, s)(-\Delta_{g_Y} + \partial_s)\, e^{-s\Delta} f_\delta v_{g_Y}\, ds = 0.$$

The integral is absolutely convergent for $0 < \varepsilon < t < \infty$ and $R < \infty$. Integrating by parts in s, using $(\Delta_{g_Y} + \partial_s)\, K_{g_Y} = 0$, and applying Green's second identity on Y_R gives

$$\int_{Y_R(x)} K_{g_Y}(\,\cdot, \varepsilon)\, e^{-\varepsilon\Delta} f_\delta v_{g_Y} = \int_{Y_R(x)} K_{g_Y}(\,\cdot, t)\, e^{-t\Delta} f_\delta v_{g_Y}$$
$$+ \int_\varepsilon^t \int_{\partial Y_R(x)} [K_{g_Y}(\,\cdot, s)(\hat{n} e^{-s\Delta} f_\delta)$$
$$- (\hat{n} K_{g_Y}(\,\cdot, s))\, e^{-s\Delta} f_\delta]\, v_{\partial Y_R}\, ds. \qquad (5)$$

To prove the proposition we let $\varepsilon \to 0^+$. To estimate the integral over ∂Y_R we note that since f_δ is an eigenfunction of Δ, it is clear that $|e^{-s\Delta} f|$, resp. $|\hat{n} e^{-s\Delta} f|$, is uniformly bounded on $(0, \infty) \times Y_R$, resp. $(0, \infty) \times \partial Y_R$. For the heat kernel K_{g_Y} we have the estimate

$$\text{for all } (s, \zeta) \in (0, t] \times Y, \qquad \| d_s^i \, d_\zeta^j K_{g_Y}(\zeta, s) \| \leqslant C(t) \, s^{-n/2 - i - j} e^{-d^2_{g_Y}(x, \zeta)/4s} \tag{6}$$

([1]; here x is the base point of Y, d_{g_Y} is the Riemannian distance in Y, and $\| \cdot \|$ is the g_Y norm). This shows that the integral over ∂Y_R is absolutely convergent as $\varepsilon \to 0^+$. The left side of (5) approaches $f_\delta(x)$ as $\varepsilon \to 0^+$ because f_δ is an eigenfunction and $K_{g_Y}(\cdot, \varepsilon) \, v_{g_Y}$ approaches the Dirac measure at the base point x as $\varepsilon \to 0$. ∎

The following lemma gives the control over the growth of the joint eigenfunctions in the complex domain needed to prove the inversion formula for joint eigenfunctions. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{u}_{\mathbf{C}}$ and an ordering of the roots, and let λ denote the highest weight of δ (extended to a representation of $\mathfrak{u}_{\mathbf{C}}$).

LEMMA 1. *There is an $M > 0$ (depending only on the metric g) such that for all $\delta \in \hat{U}_K$, $f_\delta \in C_\delta(U/K)$ analytically continued to $U_{\mathbf{C}}/K_{\mathbf{C}}$, $\zeta \in Y$,*

$$|f_\delta(\zeta)| \leqslant d(\delta)^{1/2} \, \|f_\delta\|_{L^2(U/K)} \, e^{|\lambda|_K M d_{g_Y}(x, \zeta)}$$

$$|\hat{n} f_\delta(\zeta)| \leqslant M \, |\lambda|_K \, d(\delta)^{1/2} \, \|f_\delta\|_{L^2(U/K)} \, e^{|\lambda|_K M d_{g_Y}(x, \zeta)},$$

where d_{g_Y} is the distance function on Y, $d(\delta)$ is the dimension of V_δ, and $|\cdot|_K$ is the Killing form norm.

Proof. Since U is simply connected, each representation δ extends to a holomorphic representation of $U_{\mathbf{C}}$, still denoted by δ, and $\delta(K_{\mathbf{C}}) \, \mathbf{e} = \mathbf{e}$ (since this holds on the real part K). The analytic continuation of $f_\delta(uK) = \langle \mathbf{v}, \delta(u) \, \mathbf{e} \rangle$ to $U_{\mathbf{C}}/K_{\mathbf{C}}$ is given explicitly by $f_\delta(gK_{\mathbf{C}}) = \langle \mathbf{v}, \delta(\bar{g}) \, \mathbf{e} \rangle$, where the bar denotes complex conjugation of $U_{\mathbf{C}}$ about U (note that $\langle , \rangle$ is conjugate linear in the second factor). By (4), given a coset $\zeta = gK_{\mathbf{C}}$ in $Y(x)$ there is a representative g such that $\bar{g} = u \exp V$ with $u \in U$, $V \in \mathfrak{p}$. Let $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Since $\mathrm{Ad}(K) \, \mathfrak{a} = \mathfrak{p}$, we may assume $\bar{g} = u \exp H$ with $u \in U$, $H \in \mathfrak{a}$. Since $|\mathbf{e}| = 1$ and δ is unitary, we have

$$\begin{aligned} |f_\delta(gK_{\mathbf{C}})| = |\langle \mathbf{v}, \delta(u \exp H) \, \mathbf{e} \rangle| &\leqslant |\mathbf{v}| \;\; |e^{\delta(H)}| \\ &= |\mathbf{v}| \max_{\lambda_i \text{ weights}} e^{|\lambda_i(H)|} \\ &\leqslant |\mathbf{v}| \, e^{|\lambda|_K |H|_K}, \end{aligned}$$

where λ is the highest weight, the last inequality by [8, Eq. (7), p. 498]. By the Schur orthogonality relations, $\|f_\delta\|^2_{L^2(U/K)} = |\mathbf{v}|^2\, d(\delta)^{-1}$, where $d(\delta)$ is the dimension of V_δ (here we have used the normalization $\mathrm{vol}(U/K)=1$). Since the Killing form norm on $\mathfrak{p}$ is comparable to the $\mathrm{Ad}(K)$-invariant inner product induced by $g_{Y(o)}$, there is a positive constant M (depending only on g) such that $|H|_K \leqslant M\,|H|_{g_{Y(o)}}$. The unique unit speed geodesic from x to ζ is $\gamma(t) = u\exp(-tH/|H|_{g_{Y(o)}})$, so $d_{g_{Y(x)}}(x,\zeta) = |H|_{g_{Y(o)}}$ (note also this shows $\phi(\zeta) = d^2_{g_{Y(x)}}(x,\zeta)$). This gives the estimate for f_δ.

The unit normal to ∂Y_R at $\zeta = u\exp -HK_{\mathbf{C}}$ is $\dot\gamma(R)$, $R = |H|_{g_{Y(o)}}$. Then

$$\begin{aligned}\hat n f_\delta(\zeta) &= \frac{d}{dt}\Big|_{t=R} \langle \mathbf{v}, \delta(u\exp\, -tH/R)\,\mathbf{e}\rangle \\ &= \langle \mathbf{v}, \delta(u\exp\, -H)\,\delta(-H)\,\mathbf{e}\rangle / R.\end{aligned}$$

The estimate for $|\hat n f_\delta(\zeta)|$ follows similarly. ∎

We can now prove the inversion formula for the Segal–Bargmann transform acting on the joint eigenfunctions.

Proposition 3. *For all $f_\delta \in C_\delta(U/K)$, $t>0$, $x \in U/K$,*

$$f_\delta(x) = \int_{Y(x)} e^{-t\Delta} f_\delta(\,\cdot\,)\, K_{g_Y}(\,\cdot\,, t)\, v_{g_Y}$$

and the integral is absolutely convergent.

Proof. We let $R\to\infty$ in Proposition 1. In the proof of Lemma 1 we showed that $d^2_{g_Y}(x,\zeta) = \phi(\zeta)$. So for $\zeta \in \partial Y_R$, $d_{g_Y}(x,\zeta) = R$. By Lemma 1 and (6) there are positive constants C_i (depending only on δ, f_δ, t, and g) such that for all $(s,\zeta) \in (0,t] \times \partial Y_R$,

$$\begin{aligned} &|e^{-s\Delta} f_\delta(\zeta)| \leqslant C_1 e^{-sc_\delta(\Delta) + C_2 R}, && |\hat n e^{-s\Delta} f_\delta(\zeta)| \leqslant C_3 e^{-sc_\delta(\Delta) + C_2 R} \\ &|K_{g_Y}(\zeta, s)| \leqslant C_4 s^{-n/2} e^{-R^2/4s}, && |\hat n K_{g_Y}(\zeta, s)| \leqslant C_4 s^{-n/2-1} e^{-R^2/4s}. \end{aligned}$$

Then we can estimate the boundary integral in Proposition 1 as $R\to\infty$ by

$$\begin{aligned} &\int_0^t \int_{\partial Y_R} |K_{g_Y}(\,\cdot\,, s)(\hat n e^{-s\Delta} f_\delta) - (\hat n K_{g_Y}(\,\cdot\,, s))\, e^{-s\Delta} f_\delta|\, v_{\partial Y_R}\, ds \\ &\qquad \leqslant C_5 \,\mathrm{vol}(\partial Y_R)\, e^{C_2 R} \int_0^t s^{-n/2-1} e^{-R^2/4s}\, ds \\ &\qquad \leqslant C_6\, \mathrm{vol}(\partial Y_R)\, e^{C_2 R - R^2/8t}. \end{aligned}$$

The Ricci curvature of g_Y is bounded below by a negative constant (since a symmetric space has a transitive group of isometries). Following the proof of [2, Theorem 6], we have $\mathrm{vol}(\partial Y_R) \leqslant C_7 e^{C_8 R}$. This shows that the boundary integral in Proposition 1 goes to zero as $R \to \infty$. Similarly one can show that the integral over Y converges absolutely. ∎

3. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. It is possible to prove the inversion formula for functions in the range of $e^{-\varepsilon \Delta}$ for some $\varepsilon > 0$ by brute force using the estimates above. But to prove Theorems 1 and 2 in this fashion would require unreasonably sharp estimates on $e^{-t\Delta} f$ and K_{g_Y}. We will instead take the following approach, which was suggested to us by B. Hall. For fixed $t > 0$, consider the linear operator

$$M_R f(x) = \int_{Y_R(x)} e^{-t\Delta} f(\,\cdot\,)\, K_{g_Y}(\,\cdot\,, t)\, v_{g_Y}. \tag{7}$$

The integral is absolutely convergent for $R < \infty$ and maps $L^2(U/K)$ into smooth functions. M_R commutes with the action of U on $L^2(U/K)$, since the action commutes with the heat operator and with analytic continuation and acts by isometries among the fibers (Y, g_Y). The kernel of M_R is a U-invariant subspace, so the restriction of M_R to an irreducible representation space is either a linear isomorphism onto its image or identically zero. Since the representations $C_\delta(U/K)$ in the decomposition (3) are irreducible and inequivalent, there is a constant $m_{R,\delta}$ such that

$$M_R|_{C_\delta(U/K)} = m_{R,\delta} I.$$

We claim that the $m_{R,\delta}$ are positive and increase with R. Applying M_R to $f_\delta(\,\cdot\,) = \langle \mathbf{e}, \delta(\,\cdot\,)\, \mathbf{e} \rangle$ and evaluating at o gives

$$m_{R,\delta} = e^{-tc_\delta(\Delta)} \int_{gK_{\mathbf{C}} \in Y_R(o)} \langle \mathbf{e}, \delta(\bar{g})\, \mathbf{e} \rangle\, K_{g_Y}(gK_{\mathbf{C}}, t)\, v_{g_Y}.$$

Each coset $gK_{\mathbf{C}} \in Y(o)$ has a representative g with $\bar{g} = \exp \sqrt{-1}\, V$, $V \in \mathfrak{p}_*$. Since $\delta(\mathfrak{u})$ consists of skew-symmetric matrices, $\delta(\sqrt{-1}\, V)$ is symmetric with real eigenvalues and so $\delta(\bar{g}) = e^{\delta(\sqrt{-1}\, V)}$ is positive definite. Since K_{g_Y} is positive, the claim follows. Proposition 3 shows that $\lim_{R \to \infty} m_{R,\delta} = 1$.

To prove the theorem we argue as in the proof of [6, Theorem 1]. If F is a holomorphic function on $U_{\mathbf{C}}/K_{\mathbf{C}}$ and $F|_{U/K} = \sum_{\delta \in \hat{U}_K} (F|_{U/K})_\delta$ is the decomposition (3), then the termwise analytic continuation of this series converges uniformly and absolutely on compacta in $U_{\mathbf{C}}/K_{\mathbf{C}}$ to F (cf. [5, Lemma 9]). It follows that $e^{-t\Delta} f = \sum_{\delta \in \hat{U}_K} e^{-tc_\delta(\Delta)} f_\delta$ with uniform and absolute convergence on compacta in $U_{\mathbf{C}}/K_{\mathbf{C}}$. Inserting this into (7) gives $M_R f = \sum_{\delta \in \hat{U}_K} m_{R,\delta} f_\delta$. Since the series is orthogonal, its convergence in $L^2(U/K)$ to f follows from monotone convergence. ∎

Proof of Theorem 2. It suffices to show that the integral is absolutely convergent, for then the pointwise limit $\lim_{R \to \infty} \int_{Y_R(x)} e^{-t\Delta} f K_{g_Y} v_{g_Y} = \int_{Y(x)} e^{-t\Delta} f K_{g_Y} v_{g_Y}$ is equal to f in $L^2(U/K)$ by Theorem 1. Since the integral depends continuously on parameters they must be equal everywhere.

In the proof of Lemma 1 we observed that the analytic continuation of f_δ is $f_\delta(gK_{\mathbf{C}}) = \langle \mathbf{v}, \delta(\bar{g})\, \mathbf{e} \rangle$. Write $e^{-t\Delta} f(\cdot) = \sum e^{-tc_\delta(\Delta)} \langle \mathbf{v}, \delta(\bar{\cdot})\, \mathbf{e} \rangle$. We will fix x and estimate $|\langle \mathbf{v}, \delta(\bar{\zeta})\, \mathbf{e} \rangle|$ for $\zeta \in Y(x)$. Define $\mathbf{w} \in V_\delta$ by $\mathbf{w} = \delta(u)\, \mathbf{e}$ where $x = uK$, and note that $\mathbf{w}$ does not depend on the choice of representative u. Writing $\zeta \in Y(x)$ as $\zeta = u \exp \sqrt{-1}\, V K_C$ with $\sqrt{-1}\, V \in \mathfrak{p}$, we see that $\langle \mathbf{w}, \delta(\bar{\zeta})\, \mathbf{e} \rangle$ is strictly positive on $Y(x)$. We claim that for all $\zeta \in Y(x)$ and $\mathbf{v} \in V_\delta$,

$$|\langle \mathbf{v}, \delta(\bar{\zeta})\, \mathbf{e} \rangle| \leqslant |\mathbf{v}|\, \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho))^{-1} \langle \mathbf{w}, \delta(\bar{\zeta})\, \mathbf{e} \rangle,$$

where $\mathbf{c}$ is the Harish–Chandra $\mathbf{c}$-function, $\bar{\lambda}$ is the restriction of the highest weight of δ to $\mathfrak{a}$, and ρ is one-half the sum of the positive restricted roots (with multiplicities; we have fixed a Weyl chamber $\mathfrak{a}^+$). To prove this we write $\bar{\zeta} = u' \exp H K_C$ with H in the closure of $\mathfrak{a}^+$, $u' \in U$, and $u'K = x$. Then

$$|\langle \mathbf{v}, \delta(\bar{\zeta})\, \mathbf{e} \rangle| \leqslant |\mathbf{v}| \max_{\bar{\lambda}_i \text{ restricted weights}} e^{\bar{\lambda}_i(H)} \leqslant |\mathbf{v}|\, e^{\bar{\lambda}(H)}$$

(the last inequality follows from the expression $\bar{\lambda}_i = \bar{\lambda} - \sum m_k \bar{\alpha}_k$ where $m_k \in \mathbf{Z}_+$ and $\bar{\alpha}_k$ are positive restricted roots). Let $\mathbf{u}_1, ..., \mathbf{u}_d$ be an orthonormal basis of V_δ consisting of weight vectors with corresponding weights λ_i, and let $\lambda_d = \lambda$ be the highest weight. Then $|\langle \mathbf{e}, \mathbf{u}_d \rangle|^2 = \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho)) > 0$ (see [8, Chap. V, Sect. 4, Eqs. (7), (8)]). Then

$$\begin{aligned}
|\langle \mathbf{v}, \delta(\bar{\zeta})\, \mathbf{e} \rangle| &\leqslant |\mathbf{v}|\, \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho))^{-1} |\langle \mathbf{e}, \mathbf{u}_d \rangle|^2 e^{\lambda_d(H)} \\
&\leqslant |\mathbf{v}|\, \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho))^{-1} \sum_{k=1}^{d} |\langle \mathbf{e}, \mathbf{u}_k \rangle|^2 e^{\lambda_k(H)} \\
&= |\mathbf{v}|\, \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho))^{-1} \langle \delta(u')\, \mathbf{e}, \delta(u' \exp(H))\, \mathbf{e} \rangle \\
&= |\mathbf{v}|\mathbf{c}(-\sqrt{-1}\,(\bar{\lambda} + \rho))^{-1} \langle \mathbf{w}, \delta(\bar{\zeta})\, \mathbf{e} \rangle
\end{aligned}$$

which proves the claim.

Keeping x and $\mathbf{w}$ fixed, we apply Proposition 3 to the joint eigenfunction $y \in U/K \to \langle \mathbf{w}, \delta(y)\,\mathbf{e}\rangle$ to obtain

$$e^{-tc_\delta(\Delta)} \int_{Y(x)} \langle \mathbf{w}, \delta(\bar{\cdot})\,\mathbf{e}\rangle\, K_{g_Y}(\,\cdot, t)\, v_{g_Y} = \langle \mathbf{w}, \delta(x)\,\mathbf{e}\rangle = 1,$$

and so, by the claim above,

$$\int_{Y(x)} |e^{-t\Delta} f(\,\cdot\,)|\, K_{g_Y}(\,\cdot, t)\, v_{g_Y} \leqslant \sum_{\delta \in \hat{U}_K} |\mathbf{v}|\, \mathbf{c}(-\sqrt{-1}\,(\bar{\lambda}+\rho))^{-1}. \tag{8}$$

There are positive constants C_1, C_2, p such that

$$\mathbf{c}(-\sqrt{-1}\,(\bar{\lambda}+\rho))^{-1} \leqslant C_1 + C_2\, |\bar{\lambda}+\rho|_K^p$$

([8, Propositon 7.2, Chap. IV]; since λ is a highest weight, $\bar{\lambda}+\rho$ is in $\mathfrak{a}_+^*$). If $f \in C^\infty(U/K)$, then the coefficients $|\mathbf{v}|$ decrease faster than any polynomial in $|\bar{\lambda}|_K$, and the sum in (8) converges. ∎

4. PROOF OF THEOREM 3

Let η: $U_{\mathbf{C}}/K_{\mathbf{C}} \to U_{\mathbf{C}}/K_{\mathbf{C}}$ be the diffeomorphism corresponding to multiplication by 2 in the fibers of TU/K under the identification (4); i.e., $\eta(uyK_{\mathbf{C}}) = uy^2K_{\mathbf{C}}$ for $u \in U$, $y \in \exp \mathfrak{p}$. By the Riesz Representation Theorem there is a unique measure ω_t on $U_{\mathbf{C}}/K_{\mathbf{C}}$ such that for $F \in C_c(U_{\mathbf{C}}/K_{\mathbf{C}})$,

$$\int_{U_{\mathbf{C}}/K_{\mathbf{C}}} F\omega_t = \int_{x \in U/K} \int_{Y(x)} F(\,\cdot\,)\, \eta^*\, (K_{Y(x)}(\,\cdot, 2t)\, v_{Y(x)})\, v_{U/K},$$

where $v_{U/K}$ is the Riemannian measure on U/K (we normalize the metric so $\mathrm{vol}(U/K) = 1$). Let $\pi_{\mathbf{C}}$: $U_{\mathbf{C}} \to U_{\mathbf{C}}/K_{\mathbf{C}}$ be the (holomorphic) coset projection. Since $\pi_{\mathbf{C}}$ restricted to $\exp \mathfrak{p}$ is a diffeomorphism onto $Y(o)$, this can be written as

$$\int_{U_{\mathbf{C}}/K_{\mathbf{C}}} F\omega_t = \int_{y \in \exp \mathfrak{p}} \int_{u \in U} F \circ \pi_{\mathbf{C}}(uy)\, du (\eta \circ \pi_{\mathbf{C}}|_{\exp \mathfrak{p}})^*\, (K_{Y(o)}(\,\cdot, 2t)\, v_{Y(o)}).$$

Here du is the Haar measure on U normalized by $\mathrm{vol}(U) = 1$, and we have used [8, Theorem 1.9, Chap. I], the fact that $\tau(u)$: $Y(o) \to Y(uK)$ is an isometry, and $\tau(u)$ commutes with η for $u \in U$. Let $\mathscr{H}_t$ denote the subspace of holomorphic functions in $L^2(U_{\mathbf{C}}/K_{\mathbf{C}}, \omega_t)$.

Proof of Theorem 3. Let V be the algebraic span of the joint eigenspaces $C_\delta(U/K)$, $\delta \in \hat{U}_K$. We argue as in the idea of the proof of [6, Theorem 2], to show that the Segal–Bargmann transform is an isometry from V into $\mathscr{H}_t$: for $f \in V$,

$$\|e^{-t\Delta} f\|^2_{\mathscr{H}_t} = \int_{y \in \exp \mathfrak{p}} \int_{u \in U} |e^{-t\Delta} f|^2 \circ \pi_{\mathbf{C}}(uy)\, du$$

$$\times\; (\eta \circ \pi_{\mathbf{C}}|_{\exp \mathfrak{p}})^* \,(K_{Y(o)}(\,\cdot, 2t)\, \nu_{Y(o)})$$

$$= \int_{y \in \exp \mathfrak{p}} \int_{u \in U} e^{-t\Delta} f \circ \pi_{\mathbf{C}}(uy)\, e^{-t\Delta} \bar{f} \circ \pi_{\mathbf{C}}(uy^{-1})\, du$$

$$\times\; (\eta \circ \pi_{\mathbf{C}}|_{\exp \mathfrak{p}})^* \,(K_{Y(o)}(\,\cdot, 2t)\, \nu_{Y(o)}).$$

Here $\bar{f}(\,\cdot\,)$ denotes the holomorphic function on $U_{\mathbf{C}}/K_{\mathbf{C}}$ whose restriction to U/K is $\bar{f}$. Using [6, Lemma 9], to make the "holomorphic change of variables" $w = uy^{-1}$ and Fubini's Theorem, we find that the above is equal to

$$\int_{u \in U} e^{-t\Delta} \bar{f}(uK) \int_{y \in \exp \mathfrak{p}} e^{-t\Delta} f \circ \pi_{\mathbf{C}}(uy^2) (\eta \circ \pi_{\mathbf{C}}|_{\exp \mathfrak{p}})^* \,(K_{Y(o)}(\,\cdot, 2t)\, \nu_{Y(o)})\, du.$$

Since $f \in V$, there is a $g \in V$ such that $e^{-t\Delta} g = f$. Changing variables in the inner integral by $\tau(u) \circ \eta \circ \pi_{\mathbf{C}}|_{\exp \mathfrak{p}}\colon \exp \mathfrak{p} \to Y(uK)$ gives

$$\|e^{-t\Delta} f\|^2_{\mathscr{H}_t} = \int_{u \in U} e^{-t\Delta} \bar{f}(uK) \int_{Y(uK)} e^{-2t\Delta} g(\,\cdot\,)\, K_{Y(uK)}(\,\cdot, 2t)\, \nu_{Y(uK)}\, du.$$

Proposition 3 and the fact that $e^{-t\Delta}$ is formally self-adjoint give

$$\|e^{-t\Delta} f\|^2_{\mathscr{H}_t} = \int_{u \in U} e^{-t\Delta} \bar{f}(uK)\, g(uK)\, du = \int_{U/K} \bar{f} f\, \nu_{U/K}.$$

To show that the Segal–Bargmann transform is an isometry from $L^2(U/K)$ onto $\mathscr{H}_t$, it suffices to show that the image of V is dense in $\mathscr{H}_t$. We will briefly indicate how the arguments in [5, Lemmas 9, 10], can be adapted to our situation. The projection of $f \in L^2(U/K)$ onto $C_\delta(U/K)$ can be written as

$$f_{\delta}(x) = d(\delta)\, \mathrm{Tr}(\delta(x)\, A_{\delta, f}), \qquad \text{where} \quad A_{\delta, f} = \int_U \delta(u^{-1}) f(u)\, du$$

(here $\check{\delta}$ is the representation contragredient to δ; see [8, Lemma 1.7, Corollary 1.8, Chap. IV]). Given a holomorphic function F on $U_{\mathbf{C}}/U_{\mathbf{C}}$, consider the "holomorphic Fourier series" (HFS)

$$F(gK_{\mathbf{C}}) = \sum_{\delta \in \hat{U}_K} d(\delta)\, \mathrm{Tr}(\delta(g)\, A_{\delta, F|_{U/K}}).$$

The proof of [5, Lemma 9] shows that the HFS converges uniformly and absolutely on compact subsets of $U_{\mathbf{C}}/K_{\mathbf{C}}$. Using the $\tau(U)$-invariance of ω_t, one shows exactly as in [5, Lemma 10] that the terms in the HFS are orthogonal in $L^2(U_{\mathbf{C}}/K_{\mathbf{C}}, \omega_t)$ and remain orthogonal when integrated over any $\tau(U)$-invariant subset of $U_{\mathbf{C}}/K_{\mathbf{C}}$. The rest of the proof of [5, Lemma 10] goes through unchanged to show that the HFS for F converges to F in $L^2(U_{\mathbf{C}}/K_{\mathbf{C}})$. Since each term in the HFS is in the image of V, the image is dense in $\mathscr{H}_t$. ∎

5. COMPARISON WITH B. HALL'S INVERSION FORMULA

Let H be a compact, simply connected semisimple Lie group with a bi-invariant metric g_H (normalized to unit volume) and Laplacian $\Delta_H \geqslant 0$. Let

$$P = \{\exp \sqrt{-1}\; V : V \in \mathfrak{h}\} \subset H_{\mathbf{C}}$$

with the metric g_P obtained by identifying P with the symmetric space $H_{\mathbf{C}}/H$ by the quotient map $q: H_{\mathbf{C}} \to H_{\mathbf{C}}/H$. Let Δ_P be the (nonnegative) Laplacian on P, let $\sigma_t(p)$ be the fundamental solution at the identity of the equation $(\partial_t + \Delta_P)\, u = 0$, and let dp be the Riemannian measure on P. With these conventions, B. Hall's inversion formula for the Segal–Bargmann transform is

$$f(h) = \int_P e^{-t\Delta_H} f(hp^2)\, \sigma_{t/4}(p)\, dp \qquad (f \in C^\infty(H)).$$

Put $U = H \times H$, let K be the diagonal in $H \times H$, and let θ be the involutive automorphism of $\mathfrak{u}$ given by $\theta(X, Y) = (Y, X)$. Then $(\mathfrak{u}, \theta)$ is an orthogonal symmetric Lie algebra, and U/K is a symmetric space of the compact type with respect to any U-invariant metric. We identify U/K with H by the U-equivariant map $\alpha((g, h)\, K) = gh^{-1}$ and give U/K the metric that makes α an isometry. There is a unique biholomorphic map $\alpha_{\mathbf{C}}: U_{\mathbf{C}}/K_{\mathbf{C}} \to H_{\mathbf{C}}$ extending α, and it is not hard to see that $\alpha_{\mathbf{C}}(Y(x))$ is $L_{\alpha(x)}P$, the left translate of P by $\alpha(x)$. We reconcile our inversion formula with Hall's in the following proposition.

PROPOSITION 4. *For all* $f \in C^\infty(H)$, $x \in U/K$, $t > 0$,

$$\int_{Y(x)} e^{-t\Delta} \alpha^* f(\,\cdot\,)\, K_{g_Y}(\,\cdot, t)\, v_{g_Y} = \int_P e^{-t\Delta_H} f(\alpha(x)\, p^2)\, \alpha_{t/4}(p)\, dp.$$

Proof. Changing variables by $\tau(\alpha(x), e)\colon Y(o) \to Y(x)$ we may assume that $x = o$. Let sq: $P \to P$ be the diffeomorphism $\mathrm{sq}(p) = p^2$ and let sqrt be its inverse. We will use the change of variables $\alpha_{\mathbf{C}}^{-1} \circ \mathrm{sq}\colon P \to Y(o)$. Since $\alpha_{\mathbf{C}}$ is the biholomorphic extension of the isometry α, we have for the analytic continuations $(\alpha_{\mathbf{C}}^{-1})^* (e^{-t\Delta} \alpha^* f) = e^{-t\Delta_H} f$. To prove the proposition it suffices to show that $(\mathrm{sqrt} \circ \alpha_{\mathbf{C}})^*\, g_P = \frac{1}{4} g_{Y(o)}$. The main point is that $\mathrm{sqrt} \circ \alpha_{\mathbf{C}}$ is equivariant with respect to the group actions on P and $Y(o)$, as we now explain.

As above let G be the connected subgroup of $U_{\mathbf{C}}$ with Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Here $\mathfrak{p} = \{(\sqrt{-1}\, V, -\sqrt{-1}\, V)\colon V \in \mathfrak{h}\}$, so

$$G = \{(z, \bar{z})\colon z \in H_{\mathbf{C}}\}, \qquad \text{and} \qquad Y(o) = \{(z, \bar{z})\, K_{\mathbf{C}}\colon z \in H_{\mathbf{C}}\}.$$

We will identify G with $H_{\mathbf{C}}$ by projection onto the first factor. G acts on $Y(o)$ as a subgroup of $U_{\mathbf{C}}$, and $H_{\mathbf{C}}$ acts on P through its identification with $H_{\mathbf{C}}/H$ (*not* through the left action of $H_{\mathbf{C}}$ on $H_{\mathbf{C}}$). The action of $H_{\mathbf{C}}$ on P is given explicitly by $\tau_P(z)(p) = \mathrm{sqrt}(z p^2 \bar{z}^{-1})$ (note that complex conjugation of $H_{\mathbf{C}}$ about H is a group automorphism). To show that $\mathrm{sqrt} \circ \alpha_{\mathbf{C}}$ is equivariant with respect to the $G_{\mathbf{C}} \cong H_{\mathbf{C}}$ action, let $(z, \bar{z}) \in G \cong H_{\mathbf{C}}$, $(w, \bar{w})\, K_{\mathbf{C}} \in Y(o)$. Then

$$\begin{aligned} \mathrm{sqrt} \circ \alpha_{\mathbf{C}} \circ \tau((z, \bar{z}))((w, \bar{w})) &= \mathrm{sqrt}(z w \bar{w}^{-1} \bar{z}^{-1}) \\ &= \tau_P(z)(\mathrm{sqrt}(w \bar{w}^{-1})) = \tau_P(z) \circ \mathrm{sqrt} \circ \alpha_{\mathbf{C}}((w, \bar{w})\, K_{\mathbf{C}}), \end{aligned}$$

which is the desired equivariance property.

Since $g_{Y(o)}$ and g_P are $G \cong H_{\mathbf{C}}$ invariant, it suffices to check that $(\mathrm{sqrt} \circ \alpha_{\mathbf{C}})^*\, g_P = \frac{1}{4} g_{Y(o)}$ at the identity coset. The identification of U/K with H determines an inner product on $\mathfrak{p}_*$; the metric g_Y on $Y(o)$ at the identity coset is given by (minus one times) this inner product under the usual identification of $T_o Y(o)$ with $\mathfrak{p}$. One computes

$$g_{Y(o)}((\sqrt{-1}\, V, -\sqrt{-1}\, V), (\sqrt{-1}\, W, -\sqrt{-1}\, W))_o = 4(V, W)_{\mathfrak{h}},$$

where $(\,\cdot, \cdot)_{\mathfrak{h}}$ is the $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{h}$ determined by g_H. The metric on P at the identity coset is given by $-(\,\cdot, \cdot)_{\mathfrak{h}}$ under the identification of $T_e P$ with $\sqrt{-1}\, \mathfrak{h}$. One computes easily that the tangent map at the identity coset in $Y(o)$ to $\mathrm{sqrt} \circ \alpha_{\mathbf{C}}$ is simply projection onto the first factor, so

$$(\mathrm{sqrt} \circ \alpha_{\mathbf{C}})^* g_P((\sqrt{-1}\, V, -\sqrt{-1}\, V), (\sqrt{-1}\, W, -\sqrt{-1}\, W))_o = (V, W)_{\mathfrak{h}}.$$

This proves that $(\mathrm{sqrt} \circ \alpha_{\mathbf{C}})^* g_P = \frac{1}{4} g_{Y(o)}$. ∎

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