Ricci-flat metrics on the complexification of a compact rank one symmetric space

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We construct a complete Ricci-flat Kähler metric on the complexification of a compact rank one symmetric space. Our method is to look for a Kähler potential of the form $\psi = f(u)$, where u satisfies the homogeneous Monge-Ampère equation. We use the high degree of symmetry present to reduce the non-linear partial differential equation governing the Ricci curvature to a simple second-order ordinary differential equation for the function f. To prove that the resulting metric is complete requires some techniques from symplectic geometry.

1. Introduction

Let M be a compact Kähler manifold whose first Chern class is zero. By Yau's solution of the Calabi conjecture, there is a unique Kähler metric in the original Kähler class whose Ricci curvature is zero. If M is *not* compact, the situation is completely different. There is at the moment no completely general existence theorem for complete Ricci-flat Kähler metrics on non-compact Kähler manifolds (such a metric need not be unique, even if we specifiy its Kähler class and volume form [LeB]). The most general existence theorems to date are due to Tian and Yau [T-Y1,2] and S. Bando and R. Kobayashi [B-K], [Ko]. Typical of their results is that if $M = \overline{M} \backslash D$, where $\overline{M}$ is a compact Kähler manifold with $c_1(\overline{M}) > 0$ and D is a smooth hypersurface with $c_1(\overline{M}) = \alpha\, c_1(L_D)$ for $\alpha \geq 1$, then M has a complete Ricci-flat Kähler metric. Their results are non-constructive in nature and rely on sophisticated non-linear analysis.

In this paper we consider the case where M is the "complexification" of a compact rank one globally symmetric space. On these complex manifolds we give an explicit and fairly elementary construction of complete, Ricci-flat (but not flat) Kähler metrics. Our technique is to use the large symmetry group of these manifolds to reduce the problem to solving an ordinary differential equation. In fact, our results can be seen as an illustration of the classical principle that the group of motions of a space can sometimes be used to reduce partial differential equations to ordinary differential equations.

Let us now describe our assumptions and results. Let G be a compact Lie group and let $X = G/K$. Turn G/K into a Riemannian manifold by giving it the normal metric. Decompose the Lie algebra of G as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the isotropy group K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. We will make the following assumption:

Assumption. The linear isotropy group $\mathrm{Ad}_G(K)$ acts transitively on the unit sphere in $\mathfrak{p}$.

The assumption we make is quite strong. An equivalent assumption is that X is a compact, rank one, globally symmetric space. These admit a well-known classification: they are the spheres, real, complex, and quaternionic projective spaces, and the Cayley plane. Under this assumption we will prove the following theorem.

Theorem 1. *The cotangent and tangent bundles of X have a complete, Ricci-flat Kähler metric.*

The cotangent and tangent bundles of X have complex structures canonically associated to the Riemannian symmetric metric on X (see [Ste], [G-S], [L-S] and [Szö]). These are the "complexifications" of X mentioned in the title. For explicit realizations of these as quasi-projective varieties, see [P-W]. Notice that these are *not* complex vector bundles. It should be emphasized that in the case that X is a complex manifold, our complex structure is not the standard complex structure on T^*X. In particular, in our complex structure the zero section is a totally real submanifold of T^*X. When X is the standard S^2, our metric is the Eguchi-Hanson metric (see section 7).

Note added in proof. H. Azad and R. Kobayashi have recently proven the existence of complete, Ricci-flat Kähler metrics on complexifications of higher rank compact symmetric spaces.

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2. The cotangent bundle of G/K is a Stein manifold

Let G be a compact connected semisimple Lie group, K a closed subgroup of G. There is a natural way of realizing T^*G/K as a Stein complex manifold, which we will briefly describe. For any compact connected Lie group G, there exists a unique complex connected Lie group $G_{\mathbb{C}}$ whose Lie algebra is the complexification of the real Lie algebra of G, and such that G is a maximal compact subgroup of $G_{\mathbb{C}}$ [G]. If G is not connected, then $G_{\mathbb{C}}$ is no longer connected, but each component of $G_{\mathbb{C}}$ contains only one component of G. If K is a closed (possibly not connected) subgroup of G, then $K_{\mathbb{C}}$ is isomorphic to a closed complex subgroup of $G_{\mathbb{C}}$. If G is connected and semisimple, one can show that the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ is real analytically diffeomorphic to T^*G/K, equivariantly so with respect to the G action and preserving the natural inclusion of

G/K in each [Ste]. The complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ is Stein; this fact is due to Matsushima [Mat], or follows from the results of [Ste].

A compact Riemannian manifold X is a rank one, globally symmetric space if and only if the linear isotropy group at some point p acts transitively on the unit sphere in T_p^*X (see [Hel, pg. 535] and the references given there). If G/K is a compact, rank one, globally symmetric space, the orbits of G in the Stein manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ are very easy to describe: they are hypersurfaces diffeomorphic to the sphere bundle in T^*G/K, and an exceptional orbit diffeomorphic to G/K. On the other hand, a theorem of Morimoto and Nagano implies that if the connected component of the holomorphic automorphism group of a Stein manifold M has a compact, simply connected hypersurface orbit, then M is either diffeomorphic to the cotangent bundle of a compact, rank one, globally symmetric space, or "pseudo-conformally" equivalent to the unit sphere in $\mathbb{C}^n$ [M-N].

For the convenience of the reader we include the following table, which summarizes the well-known classification of compact, rank one, globally symmetric spaces.

Table 1. Compact rank one symmetric spaces

Geometric form	G	K	dim G/K	Helgason's type
$\mathbb{CP}^n$	$SU(n+1)$	$S(U(1)\times U(n))$	$2n$	$AIII$
S^n	$SO(n+1)$	$SO(n)$	n	BDI
$\mathbb{RP}^n$	$SO(n+1)$	$O(n)$	n	(n/a)
$\mathbb{HP}^n$	$Sp(n+1)$	$Sp(1)\times Sp(n)$	$4n$	CII
$Ca\mathbb{P}^2$	F_4	$SO(9)$	16	FII

3. Trivialization of $\Lambda^{(n,0)}(M)$

On any complex manifold M, the group of holomorphic line bundles on M is isomorphic to $H^1(M,\mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of germs of non-vanishing holomorphic functions on M. If M is a Stein manifold, then this group is isomorphic to $H^2(M,\mathbb{Z})$. In our situation, G/K is a strong deformation retract of M, so $H^2(M,\mathbb{Z})$ is isomorphic to $H^2(G/K,\mathbb{Z})$. Thus if for example $H^2_{DR}(G/K)=0$, we can conclude that the canonical bundle $\Lambda^{(n,0)}(M)$ is holomorphically trivial. The following lemma gives the same conclusion about the canonical bundle of $G_{\mathbb{C}}/K_{\mathbb{C}}$ whenever K is connected. More importantly, it gives in that case a $G_{\mathbb{C}}$-equivariant trivialization of $\Lambda^{(n,0)}(M)$. This will be essential in the construction of our metrics.

Lemma 2. *Let $M = G_{\mathbb{C}}/K_{\mathbb{C}}$, where G is a compact, semisimple Lie group and K a closed, connected subgroup of G. Then there exists a $G_{\mathbb{C}}$-invariant, nonvanishing holomorphic section of $\Lambda^{(n,0)}(M)$.*

Proof. The complex structure of $G_{\mathbb{C}}/K_{\mathbb{C}}$ is defined by using the canonical projection $\pi: G_{\mathbb{C}} \to G_{\mathbb{C}}/K_{\mathbb{C}}$ to identify the tangent space to the identity coset with $\mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$ (which is negative definite since G is compact). The isotropy representation of $K_{\mathbb{C}}$ corresponds to the adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$. Choose a nonzero element Ω_o of the one complex dimensional vector space $\Lambda^{(n,0)}(\mathfrak{p}^*_{\mathbb{C}})$, which we will identify with the complex determinant of an n-tuple of vectors in $\mathfrak{p}_{\mathbb{C}}$ with respect to a basis. We need to show that Ω_o is invariant under the co-adjoint action of $K_{\mathbb{C}}$, so that Ω_o descends to a $G_{\mathbb{C}}$-invariant, nonvanishing section of $\Lambda^{(n,0)}(M)$. So we must show that the automorphisms $\mathrm{Ad}(k_{\mathbb{C}})$, $k_{\mathbb{C}} \in K_{\mathbb{C}}$, have determinant one.

The restriction of the Killing form of $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{p}_{\mathbb{C}}$ is a non-degenerate, complex bilinear symmetric form. This form is preserved by the adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$. Hence the adjoint representation is contained in $O(\mathfrak{p}_{\mathbb{C}})$; we must show it is actually in $SO(\mathfrak{p}_{\mathbb{C}})$. Since K is connected, $K_{\mathbb{C}}$ is as well. So the determinant is constant on $K_{\mathbb{C}}$, and must be equal to one.

It remains to remark that any $G_{\mathbb{C}}$-invariant $(p,0)$ form Ω on $G_{\mathbb{C}}/K_{\mathbb{C}}$ is holomorphic. To see this, we note that it suffices to show that the pullback of Ω to $G_{\mathbb{C}}$ is holomorphic. But any left $G_{\mathbb{C}}$-invariant $(p,0)$ form on $G_{\mathbb{C}}$ is holomorphic, since $G_{\mathbb{C}}$ is holomorphically parallelizable by left invariant vector fields. □

Notice that K is connected in all the examples except $\mathbb{RP}^n$. It is easy to see that the conclusion of lemma 2 is false for the even dimensional real projective spaces, since they are not orientable.

4. Reduction to an O.D.E.

Suppose $\psi: M \to \mathbb{R}$ is a strictly plurisubharmonic function. Then $\sqrt{-1}\,\partial\overline{\partial}\,\psi$ is a real, closed, non-degenerate two form. Let J be the automorphism of the real tangent bundle of M which defines its complex structure, and let $b(\cdot,\cdot)$ be the symmetric tensor $\sqrt{-1}\,\partial\overline{\partial}\,\psi(\cdot, J\cdot)$. Since ψ is strictly plurisubharmonic, b is positive definite. In other words, to every such ψ there is a canonically associated (Riemannian) Kähler metric b and an exact symplectic form $\sqrt{-1}\,\partial\overline{\partial}\,\psi = -d(\mathrm{Im}\overline{\partial}\psi)$.

To construct a Ricci-flat Kähler metric on M we will look for a Kähler potential ψ such that the Ricci curvature of the metric b is zero. This means that ψ has to satisfy the non-linear partial differential equation of Monge-Ampère type,

$$\mathrm{Ric}(\psi) \stackrel{def}{=} -\sqrt{-1}\,\partial\overline{\partial}\,\log\det\frac{\partial^2\psi}{\partial z_i \partial\overline{z}_j} = 0 \tag{1}$$

(see e.g. [K-N], p. 158). The left hand side is the Ricci form; it does not depend on the choice of coordinates.

Let $M = G_{\mathbb{C}}/K_{\mathbb{C}}$ with G compact, connected and semisimple, K a closed subgroup of G. Let $n = \dim G/K$. The main result of our paper [Ste] is the following.

Theorem 3. *There exists a G invariant, real analytic, strictly plurisubharmonic exhaustion $\rho\colon M \to [0,\infty)$ such that $u = \sqrt{\rho}$ satisfies*[1] *the homogeneous Monge-Ampère equation,*

$$(\partial\overline{\partial}\, u)^n = 0. \tag{2}$$

Now let us assume that G/K is a compact, rank one, globally symmetric space with K connected. We will show that equation (1) can be reduced to an ordinary differential equation; for the time being, we will not worry about whether the solution is strictly plurisubharmonic. The function $S(x)$ that appears in the following proposition is explained in its proof (it is the ratio of the volume form of the Kähler metric associated with ρ and the $G_{\mathbb{C}}$-invariant volume form).

Proposition 4. *Suppose f is a solution of the ordinary differential equation*

$$\frac{d}{dx}(f'(x))^n = cx^{n-1}/S(x) \qquad (c > 0). \tag{3}$$

Then $f(u)$ is a solution of the Ricci equation (1) on $M \backslash u^{-1}(0)$.

Proof. We will first find an expression for the Ricci form of the Kähler metric with potential ρ on M. Let Ω be the $G_{\mathbb{C}}$-invariant, non-vanishing holomorphic section of $\Lambda^{(n,0)}(M)$ constructed in lemma 2. Then there is a constant ϵ_n such that $\epsilon_n \Omega \wedge \overline{\Omega}$ is a volume form compatible with the orientation defined by the Kähler form $\sqrt{-1}\,\partial\overline{\partial}\,\rho$. So there is a positive, real analytic function F such that

$$(\sqrt{-1}\,\partial\overline{\partial}\,\rho)^n = F\epsilon_n \Omega \wedge \overline{\Omega}.$$

Since ρ and Ω are invariant under the action of G, F is constant on the connected orbits of G. Recalling the discussion in section 2, these orbits are precisely the level sets of u, so F must be a function of u. Then there exists a real analytic, even function $S\colon \mathbb{R} \to (0,\infty)$ such that

$$(\sqrt{-1}\,\partial\overline{\partial}\,\rho)^n = S(u)\;\epsilon_n \Omega \wedge \overline{\Omega}. \tag{4}$$

We can read off from this equation that in local holomorphic coordinates,

$$\det \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} = S(u)|h|^2$$

[1] Strictly speaking, the left hand side of the equation only makes sense on the open dense set where $\rho \neq 0$.

for some locally defined, non-vanishing holomorphic function h. Thus the Ricci form of the Kähler metric with potential ρ is $-\sqrt{-1}\,\partial\overline{\partial}\,\log S(u)$.

Now let us look for a solution of the Ricci equation (1) on M of the form $f(u)$ for a yet to be determined smooth function $f\colon \mathbb{R} \to \mathbb{R}$. Away from the set $u = 0$ we have, using the fact that $(\partial\overline{\partial}\,u)^n = 0$,

$$(\partial\overline{\partial}\,f(u))^n = nf''(u)(f'(u))^{n-1}\partial u \wedge \overline{\partial} u \wedge (\partial\overline{\partial}\,u)^{n-1}.$$

On the other hand,

$$(\partial\overline{\partial}\,\rho)^n = n\,2^n u^{n-1}\partial u \wedge \overline{\partial} u \wedge (\partial\overline{\partial}\,u)^{n-1}.$$

Thus

$$(\sqrt{-1}\,\partial\overline{\partial}\,f(u))^n = 2^{-n} f''(f')^{n-1} u^{1-n} S(u)\ \epsilon_n \Omega \wedge \overline{\Omega},$$

and the Ricci form associated with $f(u)$ is

$$\mathrm{Ric}(f(u)) = -\sqrt{-1}\,\partial\overline{\partial}\,\log f''(f')^{n-1}u^{1-n}S(u).$$

If f satisfies the equation (3), then $\mathrm{Ric}(f(u)) = 0$. □

5. Solving the O.D.E.

The equation (3) can be solved by elementary means. We need to check that the function $f(u)$ is strictly plurisubharmonic, so that it defines a Kähler metric.

Lemma 5. *There is a unique real analytic solution of the equation (3) vanishing to second order at zero. The solution is necessarily an even function.*

Proof. Integrating the equation we find that $(f'(x))^n = x^n h(x)$, where $h(x)$ is a positive, real analytic, even function of $x \in \mathbb{R}$ (since $S(x)$ is positive and even). Taking n-th roots and integrating once more gives the unique solution with the advertised properties. □

Proposition 6. *$f(u)$ is a strictly plurisubharmonic exhaustion of M.*

Proof. As above let $\rho = u^2$ and let $f(u) = g(\rho)$. Then $g(\rho)$ is a real analytic function on M. To show that it is strictly plurisubharmonic, we need to check that $\partial\overline{\partial}\,g(\rho)(V, \overline{V}) > 0$ for all non-zero V in $T^{(1,0)}M$. It suffices to check this when $\rho > 0$, since $g'(0) > 0$ and ρ vanishes to second order when $\rho = 0$.

Let $\rho(z) > 0$ and consider the subspace of $T_z^{(1,0)}M$ given by the annihilator of $\partial\rho$ at z. Pulled back to this $n-1$ (complex) dimensional subspace, $\partial\overline{\partial}\,g(\rho)$ is equal to $g'(\rho)\partial\overline{\partial}\,\rho$. From the proof of lemma 5, $g'(\rho) > 0$, so $\partial\overline{\partial}\,g(\rho)$ is positive on this subspace. Let Z be the complex gradient of ρ at z, defined by the equation

$$\partial\overline{\partial}\,\rho(Z, \cdot) = \overline{\partial}\rho(\cdot).$$

Then Z is orthogonal to the annihilator of $\partial\rho$, and together they span $T^{(1,0)}(M)$ (over $\mathbb{C}$) at any point where $\rho > 0$. We have to show that $\partial\overline{\partial}\,g(\rho)(Z, \overline{Z}) > 0$.

We recall that $\sqrt{\rho}$ satisfies the homogeneous Monge-Ampère equation if and only if $Z\rho = 2\rho$ (see [P-W], equation (3.6)). It follows that $\partial\overline{\partial}\,\rho(Z,\overline{Z}) = 2\rho$ and $\partial\overline{\partial}\,g(\rho)(Z,\overline{Z}) = \rho f''(u)$, the positivity of which follows from the ordinary differential equation f satisfies.

To show that $f(u)$ is an exhaustion, we need only show that $f(x)$ is a homeomorphism from $[0,\infty)$ to itself. But this is clear since $f'(x)$ and $f''(x)$ are positive for $x > 0$. □

Summing up, we have proved[2]: *the exhaustion $\psi = f(u)$ is a Kähler potential for a Ricci-flat Kähler metric on the cotangent bundle of a compact, rank one, globally symmetric space.*

6. Completeness of the metric

To prove that our metric is complete we will need to know something about the behavior of the function $S(x)$ appearing in equation (3). It will suffice to show that $x^n/S(x)$ is monotonically increasing. The most direct approach is to compute the Lie derivative of equation (4) with respect to the real gradient vector field of the potential ρ, which we will denote by $2\,\Xi$. To compute the Lie derivative of $\Omega \wedge \overline{\Omega}$ we will decompose Ξ into a sum (over $C^\infty(M)$) of "infinitesimal actions" of the complex Lie group, $G_{\mathbb{C}}$. This will require some techniques from symplectic geometry developed by Guillemin and Sternberg in [G2], and a short digression to describe the momentum map on M. Along the way we will obtain an interesting interpretation of the solution of the homogeneous Monge-Ampère equation in terms of the momentum map.

Let us first assume that $x^n/S(x)$ is monotonically increasing and prove our main result.

Proposition 7. *The Ricci-flat Kähler metric constructed above is complete.*

Proof. Let $F(z)$ denote the geodesic distance from z to the compact minimum set, $\{\psi = 0\}$. It suffices to show that the distance tubes $F^{-1}([0,T))$ have compact closure for all $T \in \mathbb{R}$. For a Kähler metric defined by a strictly plurisubharmonic exhaustion of the form $\psi = r^{-1}(u)$, where u satisfies the homogeneous Monge-Ampère equation (2), the geodesic distance from z to the minimum set of ψ depends only on $\psi(z)$; in fact,

$$\mathrm{dist}(z, \{\psi = \psi_{\mathrm{min}}\}) = \frac{1}{\sqrt{2}} \int_{\psi_{\mathrm{min}}}^{\psi(z)} \sqrt{\frac{-r''(t)}{r'(t)}}\; dt$$

(see [P-W], theorem 3.3). Our Kähler metric is defined by the potential $\phi = f(u)$, where f satisfies equation (3) (we may without loss of generality assume that $c = 1$). This means that $F(z) = G(u(z))$, where

[2] The case of the real projective spaces must be handled separately (since K is not connected). See section 7.

$$G(u) = \frac{1}{\sqrt{2}} \int_0^u \sqrt{f''(x)}\, dx$$
$$= \frac{1}{\sqrt{2n}} \int_0^u \sqrt{f'(x)} \sqrt{\frac{x^{n-1}/S(x)}{f'(x)^n}}\, dx.$$

Since the sets $u^{-1}([0,T))$ have compact closure, we only need to show that $G(u)$ is unbounded as $u \to \infty$. We first estimate $f'(x)^n$ from above, for x sufficiently large:

$$f'(x)^n = \int_0^x t^{n-1}/S(t)\, dt$$
$$\leq c \int_0^x t^n/S(t)\, dt$$
$$\leq c\, x^{n+1}/S(x)$$

(in the last inequality we used the fact that $x^n/S(x)$ is monotonically increasing). Since f' is monotonically increasing, we have, for u greater than some large u_o and some $c > 0$,

$$G(u) \geq c \int_{u_o}^u \frac{dx}{x}$$

which is clearly unbounded as $u \to \infty$. □

We will now review the symplectic ideas we will need to prove that $x^n/S(x)$ is monotonically increasing. Let M, ω be a symplectic manifold on which a Lie group G acts by symplectomorphisms. We get a homomorphism from the Lie algebra of G to the Lie algebra of vector fields on M by considering the corresponding one parameter subgroups of symplectomorphisms. We will refer to these vector fields as the "infinitesimal" action of G on M, and denote by $\xi^{\#}$ the infinitesimal action corresponding to $\xi \in \mathfrak{g}$. A vector field V on M is said to be Hamiltonian if the one form $\iota(V)\omega$ is exact; a Hamiltonian function for V is a function f such that $\iota(V)\omega = df$. Suppose we can find a homomorphism from $\mathfrak{g}$ to the Lie algebra of smooth functions on M (with Poisson bracket) assigning Hamiltonian functions to the infinitesimal action. The momentum map $\Phi: M \to \mathfrak{g}^*$ is then defined by $< \Phi(z), \xi >= \phi^{\xi}(z)$ where ϕ^{ξ} is the Hamiltonian function for the infinitesimal action corresponding to $\xi \in \mathfrak{g}$ (ϕ^{ξ} is called the ξ-th component of Φ). If a momentum map exists, then the action of G is said to be Hamiltonian. There are cohomological criteria for the existence and uniqueness of a momentum map; see for example [G2] and the references given there. If G is semisimple, then there exists a unique momentum map, and it is equivariant[3] with respect to the co-adjoint action of G on $\mathfrak{g}^*$.

Now let $M = G_{\mathbb{C}}/K_{\mathbb{C}}$ where G is a compact, connected, semisimple Lie group and K a closed subgroup. As in section 2, M can be G-equivariantly identified with T^*G/K, and so is equipped with a Hamiltonian G action and a unique G equivariant momentum map, Φ. Let $|\cdot|$ denote the norm on $\mathfrak{g}^*$ induced by the Killing form of $\mathfrak{g}$ and let $H: M \to \mathbb{R}$ be the G invariant function function, $|\Phi|^2$.

[3] This property of the momentum map is made part of the definition by some authors.

Lemma 8. *$H = \rho$, where ρ is the "Monge-Ampère" function in theorem 3.*

Proof. Under the identification of M with T^*G/K, ρ gets pulled back to the quadratic function "length squared of a covector" induced by the normal metric on G/K (see [Ste]). It suffices then to show that the norm squared of the momentum map on T^*G/K is this quadratic function. The symplectic form $\omega_o = -d\alpha_o$ on T^*G/K is exact, and α_o is invariant under the G action. So the G-equivariant momentum map is given over the identity coset o by (see [A-M], corollary 4.2.11)

$$\begin{aligned} < \Phi_o(p_o), \xi > &= < p_o, \xi^{\#}_{G/K}(o) > \\ &= < p_o, d\pi_e(\xi) > \\ &= < d\pi_e^*(p_o), \xi > . \end{aligned}$$

Let ξ_i be an orthonormal basis for $\mathfrak{p}$ (with respect to the Killing form), and let $v_i = d\pi_e(\xi_i)$. Let ξ_i^*, v_i^* denote the dual bases. Then $|d\pi_e^*(p_o)|^2 = \sum b_k^2$ where $< d\pi_e^*(p_o), \xi_i >= b_i$, and $|p_o|^2 = \sum a_k^2$ where $< p_o, v_i >= a_i$. The two are clearly equal. ☐

Extend the ξ_i to an orthonormal basis of $\mathfrak{g}$. Let ϕ_i be the ξ_i-th component of the momentum map on M. Then as a corollary of the above,

$$\rho = \sum_{i=1}^{\dim \mathfrak{g}} \phi_i^2 .$$

Let J be the automorphism of the real tangent bundle of M which defines its complex structure and let $\eta_i^{\#} = J\xi_i^{\#}$. Then (see [G2], lemma 5.2)

$$\text{grad } \rho = 2 \sum \phi_i \eta_i^{\#}$$

(where grad ρ is the real gradient field of ρ with respect to the Kähler metric defined by ρ). Note that the action of G extends naturally (and uniquely, by the universal property of the complexification) to an action of $G_{\mathbb{C}}$ on M. It is not hard to see that $\eta^{\#} = (\sqrt{-1}\,\xi)^{\#}$. Thus the flow of $\eta^{\#}$ is not only holomorphic, but consists of orbits of one parameter subgroups of $G_{\mathbb{C}}$.

Let $\alpha = \text{Im } \overline{\partial}\rho$ and define the "radial" vector field Ξ by $\iota(\Xi) d\alpha = \alpha$.

Lemma 9. *$\Xi = \frac{1}{2}\text{grad } \rho$ and $\Xi\rho = 2\rho$.*

Proof. See [G-S2], appendix [4]. Note that Ξ is the real part of the complex gradient field Z introduced in the proof of proposition 6. ☐

We are now ready to compute the Lie derivative of equation (4) and finish the proof of completeness.

Lemma 10. *The function $x^n/S(x)$ is monotonically increasing.*

[4] The difference in sign between here and [G-S2] is because the symplectic form is taken to have the opposite sign from our convention, so the sign of the gradient field is reversed.

Proof. Under the identification of M with T^*G/K, Im $\overline{\partial}\rho$ gets pulled back to the canonical symplectic one form, and Ξ to the radial vector field $\sum p_i \partial/\partial p_i$ (see [Ste] or [G-S], section 5). Thus

$$L_\Xi(\sqrt{-1}\,\partial\overline{\partial}\,\rho)^n = n(\sqrt{-1}\,\partial\overline{\partial}\,\rho)^n.$$

On the other hand,

$$L_\Xi(\Omega \wedge \overline{\Omega}) = \sum d\phi_i \wedge \iota(\eta_i^\#)(\Omega \wedge \overline{\Omega}) + \phi_i L_{\eta_i^\#}(\Omega \wedge \overline{\Omega}).$$

Since the flow of $\eta_i^\#$ is an orbit of a one parameter subgroup of $G_{\mathbb{C}}$ and Ω is invariant under $G_{\mathbb{C}}$, $L_{\eta_i^\#}(\Omega \wedge \overline{\Omega}) = 0$. Noting that

$$\begin{aligned} 0 &= \iota(\eta_i^\#)(d\phi_i \wedge \Omega \wedge \overline{\Omega}) \\ &= d\phi_i(\eta_i^\#)(\Omega \wedge \overline{\Omega}) - d\phi_i \wedge \iota(\eta_i^\#)(\Omega \wedge \overline{\Omega}) \end{aligned}$$

we see that

$$L_\Xi(\Omega \wedge \overline{\Omega}) = \sum \eta_i^\# \phi_i(\Omega \wedge \overline{\Omega}).$$

Using the fact that $\Xi u = u$ we obtain from equation (4) the following equation for $S(u)$:

$$u\frac{d}{du}\log\frac{u^n}{S(u)} = \sum \eta_i^\# \phi_i.$$

Since ϕ_i is the ξ-th component of the momentum map,

$$\eta_i^\# \phi_i = \sqrt{-1}\,\partial\overline{\partial}\,\rho(\xi_i^\#, \eta_i^\#) = b(\xi_i^\#, \xi_i^\#).$$

This quantity is clearly non-negative, so $u^n/S(u)$ is monotonically increasing.

□

7. Examples: T^*S^n and $T^*\mathbb{RP}^n$

In this section we will demonstrate our general result by proving it for the spheres and real projective spaces. We identify the cotangent bundle of S^n with the affine quadric

$$Q^n = \{z \in C^{n+1} \colon \sum_{i=1}^{n+1} z_i^2 = 1\}$$

as in [Szö]. The cotangent bundle of $\mathbb{RP}^n$ is identified with the affine algebraic manifold $\mathbb{CP}^n \backslash \overline{Q}^{n-1}$, where $\overline{Q}^{n-1}$ is the compact quadric in $\mathbb{CP}^n$. It is not hard to see that $\mathbb{CP}^n \backslash \overline{Q}^{n-1}$ is $Q^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by the "antipodal" map $z \to -z$. The metric we construct on Q^n will be invariant under this covering transformation and so will descend to a complete metric on $\mathbb{CP}^n \backslash \overline{Q}^{n-1}$ with the same Ricci curvature.

Let τ be the restriction to Q^n of the function $\sum_{i=1}^{n+1} z_i \overline{z_i}$. All partial derivatives of τ will be with respect to coordinates on Q^n. We will look for a Kähler potential for our metric of the form $\phi = f \circ \tau$. Computing $\det (f \circ \tau)_{i\bar{j}}$ in terms of $\tau_{i\bar{j}}$ we obtain (see [P-W])

$$\det (f \circ \tau)_{i\bar{j}} = ((f' \circ \tau)^n + (f' \circ \tau)^{n-1} (f'' \circ \tau) \tau^{i\bar{j}} \tau_i \tau_{\bar{j}}) \det \tau_{i\bar{j}}.$$

One can compute that $\det \tau_{i\bar{j}} = |h|^2 \tau$, where h is a non-zero locally defined holomorphic function. For example, if $n = 2$ and we use z_1, z_2 as coordinates where z_3 is not zero, then $h = {z_3}^{-1}$. We may also verify, either by direct computation or by using the fact that $\cosh^{-1} \tau$ satisfies the homogeneous Monge-Ampère equation, that $\tau^{i\bar{j}} \tau_i \tau_{\bar{j}} = \tau^{-1}(\tau^2 - 1)$. Thus any solution to the ordinary differential equation

$$x(f')^n + f''(f')^{n-1}(x^2 - 1) = c > 0, \tag{5}$$

real analytic in the interval $[1, \infty)$ and such that $(f \circ \tau)_{i\bar{j}} > 0$, will give us a Kähler potential for a Ricci-flat Kähler metric on Q^n. Making the change of variable $w = \cosh^{-1} x$ turns this into the exact equation

$$\frac{d}{dw}(f'(w))^n = nc(\sinh w)^{n-1},$$

i.e., $S(w) = (w/\sinh w)^{n-1}$. Clearly $w^n/S(w)$ is monotonically increasing, which confirms our previous computation.

When $n = 2$ one can easily solve this equation with the initial condition $f'(0) = 0$ to obtain $f(w) = 4\sqrt{c} \cosh w/2$. Then $g(x) := f(\cosh^{-1} x)$ solves the equation (5). Hence we obtain the following corollary.

Corollary 11. *The function $(\tau + 1)^{1/2}$ is the potential function for a complete, Ricci-flat Kähler metric on Q^2.*

Let us now show that this is the metric constructed by Eguchi and Hanson in [E-H]. We realize the cotangent bundle of S^2 as

$$T^* S^2 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon |x| = 1, x \cdot \xi = 0\}.$$

The group $G = SO(3)$ acts simply transitively on the sets $|\xi| = c > 0$ by the linear action,

$$g \cdot (x, \xi) = (gx, g\xi)$$

(which is the action of G on S^2 lifted to T^*S^2). Identify $(0, \infty) \times G$ with T^*S^2 minus the zero section by the map

$$(r, g) \mapsto (g \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, rg \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}).$$

Take as a basis of the Lie algebra $\mathfrak{so}(3)$:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Also denote by X_i the corresponding left-invariant vector fields on $(0,\infty)\times G$. Identify S^2 with $SO(3)/SO(2)$ together with the metric induced by (minus) the Killing form on $\mathfrak{so}(3)$ (this gives S^2 the round metric of radius $\sqrt{2}$). We can then make explicit the identification of T^*S^2 with the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$ given in [Ste2], and find the corresponding adapted complex structure to be

$$J\frac{\partial}{\partial r} = -X_3,\ JX_1 = -\tanh r\, X_2.$$

Let $dr, \omega^1, \omega^2, \omega^3$ be the dual basis. The Kähler metric with potential $f(r)$ is

$$2ds^2 = f''(dr)^2 + f'\tanh r\,(\omega^1)^2 + f'\coth r\,(\omega^2)^2 + f''(\omega^3)^2.$$

The $G_{\mathbb{C}}$-invariant holomorphic 2-form is

$$\sinh r\,\eta^1\wedge\eta^2$$

where $\eta^1 = dr+\sqrt{-1}Jdr$, $\eta^2 = \omega^1+\sqrt{-1}J\omega^1$. The ordinary differential equation is

$$f''f' = c\sinh r\cosh r,$$

so $f(r) = \cosh r$ (up to a positive multiplicative constant). The Ricci-flat Kähler metric is

$$2ds^2 = \cosh r\,(dr)^2 + \sinh r\tanh r\,(\omega^1)^2 + \cosh r\,((\omega^2)^2 + (\omega^3)^2).$$

After the change of variable $\cosh r = (t/a)^2$ this becomes

$$\frac{a^2}{2}ds^2 = (1-\frac{a^4}{t^4})^{-1}(dt)^2 + \frac{t^2}{4}(1-\frac{a^4}{t^4})(w^1)^2 + \frac{t^2}{4}((\omega^2)^2+(\omega^3)^2).$$

The right hand side is the Eguchi-Hanson metric with parameter a (see [G-P], eq. 4.17).

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