

A Reconstruction Theorem for Riemannian Symmetric Spaces of Noncompact Type

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ABSTRACT. Let f be a rapidly decreasing radial function on a Riemannian symmetric space of noncompact type whose spherical Fourier transform has compact support. We prove a reconstruction theorem which recovers f from the values of an integral operator applied to f on a discrete subset. When G/K is of the complex type we prove a sampling formula recovering f from its own values on a discrete subset. We give explicit results for three dimensional hyperbolic space.

1. Introduction

The Whittaker-Kotelnikov-Shannon sampling theorem states that a smooth function, f , whose Fourier transform is supported in the interval $(-\pi/\kappa, \pi/\kappa)$ for some $\kappa > 0$ can be exactly recovered from its “samples,” $f(\kappa m)$, $m \in \mathbb{Z}$, by the interpolating series

$$f(t) = \sum_{m=-\infty}^{\infty} f(\kappa m) \operatorname{sinc}\left(\frac{\pi}{\kappa}(t - \kappa m)\right), \quad (1.1)$$

where $\operatorname{sinc}(x) = \frac{\sin x}{x}$, with absolute and uniform convergence on \mathbb{R} . This result has found a wide variety of applications in signal processing and applied mathematics, and has been extended in a number of directions: using irregularly spaced sample points, sampling of not necessarily band limited functions, multidimensional sampling and sampling on manifolds. In this

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paper we will prove a reconstruction formula for radially symmetric functions which are band limited with respect to the spherical transform, a non-Euclidean analog of the Fourier transform on a symmetric space of noncompact type (e.g., real or complex hyperbolic space). Our formula reconstructs such a function from samples of an operator applied to it (see Theorem 1 and Definitions 2 and 3). We give a fairly explicit integral formula for this operator and for the analog of the sinc function in (1.1). It is not a direct analog of the sampling theorem, since it does not reconstruct from samples of the function itself. In Theorem 2 we prove a direct analog of the sampling theorem under more restrictive symmetry assumptions.

There is a well developed theory of sampling on Riemannian manifolds; see [9], [18], [19]. In these papers band limited functions on homogeneous spaces (or more generally manifolds with bounded geometry) are reconstructed from irregularly spaced samples using Lagrangian splines or iterative algorithms. Sampling theorems on symmetric spaces have also been obtained; see [4], [5], [10], [17]. In [10] the iterative reconstruction algorithm of [9] is made explicit for the hyperbolic plane. In [17] a band-limited function on a noncompact symmetric space is recovered from samples of its convolution with certain bi-invariant distributions of compact support, using either abstractly defined frames or polyharmonic splines as the analog of the sinc function. Our work is perhaps closest in spirit to [4], [5], in which a function is reconstructed from samples of its Radon transform.

We now describe our results in more detail. The goal of this paper is to give a reconstruction formula for radially symmetric functions (i.e., functions invariant under the left action of K) with compactly supported spherical transform on a Riemannian symmetric space, G/K , of Helgason's noncompact type. We recall the spherical Fourier transform maps K -invariant rapidly decreasing functions on G/K to rapidly decreasing Weyl group invariant functions on \mathfrak{a}^* , the dual of the tangent space at the identity coset of a maximal flat submanifold of G/K . Our main result is the following (Theorem 1): Suppose f is a rapidly decreasing, K -invariant function whose spherical Fourier transform is supported in the set $\Omega^b = \{\eta \in \mathfrak{a}^* : |\alpha(\eta^\sharp)| < \frac{\pi}{2} \text{ for all roots } \alpha\}$ where $\sharp: \mathfrak{a}^* \rightarrow \mathfrak{a}$ is the metric identification. Then f can be reconstructed from the values on a discrete subset of G/K of a certain operator, S , applied to f and summed against a kernel analogous to the sinc function in (1.1). The series converges to f in $L^2(G/K)$ and pointwise, uniformly on G/K . Since the set Ω^b depends on the metric we can accommodate arbitrary compact supports by scaling the metric. When G/K is of complex type we show that S is invertible and preserves the support condition on the spherical Fourier transform, and that f can be reconstructed from its own values on a discrete subset (Theorem 2).

Our strategy is to follow the proof of the Euclidean sampling theorem as much as possible. There is a natural identification of smooth, Weyl group in-

variant functions on \mathfrak{a}^* supported in Ω^b with smooth, K -invariant functions supported in a certain open set in the compact dual space U/K (Lemma 2). We denote this identification by τ . If the spherical Fourier transform, $\mathcal{F}_s f$, of a rapidly decreasing, K -invariant function on G/K is supported in Ω^b , then we consider the corresponding function $\tau\mathcal{F}_s f$ on U/K . This function is K -invariant and can be expanded in a Fourier series consisting of multiples of spherical functions, ψ_μ , on U/K , parametrized by the highest weights μ of the irreducible K -spherical unitary representations of U . The spherical function ψ_μ can be identified with the spherical function $\phi_{i(\mu+\rho)}$ for the noncompact dual G/K , analytically continued to the complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ and restricted to U/K . We now apply the inverse of τ to this Fourier series to obtain a series expansion of $\mathcal{F}_s f$ in terms of the functions $\phi_{i(\mu+\rho)}$, analytically continued and restricted to U/K . We then apply the inverse spherical Fourier transform to this series to obtain a series expansion for f . We interpret the coefficients in this series as the values on a discrete subset of G/K (parametrized by the highest weights μ) of the image of f under an integral operator S . This operator, which we call the “sampling operator,” is defined by integrating $\mathcal{F}_s f$, multiplied by an analytically continued spherical function and a Jacobian factor, over \mathfrak{a}^* (Definitions 2 and 3). The analog of the sinc function is given in Definition 4. It is essentially the inverse spherical Fourier transform of a spherical function multiplied by the characteristic function of Ω^b , in close analogy with the Euclidean case. The uniform convergence of the series on G/K follows from well-known estimates on the positive definite spherical functions.

The proof of the sampling theorem in the Euclidean case depends in part on the fact that the functions $\phi_\xi(x) = e^{ix\xi}$ are symmetric in the x and ξ variables. In the non-Euclidean setting this is not true: the analogous functions are integrals over K of $e^{(i\lambda+\rho)\mathcal{A}(k \exp \omega^\sharp)}$ and these are not symmetric in λ and ω (see Section 2 for an explanation of the notation). This is why our reconstruction theorem involves the values of Sf instead of f itself. When G/K is of the complex type we are able to find a simple relationship between $\phi_\lambda(\exp \omega^\sharp)$ and $\phi_{-i\omega}(\exp i\lambda^\sharp)$ (Lemma 4). This allows us to show that the sampling operator, rescaled by a Jacobian factor, is an invertible, bounded operator on the relevant Paley-Wiener spaces (Corollary 1), and to prove a sampling theorem involving only the values of f itself (Theorem 2).

A radially symmetric function f on G/K can be identified with a Weyl group invariant function on the Euclidean space \mathfrak{a} . One can then consider the Euclidean Fourier transform, \mathcal{F}_o , of f . In general the compactness of the support of $\mathcal{F}_s f$ does not imply that the support of $\mathcal{F}_o f$ is compact. If G/K is complex, then we show in Section 4.2 that if $\mathcal{F}_s f$ has compact support, then there is a continuous function, Ξ , on \mathfrak{a} such that the Euclidean Fourier transform of Ξf has compact support. The simplest example is hyperbolic three space where Ξ is essentially the hyperbolic sine. In Section

4.4 we consider an explicit function on hyperbolic three space and compare its reconstruction using a partial sum of our sampling series with its reconstruction using the corresponding partial sum in the Euclidean sampling series. The reconstruction based on Theorem 1 gives the smallest relative error.

Another approach to reconstruction theorems on symmetric spaces is to use the Abel transform. The Abel transform, \mathbf{A} , is a bijection from $\mathcal{S}^2(G/K)$ onto $\mathcal{S}_W(A)$ with the property that $\mathcal{F}_o \mathbf{A}f = \mathcal{F}_s f$. Thus if $\mathcal{F}_s f$ has compact support, then $\mathbf{A}f$ is a Paley-Wiener function in the Euclidean sense and has a Whittaker-Kotelnikov-Shannon sampling expansion. Inverting the Abel transform leads to a reconstruction formula for f in terms of samples $\mathbf{A}f$, similar in spirit to our Theorem 1 (we thank the referee for pointing this out). There is no explicit formula for the inverse of the Abel transform except in certain special cases (see, e.g., [21, Section 4]). The advantage of our approach is the explicit nature of the formulas in Definitions 2, 3, and 4.

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2. The Fourier Transform on a Symmetric Space of Noncompact Type

In this section we will give the background needed to introduce the Paley-Wiener sampling spaces (Definition 1). These spaces have previously been considered in [16] and [1]. Let G/K be a Riemannian globally symmetric space of Helgason's noncompact type [6, Chapter VI]. In particular, G and K are connected and G is semisimple. We will also assume that K compact, or equivalently, the center of G is finite. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G . Choose a maximal abelian subspace \mathfrak{a} of \mathfrak{p} and consider the roots, Σ , of \mathfrak{g} respect to \mathfrak{a} (the "restricted" roots). Choose an ordering on \mathfrak{a}^* and let Σ^+ denote the set of positive roots. This gives the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ and the corresponding global diffeomorphism $G = NAK$ where N and A are simply connected subgroups of G . Let $\mathcal{A}: G \rightarrow \mathfrak{a}$ be the logarithm of the Iwasawa projection onto A . The spherical functions on G/K are the smooth, K -invariant joint eigenfunctions of the G -invariant differential operators on G/K , normalized by $\phi(o) = 1$ where o is the identity coset. In the noncompact case the spherical functions are parametrized by $\mathfrak{a}_\mathbb{C}^*/W$, where W is the Weyl group, and are given by the integral formula

$$\phi_\lambda(g) = \int_K e^{(i\lambda + \rho)\mathcal{A}(kg)} dk \quad (\lambda \in \mathfrak{a}_\mathbb{C}^*) \quad (2.1)$$

(see [8], Chapter IV, Section 4)¹. Here ρ is the half-sum of the positive roots with multiplicity and the Haar measure dk of K is normalized to have unit volume. We have $\phi_\lambda = \phi_{\lambda'}$ if and only if there is a $s \in W$ such that $\lambda = s\lambda'$.

Let $\mathcal{S}^2(G/K)$ denote the Schwartz space of rapidly decreasing, K -invariant square integrable functions on G/K , i.e., the K bi-invariant smooth functions on G such that for all G -invariant differential operators \mathbf{D} and all $q \in \mathbb{Z}^+$,

$$\sup_{g \in G} |(1 + |g|)^q \phi_0(g)^{-1} \mathbf{D}f(g)| < \infty$$

(here $|g|$ is the length of the H in the polar decomposition $g = k(\exp H)k'$). Let $\mathcal{S}(\mathfrak{a}^*)^W$ denote the space of W -invariant rapidly decreasing (in the usual sense) functions on \mathfrak{a}^* . The spherical Fourier transform of a function $f \in \mathcal{S}^2(G/K)$ is the function $\mathcal{F}_s f \in \mathcal{S}(\mathfrak{a}^*)^W$ given by

$$\mathcal{F}_s f(\lambda) = \int_G f(g) \phi_{-\lambda}(g) dg.$$

The spherical Fourier transform is topological homeomorphism of Frechet spaces from $\mathcal{S}^2(G/K)$ onto $\mathcal{S}(\mathfrak{a}^*)^W$ and has a pointwise inversion formula given by

$$f(g \cdot o) = |W|^{-1} \int_{\mathfrak{a}^*} \mathcal{F}_s f(\lambda) \phi_\lambda(g) |\mathbf{c}(\lambda)|^{-2} d\lambda \quad (2.2)$$

where $|W|$ is the order of the Weyl group. It satisfies the Plancherel theorem,

$$\int_G |f(g)|^2 dg = |W|^{-1} \int_{\mathfrak{a}^*} |\mathcal{F}_s f(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

and extends to an isometry

$$\mathcal{F}_s : L^2(G/K)^K \longrightarrow L^2(\mathfrak{a}^*, |W|^{-1} |\mathbf{c}(\lambda)|^{-2} d\lambda)^W$$

(see, e.g., [20], [2]). Here we have used the Riemannian measures on \mathfrak{a} , \mathfrak{a}^* , normalized so that the inversion formula for the Euclidean Fourier transform holds without a multiplicative constant, and normalized the measure on G as in [8], Chapter IV, Exercise C.4.

Definition 1. The K -invariant Paley-Wiener sampling space associated with a relatively compact W -invariant subset ϖ of \mathfrak{a}^* is

$$PW(\varpi)^K := \{f \in \mathcal{S}^2(G/K) : \text{supp } \mathcal{F}_s f \subset \varpi\}.$$

¹We will freely identify functions on G/K with right K -invariant functions on G , and K -invariant functions on G/K with K bi-invariant functions on G .

3. Reconstruction of K -Invariant Paley-Wiener Functions

Our main result depends on realizing G/K and a dual symmetric space of compact type as embedded, totally real submanifolds of a common universal complexification. We recall that the universal complexification of a connected Lie group G is a connected complex Lie group $G_{\mathbb{C}}$ and a continuous homomorphism $\gamma: G \rightarrow G_{\mathbb{C}}$ with the following property: if $\eta: G \rightarrow H_{\mathbb{C}}$ is a continuous homomorphism into another connected complex Lie group $H_{\mathbb{C}}$, then there is a unique holomorphic homomorphism $\eta_{\mathbb{C}}: G_{\mathbb{C}} \rightarrow H_{\mathbb{C}}$ such that $\eta = \eta_{\mathbb{C}} \circ \gamma$. We will assume from now on that G is a connected, semisimple Lie group admitting a faithful finite-dimensional representation. This implies that the homomorphism γ is injective [12, p. 407]. Since G is semisimple with finite center, G can be identified with a closed subgroup of $G_{\mathbb{C}}$ [12, Lemma 1]. The Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$. Let U be the connected Lie subgroup of $G_{\mathbb{C}}$ with the Lie algebra of compact type $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Then U is compact and U/K is a Riemannian symmetric space of compact type dual to G/K .

Lemma 1. *The universal complexification of U is the inclusion of U into $G_{\mathbb{C}}$.*

Proof. The universal complexification of a Lie group G is constructed in the following way [12]. Let $p: \hat{G} \rightarrow G$ be the universal cover of G and let $F = p^{-1}(e)$. Let $\hat{G}_{\mathbb{C}}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and let $\sigma: \hat{G} \rightarrow \hat{G}_{\mathbb{C}}$ be the homomorphism whose differential is the natural inclusion of \mathfrak{g} in $\mathfrak{g}_{\mathbb{C}}$. Let F^* be the smallest closed complex subgroup of $\hat{G}_{\mathbb{C}}$ containing $\sigma(F)$. F^* is contained in the center of $\hat{G}_{\mathbb{C}}$. Then σ descends to a homomorphism $\gamma: G \rightarrow \hat{G}_{\mathbb{C}}/F^* := G_{\mathbb{C}}$. This is the universal complexification of G .

Let \hat{U} denote the connected subgroup of $\hat{G}_{\mathbb{C}}$ with Lie algebra \mathfrak{u} . Then \hat{U} is a maximal compact subgroup of $\hat{G}_{\mathbb{C}}$, so \hat{U} is simply connected and contains the center of $\hat{G}_{\mathbb{C}}$. Then \hat{U}/F^* is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra \mathfrak{u} , so $U = \hat{U}/F^*$. The realization of U as \hat{U}/F^* shows that the inclusion of U into $G_{\mathbb{C}}$ is the universal complexification of U . \square

Let $K_{\mathbb{C}}$ be the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$. Then $K_{\mathbb{C}}$ is a closed subgroup of $G_{\mathbb{C}}$ and we can form the complex homogeneous manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$. It contains G/K and U/K as embedded, totally real submanifolds. Let

$$\Omega := \left\{ H \in \mathfrak{a} : |\alpha(H)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma \right\}.$$

Then the Iwasawa projection $\mathcal{A}: G \rightarrow \mathfrak{a}$ extends to a holomorphic map

$$\mathcal{A}: G \exp(i\Omega)K_{\mathbb{C}} \rightarrow \mathfrak{a}_{\mathbb{C}}$$

with values in $\mathfrak{a} + i\Omega$. As a consequence the spherical functions ϕ_λ extend to holomorphic functions on $G \exp(i\Omega)K_{\mathbb{C}}$ (see [14], Section 1).

3.1 The Sampling Operator

Let Ω^b be the image of Ω under the metric identification $^b: \mathfrak{a} \rightarrow \mathfrak{a}^*$ with inverse $^\sharp$, i.e.,

$$\Omega^b = \left\{ \lambda \in \mathfrak{a}^* : |\alpha(\lambda^\sharp)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma \right\}.$$

Lemma 2. *There is a natural isomorphism*

$$\tau: C_c^\infty(\Omega^b)^W \xrightarrow{\cong} C_c^\infty(\mathcal{U})^K$$

where \mathcal{U} is the neighborhood of the identity coset in U/K given after (3.2) below and the superscripts indicate invariance under the corresponding group action.

Proof. The C^∞ Chevalley theorem ([3]; see also [8], Chapter II, Theorem 5.8 and Corollary 5.11) asserts that restriction from \mathfrak{p} to \mathfrak{a} induces a linear isomorphism of $C^\infty(\mathfrak{p})^K$ onto $C^\infty(\mathfrak{a})^W$ and of $C_c^\infty(\mathfrak{p})^K$ onto $C_c^\infty(\mathfrak{a})^W$. Using the polar decomposition $\mathfrak{p} = \text{Ad}(K)\mathfrak{a}$ it is easy to see that the restriction map gives a linear isomorphism

$$C_c^\infty(\Omega)^W \cong C_c^\infty(\text{Ad}(K)\Omega)^K. \quad (3.1)$$

The exponential map composed with multiplication by i ,

$$\exp \circ M_i: V \in \mathfrak{p} \rightarrow \exp(iV) \cdot o \in U/K, \quad (3.2)$$

is a K -equivariant diffeomorphism between $\text{Ad}(K)\Omega$ and a neighborhood \mathcal{U} of the origin in U/K . Thus we have a linear isomorphism

$$(\exp \circ M_i)_* : C_c^\infty(\text{Ad}(K)\Omega)^K \xrightarrow{\cong} C_c^\infty(\mathcal{U})^K. \quad (3.3)$$

Composing (3.1), (3.3) and the metric identification of \mathfrak{a} with \mathfrak{a}^* gives a linear isomorphism

$$\tau: C_c^\infty(\Omega^b)^W \xrightarrow{\cong} C_c^\infty(\mathcal{U})^K.$$

□

Remark 1. Note the inverse of τ , mapping K -invariant functions on U/K supported in \mathcal{U} to W -invariant functions on \mathfrak{a}^* supported in Ω^b , is simply $\tau^{-1}F(\lambda) = F(\exp i\lambda^\sharp \cdot o)$.

We now identify W -invariant functions on \mathfrak{a}^* with K -invariant functions on G/K . The exponential map

$$\exp: \mathfrak{p} \rightarrow G/K$$

is a global K -equivariant diffeomorphism and gives an isomorphism

$$C^\infty(\mathfrak{p})^K \cong C^\infty(G/K)^K.$$

Composing this with the metric identification $\mathfrak{a} \cong \mathfrak{a}^*$ and the C^∞ Chevalley theorem gives an isomorphism

$$\eta: C^\infty(\mathfrak{a}^*)^W \xrightarrow{\cong} C^\infty(G/K)^K.$$

For $k \in K$, $H \in \mathfrak{a}$, we have $(\eta f)(k \exp H \cdot o) = f(H^\flat)$. The inverse of η , mapping K -invariant functions on G/K to W -invariant functions on \mathfrak{a}^* , is $\eta^{-1}f(\lambda) = f(\exp \lambda^\sharp \cdot o)$.

Definition 2. For $f \in PW(\Omega^\flat)^K$ let C_f be the function on \mathfrak{a}^* defined by

$$C_f(\omega) = \frac{c_0}{|W|} \int_{\lambda \in \mathfrak{a}^*} \mathcal{F}_s f(\lambda) \phi_{-i\omega}(\exp i\lambda^\sharp) \prod_{\alpha \in \Sigma^+} |\sin \alpha(\lambda^\sharp)|^{m_\alpha} d\lambda.$$

where c_0 is given by (3.16) below.

Lemma 3. *If $f \in PW(\Omega^\flat)^K$ then C_f is a smooth, W -invariant function on \mathfrak{a}^* .*

Proof. The map $(\omega, \lambda) \mapsto \mathcal{F}_s(\lambda) \phi_{-i\omega}(\exp i\lambda^\sharp)$ is smooth on $\mathfrak{a}^* \times \mathfrak{a}^*$ because the support of $\mathcal{F}_s f$ is contained in Ω^\flat and the spherical functions can be analytically continued to $G \exp(i\Omega) K_{\mathbb{C}}$, with smooth dependence on the parameter $-i\omega$ by (2.1). Since the support of $\mathcal{F}_s f$ is compact we may differentiate under the integral sign to prove that C_f is a smooth function on \mathfrak{a}^* . C_f is W -invariant because $\phi_{-is\omega} = \phi_{-i\omega}$ on G and hence on $G \exp(i\Omega) K_{\mathbb{C}}$ by analytic continuation. □

Definition 3. The sampling operator $S: PW(\Omega^\flat)^K \rightarrow C^\infty(G/K)^K$ is the operator defined by $Sf = \eta C_f$, i.e., for $\omega \in \mathfrak{a}^*$, $Sf(\exp \omega^\sharp \cdot o) = C_f(\omega)$.

Theorem 1 below will show that f can be reconstructed from the values of Sf on the lattice $\exp(\Lambda + \rho)^\sharp \cdot o$ in G/K .

3.2 The Reconstruction Theorem

Let F be a smooth function on the compact symmetric space U/K . The Fourier series for F can be written as

$$F = \sum_{\mu \in \Lambda} d(\mu) F \times \psi_\mu \tag{3.4}$$

(see [7, Chapter III, Proposition 9.1] and (14), Chapter III.9.2, loc. cit.). The lattice $\Lambda = \{\mu \in \mathfrak{a}^*: \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+\}$ parametrizes the

complex valued finite dimensional irreducible K -spherical representations of G [8, Chapter V, Theorem 4.1]. Since G and U are embedded in a common universal complexification, the universal property implies that Λ also parametrizes the spherical representations of U . $d(\mu)$ is the complex dimension of the finite dimensional irreducible spherical representation of U with highest restricted weight μ , and ψ_μ is the spherical function on U corresponding to the spherical representation of U with highest weight $\mu \in \Lambda$. The series (3.4) converges absolutely and uniformly (and in the smooth topology; see [22]).

Definition 4. For $\mu \in \Lambda$ and $g \in G$, let

$$\begin{aligned} I(\mu, g \cdot o) &= |W|^{-1} \int_{\Omega^b} \phi_{i(\mu+\rho)}(\exp i\lambda^\sharp) \phi_\lambda(g) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= \mathcal{F}_s^{-1} \left(\phi_{i(\mu+\rho)}(\exp i(\cdot)^\sharp) \chi_{\Omega^b}(\cdot) \right) (g \cdot o) \end{aligned} \quad (3.5)$$

where χ_{Ω^b} is the characteristic function of the set $\Omega^b \subset \mathfrak{a}^*$.

Theorem 1. Let $f \in PW(\Omega^b)^K$. Then for all $g \in G$,

$$f(g \cdot o) = \sum_{\mu \in \Lambda} d(\mu) S f(\exp(\mu + \rho)^\sharp \cdot o) I(\mu, g \cdot o). \quad (3.6)$$

The sum converges to f in the $L^2(G/K)$ sense and pointwise, uniformly on G/K .

Proof. We can identify $\tau \mathcal{F}_s f$ with a smooth, K -invariant function on U/K which vanishes outside of \mathcal{U} . The function $(\tau \mathcal{F}_s f) \times \psi_\mu$ is a K -invariant vector in the spherical representation of type μ generated by the translates of ψ_μ . Since there is a one dimensional space of K -invariant vectors, $(\tau \mathcal{F}_s f) \times \psi_\mu$ must be a multiple of ψ_μ . We define the Fourier coefficient f_μ by

$$(\tau \mathcal{F}_s f) \times \psi_\mu = f_\mu \psi_\mu \quad (3.7)$$

and write the Fourier series (3.4) as

$$\tau \mathcal{F}_s f = \sum_{\mu \in \Lambda} d(\mu) f_\mu \psi_\mu. \quad (3.8)$$

The spherical function ψ_μ can be analytically continued to a holomorphic function on $G_{\mathbb{C}}$. Its restriction to G is the spherical function ϕ_λ given by (2.1) with parameter $\lambda = i(\mu + \rho)$ [7, Chapter III.9.2]². Thus (3.8) can be

²In [7, Chapter III.9.2] it is assumed that U is simply connected. The result is still true under our assumptions, since U and G share a common universal complexification.

written as

$$\tau \mathcal{F}_s f(u \cdot o) = \sum_{\mu \in \Lambda} d(\mu) f_\mu \phi_{i(\mu+\rho)}(u), \quad u \cdot o \in U/K. \quad (3.9)$$

Applying τ^{-1} to equation (3.9), using Remark 1, and noting that the support of $\mathcal{F}_s f$ is contained in Ω^b allows us to write

$$\mathcal{F}_s f(\lambda) = \sum_{\mu \in \Lambda} d(\mu) f_\mu \phi_{i(\mu+\rho)}(\exp i\lambda^\sharp) \chi_{\Omega^b}(\lambda) \quad (3.10)$$

where χ_{Ω^b} is the characteristic function of $\Omega^b \subset \mathfrak{a}^*$. The series converges uniformly to $\mathcal{F}_s f$ and hence, since the support of $\mathcal{F}_s f$ is contained in the relatively compact set Ω^b , converges in $L^2(\mathfrak{a}^*)$ with respect to the Plancherel measure. Since f is in $\mathcal{S}^2(G/K)$ we have, pointwise for all $g \in G$,

$$f(g \cdot o) = \mathcal{F}_s^{-1} \left(\sum_{\mu \in \Lambda} d(\mu) f_\mu \phi_{i(\mu+\rho)}(\exp i(\cdot)^\sharp) \chi_{\Omega^b}(\cdot) \right) (g \cdot o). \quad (3.11)$$

Since \mathcal{F}^{-1} is an L^2 isometry the series obtained by exchanging the sum and inverse Fourier transform in (3.11) converges to f in the L^2 sense. Since the series on the right hand side of (3.10) converges uniformly we may interchange the sum and integral pointwise to obtain, for all $g \in G$,

$$f(g \cdot o) = \sum_{\mu \in \Lambda} d(\mu) f_\mu I(\mu, g \cdot o) \quad (3.12)$$

where $I(\mu, g \cdot o)$ is as in Definition 4.

To prove that (3.12) converges uniformly on G/K we first note that $d(\mu) f_\mu$ is rapidly decreasing in μ (this follows from the fact that f_μ is rapidly decreasing in μ [22, Theorem 3], and $d(\mu)$ is bounded by a polynomial in $|\mu|$ by the Weyl dimension formula). Thus it suffices to show that $I(\mu, \cdot)$ is uniformly bounded on G/K , independent of $\mu \in \Lambda$. To see this we note that ϕ_λ is a positive definite spherical function on G (since $\lambda \in \mathfrak{a}^*$) [8, Exercise IV.B.9], and $\phi_{i(\mu+\rho)}$ is a positive definite spherical function on U (it is ψ_μ above) [8, Chapter IV, Theorem 4.2]. Thus for $g \in G$ and $\lambda \in \mathfrak{a}^*$,

$$|\phi_\lambda(g)| \leq \phi_\lambda(e) = 1 \quad \text{and} \quad |\phi_{i(\mu+\rho)}(\exp i\lambda^\sharp)| \leq \psi(e) = 1.$$

The uniform bound on $I(\mu, \cdot)$ follows easily and completes the proof of uniform convergence.

To complete the proof of Theorem 1 we need to show that $f_\mu = Sf(\exp(\mu + \rho)^\sharp \cdot o)$. Evaluating (3.7) at the identity coset gives an integral expression for f_μ :

$$f_\mu = \int_U (\tau \mathcal{F}_s f) \circ \pi(u) \psi_\mu(u^{-1}) du. \quad (3.13)$$

Since the support of $\tau\mathcal{F}_s f$ is contained in $\mathcal{U} \subset G(\exp i\Omega)K_{\mathbb{C}}$, we have on the support of $\tau\mathcal{F}_s f$ that

$$\psi_{\mu}(u^{-1}) = \phi_{i(\mu+\rho)}(u^{-1}) = \phi_{-i(\mu+\rho)}(u) \quad (3.14)$$

(the last equality is an elementary property of spherical functions on G which holds on $G(\exp i\Omega)K_{\mathbb{C}}$ by analytic continuation). Using (3.14) we can write (3.13) as

$$f_{\mu} = \int_{U/K} \tau\mathcal{F}_s f(u \cdot o) \phi_{-i(\mu+\rho)}(u) du_K. \quad (3.15)$$

We will express f_{μ} as an integral over \mathfrak{a}^* in the spirit of the inverse Fourier transform. The ‘‘polar coordinate map’’

$$\psi: Q_o^{\flat} \times K/M \mapsto U/K, \quad \psi(\lambda, kM) = k(\exp i\lambda^{\sharp}) \cdot o$$

is a diffeomorphism onto an open dense subset $(U/K)_r$ of U/K (see [8, Chapter I, Theorem 5.11]) where, if $\mu_1, \mu_2, \dots, \mu_l$ are the simple restricted roots and δ is the highest root, then

$$Q_o = \{H \in \mathfrak{a}: \mu_i(H) > 0, i = 1, 2, \dots, l, \text{ and } \delta(H) < \pi\}$$

(note that Q_o is the intersection of 2Ω with the positive Weyl chamber). The restriction of τF to $(U/K)_r$ is simply $(F|_{Q_o^{\flat} \times K/M}) \circ \psi^{-1}$. We can write the right hand side of (3.15) as an integral over $(U/K)_r$. Pulling back to $Q_o^{\flat} \times K/M$ using the map ψ gives

$$f_{\mu} = \int_{Q_o^{\flat} \times K/M} \mathcal{F}_s f(\lambda) \phi_{-i(\mu+\rho)}(k \exp i\lambda^{\sharp}) |\det d\psi_{(\lambda, kM)}| d\lambda dkM.$$

The Jacobian factor $|\det d\psi_{(\lambda, kM)}|$ is [8, Chapter I, Corollary 5.12]

$$c_0 \prod_{\alpha \in \Sigma^+} |\sin(\alpha(\lambda^{\sharp}))|^{m_{\alpha}} \quad \text{where} \quad c_0^{-1} = \int_{Q_o} \prod_{\alpha \in \Sigma^+} |\sin \alpha(H)|^{m_{\alpha}} dH \quad (3.16)$$

and dH is the Euclidean measure on \mathfrak{a} induced the the Killing form. Since the integrand is independent of $kM \in K/M$, the volume of K/M is one and the support of $(\mathcal{F}_s f)|_{\mathfrak{a}_+^*}$ is contained in Q_o^{\flat} we have

$$f_{\mu} = c_0 \int_{\mathfrak{a}_+^*} \mathcal{F}_s f(\lambda) \phi_{-i(\mu+\rho)}(\exp i\lambda^{\sharp}) \prod_{\alpha \in \Sigma^+} |\sin(\alpha(\lambda^{\sharp}))|^{m_{\alpha}} d\lambda.$$

From the invariance of the integrand under the action of the Weyl group we obtain $f_{\mu} = C_f(\mu + \rho) = S f(\exp(\mu + \rho)^{\sharp} \cdot o)$. This completes the proof of Theorem 1. \square

4. Symmetric Spaces of Complex Type

An irreducible Riemannian symmetric space G/K is of the complex type if G has the structure of a simple, connected complex Lie group. These are exactly the Riemannian globally symmetric spaces of Helgason's Type IV (see [6], Chapter X, Section 6.1). The complex type is also characterized as those symmetric spaces whose restricted roots form a reduced root system with all multiplicities equal to two. The symmetric space of compact type dual to G/K is $(K \times K)/K^*$, where K^* is the diagonal in $K \times K$, which can be identified with K equipped with a bi-invariant metric. For the rest of this section we will assume that G/K is of the complex type. Since a complex semisimple Lie group admits a faithful finite dimensional representation [13, Chapter XVII.3], we may apply the results of Section 3.

Lemma 4 shows one can “reverse the roles” of the \mathfrak{a}^* variables and the group variables in the spherical function $\phi_\lambda(g)$, up to a multiplicative factor. This will be used to make the reconstruction formula (3.6) much more explicit and to give a sampling formula in terms of the values of f itself (Theorem 2). In Section 4.2 we show that if $f \in PW(\Omega^b)^K$, then a closely related function satisfies the hypotheses of the Euclidean sampling theorem. Finally we will illustrate these sampling formulae with an explicit example in three dimensional hyperbolic space.

4.1 The Sampling Formula

We recall that the spherical functions for symmetric spaces of complex type take the following simple form: for $\lambda \in \mathfrak{a}_\mathbb{C}$ and $\omega \in \mathfrak{a}^*$,

$$\phi_\lambda(\exp \omega^\sharp) = \phi_0(\exp \omega^\sharp) \int_K e^{i\lambda(\text{Ad}(k)\omega^\sharp)} dk, \quad \phi_0(\exp \omega^\sharp) = \prod_{\alpha \in \Sigma^+} \frac{1}{\text{sinc } \alpha(i\omega^\sharp)} \quad (4.1)$$

([8], Theorem 4.7, §4, Chapter IV). In addition it is well known that the spherical functions can be expressed as alternating sums of exponentials:

$$\begin{aligned} \phi_\lambda(\exp \omega^\sharp) &= \frac{\pi(\rho)}{\pi(i\lambda)} \cdot \frac{\sum_{s \in W} (\det s) e^{is\lambda(\omega^\sharp)}}{\sum_{s \in W} (\det s) e^{s\rho(\omega^\sharp)}} \\ &= \frac{1}{2^{q/2}} \frac{\pi(\rho)}{\pi(i\lambda)} \sum_{s \in W} (\det s) e^{is\lambda(\omega^\sharp)} \prod_{\alpha \in \Sigma^+} (\sinh \alpha(\omega^\sharp))^{-1} \end{aligned} \quad (4.2)$$

where W is the Weyl group, $\pi(\lambda) = \prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle$, and $q = \dim \mathfrak{n}$ ([8], Chapter IV, Section 5.2 and the proof of Proposition 3.10, page 268).

Lemma 4. For all $\lambda, \omega \in \Omega^b$,

$$\phi_{-i\omega}(\exp i\lambda^\sharp) = \frac{\phi_0(\exp i\lambda^\sharp)}{\phi_0(\exp \omega^\sharp)} \cdot \phi_\lambda(\exp \omega^\sharp).$$

Proof. Expression (4.1) holds for $\omega \in \mathfrak{a}^* + i\Omega^b$ by analytic continuation. Note ϕ_0 is positive on \mathfrak{a} and on $i\Omega$. The Lemma follows directly from (4.1) since $i\lambda(\text{Ad}(k)\omega^\sharp) = i(-i\omega)(\text{Ad}(k^{-1})i\lambda^\sharp)$ and the volume element on K is invariant under the map $k \rightarrow k^{-1}$. (The Lemma also follows from (4.2) by direct computation). \square

We obtain the following simple expression for the sampling operator, rescaled by ϕ_0 , as a multiplier on the spherical Fourier transform side.

Lemma 5. *Let $f \in PW(\Omega^b)^K$. Then for all $\omega \in \mathfrak{a}^*$,*

$$\phi_0 S f(\exp \omega^\sharp \cdot o) = c_1 \mathcal{F}_s^{-1} \left(\mathcal{F}_s f(\cdot) \prod_{\alpha \in \Sigma^+} \text{sinc } \alpha \left((\cdot)^\sharp \right) \right) (\exp \omega^\sharp \cdot o) \quad (4.3)$$

where $c_1 = c_0 \prod_{\alpha \in \Sigma^+} \alpha(\rho^\sharp)^2$ and c_0 is given by (3.16).

Proof. From Definitions 2 and 3 and Lemma 4 we obtain

$$\begin{aligned} & S f(\exp \omega^\sharp \cdot o) \\ &= \frac{c_0}{|W| \phi_0(\exp \omega^\sharp)} \int_{\lambda \in \mathfrak{a}^*} \mathcal{F}_s f(\lambda) \phi_0(\exp i\lambda^\sharp) \phi_\lambda(\exp \omega^\sharp) \prod_{\alpha \in \Sigma^+} \sin^2 \alpha(\lambda^\sharp) d\lambda. \end{aligned}$$

Using the well-known expression for the \mathbf{c} -function in the complex case, $\mathbf{c}(\lambda) = \prod_{\alpha \in \Sigma^+} \frac{\langle \alpha, \rho \rangle}{\langle \alpha, i\lambda \rangle}$, noting that $\frac{\langle \alpha, \rho \rangle}{\langle \alpha, i\lambda \rangle} = \frac{\alpha(\rho^\sharp)}{i\alpha(\lambda^\sharp)}$, and using the expression for ϕ_0 in (4.1) gives

$$\phi_0(\exp i\lambda^\sharp) \prod_{\alpha \in \Sigma^+} \sin^2 \alpha(\lambda^\sharp) = |\mathbf{c}(\lambda)|^{-2} \prod_{\alpha \in \Sigma^+} \alpha(\rho^\sharp)^2 \prod_{\alpha \in \Sigma^+} \text{sinc } \alpha(\lambda^\sharp).$$

The definition of the inverse spherical Fourier transform (2.2) and of the constant c_0 in (3.16) gives the result. \square

Corollary 1. *The map $f \mapsto \phi_0 S f$ is a bijection from $PW(\Omega^b)^K$ onto itself and extends to a bounded operator on $L^2(G/K)$ with norm less than one.*

Proof. The spherical Fourier transform is a bijection from $\mathcal{S}^2(G/K)$ onto $\mathcal{S}(\mathfrak{a}^*)^W$ and restricts to a bijection from $PW(\Omega^b)^K$ onto $C_c^\infty(\Omega^b)^W$. Since $J(\lambda) := \prod_{\alpha \in \Sigma^+} \frac{\sin \alpha(\lambda^\sharp)}{\alpha(\lambda^\sharp)}$ is a smooth, non-vanishing, W -invariant function on \mathfrak{a}^* , multiplication by J is a bijection from $C_c^\infty(\Omega^b)^W$ onto itself. Then (4.3) shows that $\phi_0 S$ is a bijection from $PW(\Omega^b)^K$ onto itself. To prove that $\phi_0 S$ extends to a bounded operator on $L^2(G/K)$ of norm less than one, it suffices to observe that the Fourier transform is an isometry and multiplication by J is a bounded operator of norm less than one on $L^2(\mathfrak{a}^*, |W|^{-1} |\mathbf{c}(\lambda)|^{-2} d\lambda)$. \square

Theorem 2. For all $f \in PW(\Omega^b)^K$, $g \in G$,

$$f(g \cdot o) = \sum_{\mu \in \Lambda} d(\mu) (\phi_0^{-1} f)(\exp(\mu + \rho)^\sharp \cdot o) \phi_0 SI(\mu, \cdot)(g \cdot o) \quad (4.4)$$

where, for $H \in \mathfrak{a}$,

$$\begin{aligned} & \phi_0 SI(\mu, \cdot)(\exp H \cdot o) \\ &= c_1 \mathcal{F}_s^{-1} \left(\phi_{i(\mu+\rho)}(\exp i(\cdot)^\sharp) \chi_{\Omega^b}(\cdot) \prod_{\alpha \in \Sigma^+} \text{sinc } \alpha((\cdot)^\sharp) \right) (\exp H \cdot o) \end{aligned} \quad (4.5)$$

and c_1 is as in Lemma 5. The sum converges to f in $L^2(G/K)$ and pointwise, uniformly on G/K .

Proof. The expression (4.5) for $\phi_0 SI(\mu, \cdot)$ follows immediately from Definition 4 and Lemma 5. By Corollary 1 we can apply Theorem 1 to $(\phi_0 S)^{-1} f$ and note that $S(\phi_0 S)^{-1} f = \phi_0^{-1} f$ to obtain, pointwise and in the L^2 sense,

$$(\phi_0 S)^{-1} f(g \cdot o) = \sum_{\mu \in \Lambda} d(\mu) (\phi_0^{-1} f)(\exp(\mu + \rho)^\sharp \cdot o) I(\mu, g \cdot o). \quad (4.6)$$

We apply $\phi_0 S$ to both sides of (4.6) and justify exchanging the summation with $\phi_0 S$ in the L^2 sense by noting that $\phi_0 S$ is a bounded operator on $L^2(G/K)$, so the sum on the right hand side of (4.4) converges to f in L^2 . To show that the sum converges uniformly on G/K , we note that $\phi_0^{-1} f(\exp(\mu + \rho)^\sharp \cdot o)$ decreases faster than any polynomial in μ since $f \in S^2(G/K)$, and the integration in (4.5) takes place over the relatively compact set Ω^b . Then the same argument as in the proof of Theorem 1 shows that the convergence is uniform on G/K . \square

4.2 Reduction to the Euclidean Case

It is natural to ask if the spherical Fourier transform of a radial function f has compact support, then does the Euclidean Fourier transform of f , regarded as a function on \mathfrak{a} , also have compact support. We will show that if G/K is a noncompact symmetric space of complex type, then a multiple of f does have compactly supported Euclidean Fourier transform.

In the complex case the spherical Fourier transform can be written in ‘‘polar coordinates’’ ([8], Chapter I, Section 5.2, Theorem 5.8) as

$$\begin{aligned} & \mathcal{F}_s f(\lambda) \\ &= c \frac{1}{2^{q/2}} \frac{\pi(\rho)}{\pi(-i\lambda)} \sum_{s \in W} (\det s) \int_{\mathfrak{a}^+} f(\exp H \cdot o) e^{-is\lambda(H)} \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H)) dH. \end{aligned}$$

After making the change of variable $H' = s^{-1}H$ in each integral in the sum we see that, since the Weyl group acts simply transitively on the set of all Weyl chambers,

$$2^{q/2}\pi(-i\lambda)\mathcal{F}_s f(\lambda) = c\pi(\rho) \int_{\mathfrak{a}} f(\exp H \cdot o)\Xi(H)e^{-i\lambda(H)} dH$$

where

$$\Xi(H) = \sum_{s \in W} \chi_{s^{-1}\mathfrak{a}^+}(H)(\det s) \prod_{\alpha \in \Sigma^+} (\sinh \alpha(sH))$$

and $\chi_{s^{-1}\mathfrak{a}^+}$ is the characteristic function of the closure of $s^{-1}\mathfrak{a}^+$. Thus if the spherical Fourier transform of f has compact support in Ω^p , then so does the Euclidean Fourier transform of the function F on \mathfrak{a} given by

$$F: H \mapsto f(\exp H \cdot o)\Xi(H).$$

It is easy to see that $F \in L^2(\mathfrak{a})$ since

$$\|F\|_{L^2(\mathfrak{a})} = \int_{\mathfrak{a}^+} |f(\exp H \cdot o)|^2 \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^2 dH = \|f\|_{L^2(G/K)} < \infty.$$

F is continuous because it is a sum of continuous functions: the term in the sum corresponding to $s \in W$ is continuous on the closure of $s^{-1}\mathfrak{a}^+$ and is equal to zero on the boundary and so extends by zero to a continuous function on \mathfrak{a} . Thus F satisfies the hypotheses of the Euclidean sampling theorem on \mathfrak{a} (see, e.g., [11], Theorem 14.1).

4.3 Three dimensional hyperbolic space

We will illustrate the results above in the case when G/K is three dimensional hyperbolic space. This is the simplest example of a Riemannian symmetric space of complex type. It can be realized as G/K where $K = SU(2)$ and $G = K_{\mathbb{C}} = SL(2, \mathbb{C})$. We choose the maximal abelian subspace \mathfrak{a} of \mathfrak{p} to be

$$\mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\}$$

and let \mathfrak{a}_+ be the subspace of \mathfrak{a} where $t > 0$. The Weyl group consists of multiplication by ± 1 . There is a single positive restricted root, α , of multiplicity 2 given by

$$\alpha(H(t)) = t \text{ where } H(t) = \begin{pmatrix} t/2 & 0 \\ 0 & -t/2 \end{pmatrix}.$$

Then Ω is identified with the interval $|t| < \pi/2$. The Killing form of $\mathfrak{g}_{\mathbb{C}}$ restricted to \mathfrak{g} is positive definite and can be computed by $B(X, Y) =$

$4 \operatorname{Tr}(XY)$. The metric on G/K induced by $(2\kappa)^{-1}B$ has constant sectional curvature $-\kappa$ ($\kappa > 0$). We will use this metric in what follows. The metric identification of \mathfrak{a} and \mathfrak{a}^* is given by $\alpha^\sharp = H(\kappa)$. In this rank one case Ω is equal to the ball in \mathfrak{a} of radius $r = \pi/(2\sqrt{\kappa})$ and the corresponding set in \mathfrak{a}^* is

$$\Omega^\flat = \left\{ l\alpha : |l| < \frac{\pi}{2\kappa} \right\}.$$

The lattice Λ is $\{m\alpha : m = 0, 1, 2, \dots\}$ and if $\mu = m\alpha$, then $d(\mu) = (m+1)^2$ [7, Chapter III, Theorem 9.10]. In the following we will write $\lambda(l) = l\alpha$ and $\mu(m) = m\alpha$ ($t, l \in \mathbb{R}, m \in \mathbb{Z}^+$). Note $\alpha(\lambda(l)^\sharp) = \kappa l$. The spherical functions are $\phi_{\lambda(l)}(\exp H(t)) = \operatorname{sinc}(lt)/\operatorname{sinc}(it)$ [8, Chapter IV.5.2]. Note

$$\phi_{\pm i(\mu(m)+\rho)}(\exp i\lambda(l)^\sharp) = \frac{\operatorname{sinc}((m+1)l\kappa)}{\operatorname{sinc}l\kappa}.$$

The \mathbf{c} -function is $\mathbf{c}(\lambda(l)) = -i/l$. The spherical Fourier transform is³

$$(\mathcal{F}_s f)(\lambda(l)) = \frac{4}{l\sqrt{2\pi\kappa}} \int_0^\infty f(\exp H(t) \cdot o) \sin(lt) \sinh(t) dt \quad (4.7)$$

and the inverse spherical Fourier transform is

$$\mathcal{F}_s^{-1} F(\exp H(t) \cdot o) = \frac{1}{\sinh(t)} \sqrt{\frac{\kappa}{2\pi}} \int_0^\infty F(\lambda(l)) l \sin(lt) dl.$$

Putting all this together we find (4.3), (3.5) and (4.5) become⁴

$$\begin{aligned} \frac{t}{\sinh t} S f(\exp H(t) \cdot o) &= \frac{\kappa^2}{\pi \sinh t} \int_{-\pi/(2\kappa)}^{\pi/(2\kappa)} \mathcal{F}_s f(l\alpha) \sin(\kappa l) \sin(tl) dl, \\ I(\mu, \exp H(t) \cdot o) &= \frac{1}{2(m+1) \sinh(t)} \sqrt{\frac{\kappa}{2\pi}} \int_{-\pi/(2\kappa)}^{\pi/(2\kappa)} \frac{\sin((m+1)l\kappa)}{\sin(l\kappa)} l \sin(lt) dl, \\ \frac{t}{\sinh t} S I(\mu, \cdot)(\exp H(t) \cdot o) &= \frac{\kappa^2}{\pi(m+1) \sinh(t)} \int_{-\pi/(2\kappa)}^{\pi/(2\kappa)} \sin((m+1)l\kappa) \sin(lt) dl \\ &= \frac{\kappa}{2(m+1) \sinh(t)} \left(\operatorname{sinc}\left(\frac{\pi(t - (m+1)\kappa)}{2\kappa}\right) - \operatorname{sinc}\left(\frac{\pi(t + (m+1)\kappa)}{2\kappa}\right) \right). \end{aligned}$$

³ $d\lambda(l) = \sqrt{\kappa/(2\pi)} dl$, $dH(t) = 1/\sqrt{2\pi\kappa} dt$.

⁴ $\prod_{\alpha \in \Sigma^+} |\sin(\alpha(\lambda(l)^\sharp))|^{m_\alpha} = \sin^2(l\kappa)$ and $c_1 = \left(\int_0^\pi \sin^2(t) \frac{1}{\sqrt{2\pi\kappa}} dt \right)^{-1} \kappa^2$ (see [8], Chapter I, Corollary 5.12).

The series (3.6) becomes

$$f(\exp H(t) \cdot o) = \frac{1}{(2\pi/\kappa)^{3/2} \sinh t} \sum_{m=0}^{\infty} \int_{-\pi/(2\kappa)}^{\pi/(2\kappa)} \mathcal{F}_s f(l\alpha) \sin((m+1)l\kappa) \sin(l\kappa) dl \\ \int_{-\pi/(2\kappa)}^{\pi/(2\kappa)} \frac{\sin((m+1)l\kappa)}{\sin(l\kappa)} l \sin(lt) dl \quad (4.8)$$

and (4.4) becomes

$$f(\exp(H(t) \cdot o)) = \frac{1}{2 \sinh t} \sum_{m=-\infty}^{\infty} \sinh(m\kappa) f(\exp H(m\kappa) \cdot o) \operatorname{sinc}\left(\frac{\pi}{2\kappa}(t - m\kappa)\right). \quad (4.9)$$

To illustrate Theorem 1 we put $\kappa = 0.3$ and consider the K -invariant function, f , whose spherical Fourier transform is $\cos(l^3 - 2 \sin(3l))$ multiplied by the smooth, even cut-off function in [23, Lemma 1.10] rescaled to have support in $[-\pi/(4\kappa), \pi/(4\kappa)]$. We multiply both sides of (4.8) by $\sinh(t)$ to eliminate the exponentially decaying factor in the sum. Figure 1 shows the absolute value of the difference between $\sinh(t)f(t)$ and partial sum up to $m = 50$ in the series on the right hand side of (4.8).

4.4 Comparison with the Euclidean Sampling Formula

If we define the Euclidean Fourier transform of a function on hyperbolic three space as

$$\mathcal{F}_o f(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\exp H(t) \cdot o) e^{-ilt} dt,$$

then (4.7) shows that, since f is an even function, $\mathcal{F}_s f$ is supported in Ω^b if and only if $\mathcal{F}_o(\sinh(\cdot)f(\cdot))$ is supported in the interval $(-\pi/(2\kappa), \pi/(2\kappa))$ (see Section 4.2; here $\Xi(H(t)) = \sinh t$). If we apply the Euclidean sampling theorem to the function $t \mapsto \sinh(t)f(\exp H(t) \cdot o)$ and think of the Euclidean Fourier transform as being supported in $(-\pi/\kappa, \pi/\kappa)$, then we obtain

$$f(\exp H(t) \cdot o) = \frac{1}{\sinh t} \sum_{m=-\infty}^{\infty} \sinh(m\kappa) f(\exp H(m\kappa) \cdot o) \operatorname{sinc}\left(\frac{\pi}{\kappa}(t - m\kappa)\right). \quad (4.10)$$

This is a different reconstruction of f than (4.9) because the partial sums of (4.10) for $|m| < M$ interpolate between the values of f at $\exp H(m\kappa) \cdot o$ for $|m| < M$, but the partial sums of (4.9) do not.

Figures 2 and 3 show the same information as Figure 1 for the reconstructions from (4.9) and (4.10) with $|m| \leq 50$. The table below the figures gives the relative L^1 error over the interval $[0, 3\pi/\kappa]$ for each of the reconstructions. Equation (4.8) gives the smallest relative error in this example. All numerical computations were done with [15].

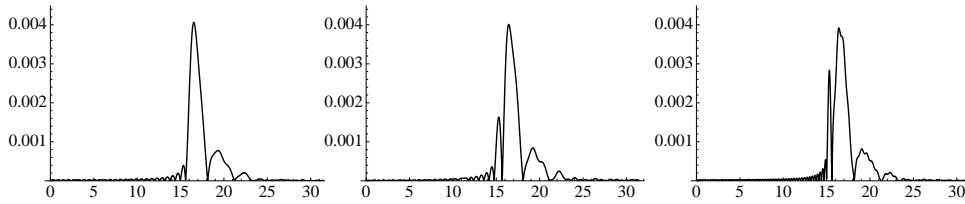


FIGURE 1 Error in reconstruction using Equation (4.8) FIGURE 2 Error in reconstruction using Equation (4.9) FIGURE 3 Error in reconstruction using Equation (4.10)

	Equation (4.8)	Equation (4.9)	Equation (4.10)
$\frac{\ S_{50}-f\ }{\ f\ }$	0.021194	0.0236768	0.0241348

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