

**ON THE p -SPIN INTERACTION MODEL
AT LOW TEMPERATURE**

Note de Michel TALAGRAND, présentée par Yves MEYER

Abstract : We give a complete proof of the existence at low (but not too low) temperature of the structure conjectured by the physicists for the p -spin interaction model with Ising spins, when p is large enough. We prove (asymptotically) the decomposition of the system in a sequence of “pure states”, and we describe the distribution of their weights for Gibbs’ measure. We prove that asymptotically the overlap of two independent configurations take only two possible values (one of which being zero).

Titre français : La structure à basse température des verres de spins à interactions multiples.

Résumé : Lorsque p est assez grand, nous donnons une description complète de la structure à basse (mais pas trop basse) température du verre de spin à interactions multiples d’ordre p dans le cas des spins d’Ising. Nous démontrons asymptotiquement la décomposition du système en une suite d’“états purs” et nous calculons la distribution des poids de ces états pour la mesure de Gibbs. Nous montrons qu’asymptotiquement, les recouvrements de deux configurations indépendantes prennent seulement deux valeurs (dont l’une est zéro).

Version française abrégée

Les physiciens ont conjecturé des comportements remarquables des modèles de verres de spins à champ moyen [M-P-V]. Même à très haute température, il est ardu de démontrer le comportement prévu qui a cependant été obtenu pour plusieurs modèles importants [T1]. Le cas de la basse température est encore bien plus difficile. Nous présentons ici une description (presque) complète du comportement du modèle à p -spins d’Ising. L’énergie d’une configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ ($\sigma_i = \pm 1$) est donnée par la formule (1.1) (les numéros des formules renvoient bien sûr à la version anglaise). On dénote par $G_{N,\beta}$ la mesure de Gibbs à température inverse β , laquelle est donnée par (1.2), et on définit le paramètre $A_N(\beta)$ du système par (1.3). Nous ne décrivons que les résultats dans le cas où p est impair, le cas p étant un peu plus compliqué à cause de l’existence d’une symétrie globale autour de zéro.

THÉORÈME.— *Il existe un nombre L tel que, si p est impair, $p \geq L$, le système d’équations (1.4), (1.5) possède une unique solution $q_N(\beta), m_N(\beta)$ et que l’on ait la relation (1.6).*

En termes plus parlants, en dehors peut-être de quelques valeurs exceptionnelles de β , le recouvrement de deux configurations indépendantes ne prend essentiellement que les valeurs 0 et $q_N(\beta)$ pour N grand. La seule faiblesse de ce résultat est qu'il ne donne pas la valeur de $A_N(\beta)$. Nous ne savons même pas démontrer qu'à β fixe cette quantité possède une limite quand $N \rightarrow \infty$.

Le théorème est ici formulé dans le langage des physiciens, et le lecteur non averti ne réalise sans doute pas que cette formulation compacte contient une foule d'informations, informations qui sont obtenues successivement lors de la preuve du théorème. La première étape, cruciale, consiste à montrer la décomposition spontanée de la mesure de Gibbs en "paquets" bien séparés, et avait été obtenue lors d'un travail précédent [T2]. L'ingrédient nouveau principal est d'arriver à montrer que ces "paquets" sont des "états purs", au sens que le recouvrement de deux configurations prises dans un même paquet est essentiellement indépendant de ces configurations. On démontre alors, par un argument de cavité et un conditionnement approprié, que la seule valeur possible de ces recouvrements satisfait les relations (1.4), (1.5). Le contrôle de la distribution des poids des états purs pour la mesure de Gibbs est essentiel dans l'argument de cavité et utilise les relations de Ghirlanda-Guerra [G-G]. Asymptotiquement, cette distribution est une distribution de Poisson-Dirichlet de paramètre $m_N(\beta)$. Signalons enfin que nous savons étendre le théorème au cas d'un champ externe $h \leq \frac{1}{L}$, à condition toutefois d'ajouter à l'Hamiltonien un terme infinitésimal approprié ayant pour objet d'obtenir toutes les relations considérées dans [G-G].

1. INTRODUCTION

The physicists have conjectured a very rich and far-reaching behavior of mean-field models for spin glasses [M-P-V]. Even at very high temperature the study of these models is rather delicate [T1]. The low temperature case is even more difficult. The simplest type of low temperature behavior, the so-called "one level of replica-symmetry breaking" is predicted at low (but not too low) temperature for the p -spin interaction model with Ising spins. In this model, for a configuration $\sigma = (\sigma_1, \dots, \sigma_N)$ ($\sigma_i = \pm 1$), its energy is given by

$$(1.1) \quad -H_N(\sigma) = \left(\frac{p!}{2N^{p-1}} \right)^{\frac{1}{2}} \sum_{i_1 < \dots < i_p} g_{i_1 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p}.$$

There, the summation is over all choice of indices $i_1 < \dots < i_p$ and $g_{i_1 \dots i_p}$ are i.i.d. $N(0, 1)$ random variables (r.v.). These variables will be called the disorder. The object of interest is Gibbs' measure $G_{N,\beta}$, that at inverse temperature β , is given by

$$(1.2) \quad G_{N,\beta}(\sigma) = \frac{\exp(-\beta H_N(\sigma))}{\sum \exp(-\beta H_N(\boldsymbol{\varrho}))},$$

where the summation is of course over all configurations $\boldsymbol{\varrho}$. A major effort towards understanding this model was done in [T2]. While it failed to give a precise picture, it

succeeded in proving the validity of several key predictions. In particular, it proved, below the critical temperature, the spontaneous decomposition of Gibbs' measure into "lumps" that are as far of each other as they can be. In the work reported here, we have succeeded to improve upon these techniques and to combine them with the information provided by the Ghirlanda-Guerra identities [G-G] to obtain a nearly complete picture of the model. We define

$$(1.3) \quad A_N(\beta) = E \left\langle \left(\frac{1}{N} \sum_{i \leq N} \sigma_i \rho_i \right)^p \right\rangle.$$

There, E denotes expectation with respect to the disorder (the r.v. g_{i_1, \dots, i_p}) and $\langle \cdot \rangle$ a double average with respect to Gibbs' measure on the two independent configurations σ, ρ . (The quantity $N^{-1} \sum_{i \leq N} \sigma_i \rho_i$ is called the overlap of σ, ρ .)

Consider a new $N(0, 1)$ r.v. g , and $X = \beta g \sqrt{pq^{p-1}/2}$. Consider the system of equations

$$(1.4) \quad q = \frac{E (\text{th}^2 X \text{ch}^m X)}{E \text{ch}^m X}$$

$$(1.5) \quad m = 1 - \frac{A_N(\beta)}{q^p},$$

where of course E denotes expectation in g . We will describe our results only when p is odd, to avoid the unessential complications arising from the global symmetry around zero when p is even. Unless β is small (in which case the system is in the high temperature phase and its behavior is well understood), the equations (1.4), (1.5) have a unique solution $q_N(\beta), m_N(\beta)$. Given $\varepsilon > 0$, consider the set

$$U = \left\{ (\sigma, \rho) ; \frac{1}{N} \sum_{i \leq N} \sigma_i \rho_i \notin [-\varepsilon, \varepsilon] \cup [q_N(\beta) - \varepsilon, q_N(\beta) + \varepsilon] \right\},$$

that is, the set of pairs of configurations with an overlap not close to zero or $q_N(\beta)$.

THEOREM 1.1.— *There exists a number L such that, if $p \geq L$, then for each $\varepsilon > 0$*

$$(1.6) \quad \lim_{N \rightarrow \infty} \int_0^{\exp(-p/L)} E G_{N,\beta}^{\otimes 2}(U) d\beta = 0.$$

In other words, for large N , except possibly for a small set of exceptional values of β , the overlap of two configurations is either close to zero or to $q_N(\beta)$. The reader who has never thought about the work of the physicists might not realise that (1.6) amounts to a very precise description of the system, and contains in a compact form a lot of information. The nature of this information will be explained in the next section. The only real weakness of Theorem 1.1 is that we do not know what is the

value of $A_N(\beta)$, or even how to prove (as one should expect) that this quantity has a limit as $N \rightarrow \infty$.

We know how to extend Theorem 1.1 to the case of an external field, i.e. when a term $h \sum_{i \leq N} \sigma_i$ is added to the right-hand side of (1.1) (with $h < \frac{1}{L}$) but, for technical reasons, our arguments require then that we add a (smaller order) suitable perturbation term to the Hamiltonian. The values 0 and $q_N(\beta)$ in the definition of U have then to be replaced respectively by the solutions of (1.5) and

$$q_0 = E \frac{E_g (\text{th} Y \text{ch}^m Y)^2}{E_g (\text{ch}^m Y)^2} ; q_1 = E \frac{E_g (\text{th}^2 Y \text{ch}^m Y)}{E_g (\text{ch}^m Y)},$$

where

$$Y = \beta \sqrt{\frac{p}{2}} (g \sqrt{q_1^{p-1} - q_0^{p-1}} + z \sqrt{q_0^{p-1}}) + \beta h,$$

where g, z are independent $N(0, 1)$ and where E_g denotes expectation at z given.

2. STRUCTURE OF THE PROOF

The first step of the proof consists in proving a priori that the overlap of two configurations is essentially always close to zero or to $q_N(\beta)$, that is, say, that

$$(2.1) \quad E G_N^{\otimes 2} \left(\left\{ \boldsymbol{\sigma}, \boldsymbol{\varrho} ; \left| \frac{1}{N} \sum_{1 \leq i} \sigma_i \rho_i \right| \in \left[\frac{1}{100}, \frac{99}{100} \right] \right\} \right)$$

is exponentially small. This was proved in [T2] for values of β not too large; small changes in the arguments allow in fact to reach values of β exponentially large in p . It is shown in [T2] that (2.1) allows to decompose the configuration space $\Sigma_N = \{-1, 1\}^N$ in disjoint sets $(C_\alpha)_{\alpha \geq 1}$ such that the set

$$\left\{ (\boldsymbol{\sigma}, \boldsymbol{\varrho}) ; \left| \frac{1}{N} \sum_{1 \leq i} \sigma_i \rho_i \right| \geq \frac{1}{100} \right\} \setminus \bigcup_{\alpha} C_\alpha^2$$

is very small (as in (2.1)) while

$$(\boldsymbol{\sigma}, \boldsymbol{\varrho}) \in C_\alpha \implies \frac{1}{N} \sum_{1 \leq i} \sigma_i \rho_i \geq \frac{99}{100}.$$

The sets C_α are the natural candidates for the “pure states” conjectured by the physicists. A quantity of crucial importance is the distribution of the sequence of the weight $w_\alpha = G_{N,\beta}(C_\alpha)$ of the sets C_α for Gibbs’ measure. In [T2] it is proved that unless some strong pathology occurs (the pathology being that the largest w_α is nearly one) the sets C_α are pure states, in the sense that the overlap of two configurations in C_α is generically a number q_α (asymptotically) independent of these configurations.

But we could not prove that this pathology does not occur. We have now discovered a new approach, and we can prove (for $\beta \leq \exp(p/L)$) that the C_α are pure states. We can also prove that if $\beta \leq \exp(p/L)$, the overlap of two configurations in different sets C_α is typically vanishing, as was already done in [T2] for β not too large. (It is this part of the argument that fails when there is an external field, making the whole approach considerably more difficult in that case.) At that stage to prove Theorem 1.1 one has to show that the numbers q_α are essentially equal to the value q given by (1.4), (1.5). The idea of the proof is, given a number q , to find a way to give a meaning “the part of the system consisting of the sets C_α for which $q_\alpha \simeq q$ ” and assuming this part of the system to be non-vanishing, to develop techniques of computing “conditionally upon the fact that the configurations belong to this part of the system”. Roughly speaking, this conditioning argument puts us in a situation where we would know a priori that the overlaps take only the values 0 and q . The essential point of the proof is that in this situation the relations of [G-G] imply that the sequence (w_α) of weights has a Poisson-Dirichlet distribution $PD(m, 0)$ where m is given by (1.5). This knowledge then allows to use the cavity method (induction upon N) to establish the relation (1.4), showing that the numbers q_α can take only this particular value q .

We conjecture that (1.6) can be improved into

$$\forall \beta \leq e^{p/L} , \quad EG_{N,\beta}^{\otimes 2}(U) \leq \exp(-N/K)$$

where K depends only upon β, ε , but the present methods (and in particular the use of the identities of [G-G]) seem powerless to achieve this.

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Équipe d'Analyse – Boîte 186
 ESA n° associée au CNRS
 Université Paris VI
 4, place Jussieu
 F-75230 Paris Cedex 05