An Introduction to Exponential Asymptotics

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Abstract

This is an expository article on exponential asymptotics, transseries and Borel summation and their applications to nonlinear differential equations. We illustrate the main features of the theory through concrete and relatively simple examples. Recent developments include singularity prediction in both ordinary and partial differential equations.

1 Introduction

Following pioneering work by Stokes[18], Dingle[7], exponential asymptotics has been a very active area of research in recent years, after fundamental advances by Berry[3], Kruskal and Ecalle[14]. It deals with terms that are smaller than any term of a classical asymptotic expansion. Besides intrinsic mathematical interest, this area is of relevance to many applications, mathematical and physical, ranging from ionization of atoms, quantum tunnelling, bending losses in an optical fiber, capillary water waves, dendritic crystal growth, viscous fingering, splitting of stable and unstable manifolds in dynamical systems and many other areas. The reader is referred to [17] and [4] for a host of such applications.

Exponential asymptotics and associated Borel summation have also become important tools in studying questions of regularity and singularity for both ODEs [9] and PDEs [13]. This may be thought of as a rather unexpected development since exponential asymptotics have usually been associated with smallness, whereas singularities are where a function and/or its derivatives are unbounded. Yet, this connection is afforded by a proper understanding of transseries in a region near *anti-Stokes* lines (these concepts are explained in the sequel).

This expository lecture is meant to introduce the reader to some key concepts of Borel transform, Borel summation and transseries in the context of simple differential equations (for other relevant material, see [1], [5] and [6]). Later on, we apply these to determine Stokes phenomena and singularities of ODEs. We end the paper with a recent application to a nonlinear PDE.

2 Classical Asymptotic Series and Their Limitations

Exponential asymptotics refers to study of terms that are smaller than any term of the classical asymptotic expansion of a function.

Classical asymptotics in the sense of Poincaré analyzes behavior of a function as $x \to x_0$ (see for instance [21] and [2]). In most cases this behavior depends on the direction of approach (*i.e* on $\arg(x - x_0)$). Having specified a point and a direction of approach, the objective of classical asymptotics is to provide a formal series representation of the function in terms of a simpler set of functions $\{\phi_k(x)\}_{k=0}^{\infty}$, with the property

$$\lim_{x \to x_0} \frac{\phi_{k+1}(x)}{\phi_k(x)} = 0 \quad \text{along a specified direction for} \quad k \geq 0 \tag{1}$$

Then, we define

$$f(x) \sim \sum_{k=0}^{\infty} a_k \phi_k(x) \quad \text{if} \quad \lim_{x \to x_0} \frac{[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)]}{\phi_n(x)} = 0 \tag{2}$$

$$\text{or} \qquad f(x) - \sum_{k=0}^{n} a_k \phi_k(x) = o\left(\phi_n(x)\right) \tag{3}$$

However, this type of asymptotics is unable to distinguish between two functions f and g, if $f(x) - g(x) = o(\phi_k(x))$ for any $k \geq 0$.

One of the simplest non-trivial example is the asymptotic expansion of the exponential integral:

$$Ei(x) = \fint_{-\infty}^{x} \frac{e^t}{t} dt \tag{4}$$

Breaking the integral as $\fint_{-\infty}^{1} + \int_1^x$ and carrying out repeated integration by parts on the second term, it can be easily seen

$$Ei(x) \sim \sum_{k=0}^{\infty} \frac{k! e^x}{x^{k+1}} \equiv \tilde{f}(x) \quad \text{as } x \to +\infty \tag{5}$$

Here, $\tilde{f}(x)$ is the formal notation for the obviously divergent asymptotic expansion and $\phi_k(x) := e^x x^{-k-1}$. Since for any constant $\tilde{C}$, $\phi_k(x) \gg \tilde{C}$ for $k \geq 0$, the classical asymptotic expansion of both $Ei(x)$ and $Ei(x) + \tilde{C}$ is given by $\tilde{f}(x)$. The constant $\tilde{C}$ in this case is exponentially small compared to all other terms of the asymptotic expansion. In particular, this means for instance

$$Ei(x; C) = \int_C^x \frac{e^t}{t} dt \sim \tilde{f}(x)$$

for any complex C in the complex plane $\mathbb{C}$. In particular (5) gives no asymptotic knowledge of $Ei(x; C) - Ei(x)$, since $\tilde{f}(x)$ cancels out and it is *not* proper to write $Ei(x; C) - Ei(x) \sim 0$. Thus, the description provided by classical asymptotics is fundamentally incomplete.

The same situation can arise in determining the asymptotics of a function $f(x;\varepsilon)$ when a small parameter $\varepsilon \to 0^+$. For instance, if

$$f(x;\varepsilon) \sim \sum_{k=0}^{\infty} \varepsilon^k f_k(x)$$

then a term that scales like $e^{-x/\varepsilon}$ is beyond all orders of the the above asymptotic expansion for fixed $x \in (0,\infty)$. Usually, such terms are neglected in many problems of physical interest; however, there exists particular problems of interest where fundamental questions of existence and symmetry cannot be answered properly without accounting for terms beyond all orders. We illustrate their importance through a relatively simple ODE.

3 Role of Exponential Asymptotics: Illustrative ODE example

Problem A: Consider the differential equation for $u(x)$:

$$\varepsilon \frac{d^2u}{dx^2} - \frac{\varepsilon}{x}\frac{du}{dx} - 4x^2 u = 1 \tag{6}$$

We require that as $x \to \infty$, with $\arg x$ in $[0, \frac{\pi}{2})$ (see shaded sector in Fig 3).

$$u \to 0 \tag{7}$$

and that on the positive real $x-$axis: $\mathbb{R}^+$:

$$\text{Im } u = 0 \tag{8}$$

This problem, as posed above will be referred to as *problem A*.

First, at a formal level, through a repeated dominant balance procedure, a consistent asymptotic behavior of such a solution, assuming it exists, for small ε is given by

$$u(x) \sim -\frac{1}{4x^2}\sum_{n=0}^{\infty}(2n)!\frac{\varepsilon^n}{x^{4n}} \tag{9}$$

Each and every term of the asymptotic series in (9) satisfies all the conditions of *problem A* including (8). However, this does not imply that $u(x)$ satisfies the same conditions, since there may be exponentially terms in ε not included in the expansion (9).

Now, we explicitly show that *Problem A* does not have a solution because of such terms. Through a routine use of variation of parameters, a solution to (6) satisfying (7) is given by

$$u(x) = \frac{1}{4\varepsilon^{1/2}}\left[e^{\varepsilon^{-1/2}x^2}\int_{\infty e^{i0}}^{x}\frac{dx'}{x'}e^{-\varepsilon^{-1/2}x'^2} - e^{-\varepsilon^{-1/2}x^2}\int_{i\infty}^{x}\frac{dx'}{x'}e^{\varepsilon^{-1/2}x'^2}\right] \tag{10}$$

This can be argued intuitively in the following manner. The argument is somewhat different depending on whether or not the exponentials $\exp\left[\pm\varepsilon^{-1/2}x^2\right]$ are large or small. In sector I of Fig 3), bounded by $\arg x = \pm\frac{\pi}{4}$, where Re $x^2 = 0$ (referred to as *anti-Stokes lines*), exponentially

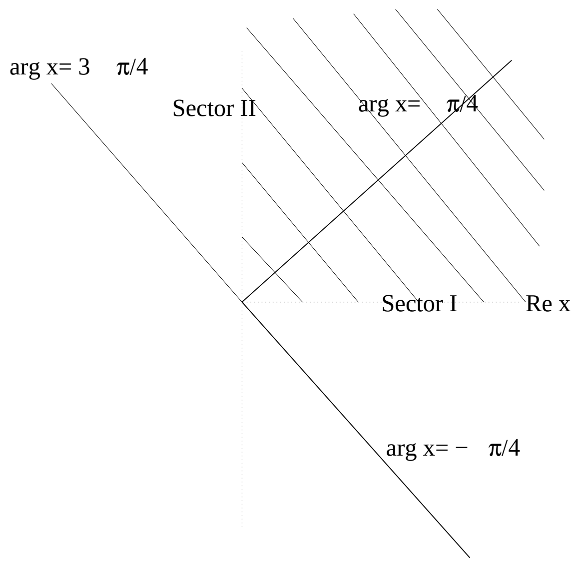

Figure 1: Stokes Sectors for *Problem A*

large contribution outside the first integral is balanced by exponentially small contribution from the integral itself (owing to the choice of lower limit) to leave terms that decay algebraically to the leading order. Again, for the second integral in sector I, the exponentially large contribution of the integral itself is being multiplied by an exponentially small contribution outside the integral to leave us with algebraically decaying factors. In sector II of Fig. 1, which is bounded by the *anti-Stokes lines* $\arg x = \frac{\pi}{4}$ and $\arg x = \frac{3\pi}{4}$, the exponentially large contribution from the first integral is balanced by the exponentially multiple outside the first integral. Again, in this sector, the exponentially small contribution of the second integral (owing to the choice of lower limit) is balanced by the transcendentally large multiple outside the second integral. Thus in Stokes sectors I and II, which include the sector of interest, $\arg x$ in $[0, \pi/2)$, it is possible to find solutions to (6) that satisfy condition (7). Further, through integration by parts in (10), we can easily establish that $u \sim -\dfrac{1}{4x^2}$ as $x \to \infty$ for $\arg x$ in $[0, \frac{\pi}{2})$. By repeated integration by parts, we can verify the the formal result (9). Further, we cannot add any nonzero linear combination of $e^{\varepsilon^{-1/2}x^2}$ and $e^{-\varepsilon^{-1/2}x^2}$ to the solution (10) as condition (7) will then be violated.

The arguments above can be made rigorous. By appropriate changes in integration variables in the first and second integral, for $\arg x$ in $(0, \frac{\pi}{2})$, we rewrite (10):

$$u(x) = \frac{1}{8\varepsilon^{1/2}} \left[\int_0^\infty dy e^{-\varepsilon^{-1/2} y} \left\{ \frac{1}{y - x^2} - \frac{1}{y + x^2} \right\} \right] \tag{11}$$

From Watson's Lemma, we can calculate the classic asymptotic expansion for $\arg x$ in $(0, \frac{\pi}{2})$, once again verifying (9). Note, in particular, that (9) is valid on the *anti-Stokes* line $\arg x = \frac{\pi}{4}$, approaching it from either side. Thus coefficients of any possible exponentials on either side of the *anti-Stokes lines*, which dominate the asymptotic expansion (9) on the *anti-Stokes* lines, must be identically zero. However, this does not imply that that there are no exponentials in the solutions for other ranges of $\arg x$. Consider what happens as we cross $\arg x = 0$ (referred to in a more general context as a *Stokes line*) and move to the region where $\arg x$ is in $(-\frac{\pi}{2}, 0)$. We notice from the exact expression (11) that in doing so we collect a residue from pole at $y = x^2$ and obtain

$$u(x) = \frac{1}{8\varepsilon^{1/2}} \left[\int_0^\infty dy e^{-\varepsilon^{-1/2} y} \left\{ \frac{1}{y - x^2} - \frac{1}{y + x^2} \right\} \right] + \frac{2\pi i}{8\varepsilon^{1/2}} e^{-\varepsilon^{-1/2} x^2} \tag{12}$$

From Watson's Lemma, the integral term in (12) still has the asymptotic expansion

$$-\frac{1}{4x^2} \sum_{n=0}^{\infty} (2n)! \frac{\varepsilon^n}{x^{4n}} \tag{13}$$

The additional contribution $\frac{2\pi i}{8\varepsilon^{1/2}} e^{-\varepsilon^{-1/2} x^2}$ is exponentially small for $\arg x \in \left(-\frac{\pi}{4}, 0\right)$; however, it becomes dominant when $\arg x$ is in $(-\frac{3\pi}{4}, -\frac{\pi}{4})$. When $\arg x = 0$, (referred to as a *Stokes line*), the coefficient of the exponentially small term is the average between the two sides. This is seen by noting that in that case

$$u(x) = \frac{1}{8\varepsilon^{1/2}} \left[\fint_0^\infty dy e^{-\varepsilon^{-1/2} y} \left\{ \frac{1}{y - x^2} - \frac{1}{y + x^2} \right\} \right] + \frac{\pi i}{8\varepsilon^{1/2}} e^{-\varepsilon^{-1/2} x^2} \tag{14}$$

Note that (14) implies that on the real $x$axis,

$$\text{Im} u(x) = + \frac{\pi}{8\varepsilon^{1/2}} e^{-\varepsilon^{-1/2} x^2} \tag{15}$$

which is nonzero. We note that transcendentally small terms that are usually subdominant now become dominant since every term of the regular perturbation expansion (13) is identically zero. Thus, from (15), *we conclude that there are no solutions to problem A*.

This problem is posed in the complex-plane. However, there are simple examples in the context in a PDE in the real domain (See for instance, [19], §6), where exponential asymptotics plays a crucial role whether or not a solution exists in the first place or in determining conditions under which solutions exist.

4 Laplace and Borel transforms

If $e^{-c|p|} F(p) \in L^1(0, \infty)$, then it known that the Laplace transform

$$[\mathcal{L}F](x) \equiv \int_0^\infty F(p) e^{-px} dp \tag{16}$$

exists and defines an analytic function of x in the half-plane

$$\mathcal{H}_c = \{x : \text{Re } x > c\}$$

and continuous in its closure $\overline{\mathcal{H}_c}$ and goes to 0 as $x \to \infty$ in $\overline{\mathcal{H}_c}$. Further, the inverse Laplace transform is well-defined for functions which are analytic in $\mathcal{H}_c$, continuous on $\overline{\mathcal{H}_c}$ and go to zero at a suitable rate as $x \to \infty$ in $\overline{\mathcal{H}_c}$ and is given explicitly by the following integral in the complex plane

$$[\mathcal{L}^{-1} f](p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx \tag{17}$$

For Re $p < 0$, it can be shown that $[\mathcal{L}^{-1} f](p) = 0$ by closing the contour for (17) with a right-half semi-circle and using Jordan's lemma to estimate the contribution of the latter contour.

Laplace transform can be generalized along a complex ray. If $e^{-c|p|} F(p) \in L^1(0, \infty e^{i\theta})$ then we can define

$$[\mathcal{L}_\theta F](x) = \int_0^{\infty e^{i\theta}} e^{-px} F(p) dp, \tag{18}$$

and this defines an analytic function of x in the half-plane Re $[x e^{i\theta}] > c$. The formula for inverse Laplace transform is similarly changed by a rotation through $-\theta$.

Consider $x^{-\beta}$ for any $\beta > 0$. We define its Borel transform in terms of a dual variable:

$$\mathcal{B}[x^{-\beta}](p) = \frac{p^{\beta-1}}{\Gamma(\beta)} \tag{19}$$

where $\Gamma(\beta)$ is the Gamma function. It is to be noted that the Laplace transform $\mathcal{L}$ applied to the right hand side of (19) gives

$$\mathcal{L}\left[\frac{p^{\beta-1}}{\Gamma(\beta)}\right](x) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-px} p^{\beta-1} dp = \frac{x^{-\beta}}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} e^{-s} ds = x^{-\beta} \tag{20}$$

More generally, Borel transform transforms one series

$$\tilde{f}(x) = x^{-\beta} \sum_{k=0}^{\infty} a_k x^{-\alpha k} \text{ with } \alpha, \beta > 0$$

into another series:

$$F(p) := [\mathcal{B}\tilde{f}](p) = \sum_{k=0}^{\infty} \frac{a_k p^{\alpha k + \beta - 1}}{\Gamma(k\alpha + \beta)} \tag{21}$$

Because of division by $\Gamma(k\alpha + \beta)$, it is clear that if $\tilde{f}$ is divergent but for large k, a_k grows at a factorial rate comparable or less than $\Gamma(k\alpha + \beta)$ (recall $\Gamma(n+1) = n!$), then $F(p)$ will be convergent. In the case when $\tilde{f}(x)$ is convergent, it is clear from term by term application of inverse Laplace transform that for $p \in \mathcal{H}^+$,

$$\mathcal{L}^{-1} \tilde{f} = \mathcal{B} \tilde{f} \tag{22}$$

However, it is to be noted that the Borel and inverse Laplace-transform are not the same. For Re $p < 0$, $\mathcal{L}^{-1} \tilde{f} = 0$; however, $\mathcal{B}\tilde{f}$ is defined by the analytic continuation of the series representation (21), which cannot be 0.

However, generally, an asymptotic series $\tilde{f}$ is rarely convergent. If the Borel transform $\mathcal{B}\tilde{f}$ satisfies the following two conditions

1. The series for $F(p) = [\mathcal{B}\tilde{f}](p)$ is convergent in a neighborhood of $p = 0$,

2. Analytical continuation of $F(p)$ along a ray $\arg p = \theta$ results in $e^{-c|p|}F(p) \in L^1(0, \infty e^{i\theta})$ for some $c > 0$,

then it is clear that

$$f(x) = [\mathcal{L}_\theta F](x) = [\mathcal{L}_\theta \mathcal{B}\tilde{f}](x) \tag{23}$$

is an actual function for $\arg x \in \left(-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right)$ for large enough $|x|$. Further, using Watson's Lemma, it is clear that

$$f(x) \sim \sum_{k=0}^{\infty} \int_0^{\infty e^{i\theta}} dp e^{-px} \frac{a_k p^{k\alpha+\beta-1}}{\Gamma(\beta + k\alpha)} = \sum_{k=0}^{\infty} a_k x^{-k\alpha-\beta} = \tilde{f}(x) \tag{24}$$

The process of Borel transform, followed by Laplace transform is called Borel summation of the formal series $\tilde{f}$. If assumptions (i) and (ii) above are valid, we call $\tilde{f}$ *Borel summable*. The association of $\tilde{f}$ with an actual function f given by the expression (23) is not unique, however because of the different values of θ possible. We can make this association unique by choosing $\theta = -\arg x \equiv -\phi$, i.e.

$$f(x) = [\mathcal{L}_{-\phi}\mathcal{B}\tilde{f}](x) \equiv \sum^{\mathcal{B}} \tilde{f} \tag{25}$$

where $\sum^{\mathcal{B}}$ is the notation for the Borel-sum $\mathcal{L}_{-\phi}\mathcal{B}$. However, it is to be noted that in general, $\mathcal{B}\tilde{f}$ will have singularities, when analytically continued in the complex p-plane. If $-\phi$ is a singular direction in the p-plane, then the definition (25) is not adequate. However, it is possible [10] to modify the definition of (25), a process called "Balanced averaging" that involve weighting over different possible paths of integration avoiding singularities, that yields good algebraic properites of the association between $\tilde{f}$ and its Borel sum f.

The rays in the complex-x-domain characterized by $\arg x = \phi$ for which $\arg p = -\phi$ is a singular direction for $[\mathcal{B}\tilde{f}]$ in the p-plane will be referred to as *Stokes lines*. If ϕ_s is a *Stokes line*, we define its associated *anti-Stokes lines* at $\arg x = \phi_s \pm \frac{\pi}{2}$. These *Stokes* and *anti-Stokes lines* play a crucial role in asymptotics, as will be seen later.

We define the Borel Transform of the product of series $\tilde{f}$ and $\tilde{g}$ through the relation:

$$\mathcal{B}[\tilde{f}\tilde{g}] = [\mathcal{B}\tilde{f}] * [\mathcal{B}\tilde{g}] \tag{26}$$

where the convolution operation $*$ is defined as

$$[F * G](p) = \int_0^p F(p-s)G(s)ds \tag{27}$$

We note that for α, $\beta > -1$,

$$p^\alpha * p^\beta = p^{\alpha+\beta+1} \int_0^1 \tau^\alpha (1-\tau)^\beta ds = p^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{28}$$

Thus, convolution of two formal series can be computed using:

$$\left[\sum_{k=0}^{\infty} a_k p^{k+\alpha}\right] * \left[\sum_{j=0}^{\infty} b_j p^{j+\beta}\right] = p^{\alpha+\beta+1} \sum_{j,k=0}^{\infty} a_k b_j p^{j+k} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+j+1)}{\Gamma(j+k+\alpha+2)} \tag{29}$$

It is not difficult to prove that that if $\tilde{f}$ and $\tilde{g}$ are *Borel summable*, so is the product $\tilde{f}\tilde{g}$.

5 Borel Transforming a simple differential equation

A well-known method to solving elementary differential equation, given in any text book, is to use a series method of solution. Frobenius theory guarantees convergence of the series, provided the point of expansion x_0 is a regular-singular point. This convergent series representation is also asymptotic as $x \to x_0$ as is true for any convergent series. The series method also works for some nonlinear differential equations also when the linearized equation in the neighborhood of $x = x_0$ is a regular singular point. For irregular singular point, there is no such theory. Indeed, consider for instance the simple differential equation

$$y' + y = \frac{1}{x^2} \tag{30}$$

where we seek to determine the asymptotic behavior of the solution as $x \to \infty$. Notice that $x = \infty$ is an irregular singulary point of the differential equation. To see this, we introduce transformed variables $x = 1/z$, $y(1/z) = Y(z)$ to find $-z^2 Y' + Y = z^2$, or $Y' - \frac{Y}{z^2} = -1$. $z = 0$ ($x = \infty$) is clearly an irregular singular point and therefore, we are not guaranteed that a series will converge. If we try

$$y(x) = \sum_{k=0}^{\infty} \frac{a_k}{x^{k+1}},$$

and plug it into (30) and equate different powers of $\frac{1}{x}$, we obtain $a_0 = 0$, $a_1 = 1$ and the recurrence relation $a_k = k a_{k-1}$ for $k \geq 2$. Thus, we obtain *formally*

$$y(x) \sim \sum_{k=1}^{\infty} \frac{k!}{x^{k+1}} \equiv \tilde{y}(x) \tag{31}$$

However, the formal asymptotic expansion (31) has no free parameters, as expected for a general solution to (30). If one solution $y = y_p$ to (30) has the asymptotic series $\tilde{y}$, so will the general solution $y = y_p + Ce^{-x}$ to (30) as Re $x \to +\infty$.

Noting that the divergence of $\tilde{y}$ is only at a factorial rate, we Borel transform the series to obtain

$$Y(p) := [\mathcal{B}\tilde{y}](p) = \sum_{k=1}^{\infty} p^k = \frac{p}{1-p}$$

Then, the *Borel sum* of $\tilde{y}$, for $\arg x = \phi \in (0, 2\pi)$ is given by

$$y_0(x) = \int_0^{\infty e^{-i\phi}} e^{-xp} Y(p) dp = \int_0^{\infty e^{-i\phi}} \frac{pe^{-px}}{1-p} dp = \frac{1}{x} \int_0^{\infty} \frac{se^{-s}}{x-s} ds \tag{32}$$

It can be directly verified from the last expression in (32) that $y_0(x)$ is a solution to the differential equation (30). Further, applying Watson's Lemma,

$$y_0(x) \sim \int_0^{\infty e^{-i\phi}} e^{-xp} p dp + \int_0^{\infty e^{-i\phi}} e^{-xp} p^2 dp + ...+ = \frac{1}{x^2} + \frac{2}{x^3} + .. = \tilde{y} \tag{33}$$

The general solution of the differential equation (30) is of course

$$y(x) = y_0(x) + Ce^{-x} \tag{34}$$

If we require that $y(x) \to 0$ as $x \to \infty$ along any ray for which $\arg x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, then that solution is uniquely given by $y(x) = y_0(x)$ since Ce^{-x} is not small for large x on such a ray, where as $y_0 \sim \tilde{y}$.

As we cross $\arg x = 0$, the representation for $y_0(x)$ changes because of the singularity of $Y(p)$ at $p = 1$. To find the analytic continuation of $y_0(x)$ we deform the integration contour in the s-plane, as shown in Fig 2,

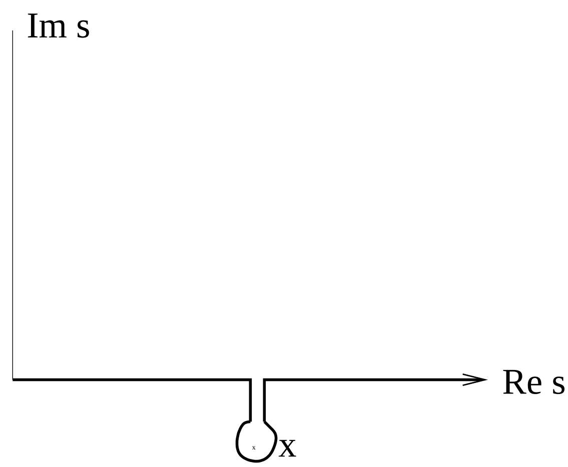

Figure 2: Contour deformation for analytic continuation of $y_0(x)$

Collecting the residue at $s = x$, we find we obtain for $\arg x \in (-2\pi, 0)$,

$$y_0(x) = \frac{1}{x}\int_0^\infty \frac{se^{-s}}{x-s}ds - 2\pi i e^{-x}$$

There is the birth of a new term e^{-x} as we move across $\arg x = 0$. This line, is therefore, a *Stokes line*. Nonetheless, despite this new term, $y_0(x) \sim \tilde{y}$ remains valid for $\arg x \in (-\frac{\pi}{2}, 0)$ since the exponential term is subdominant to every term in $\tilde{y}$. However, as we approach $\arg x = -\frac{\pi}{2}$, things begin to change. $2\pi i e^{-x}$ is no longer negligible compared to $\tilde{y}$. $\arg x = -\frac{\pi}{2}$ is an *anti-Stokes line*, where hitherto exponentially small term born at a *Stokes line* first becomes important enough so that $y_0 \sim \tilde{y}$ is no longer valid. Indeed, for $\arg x \in \left[-\frac{3}{2}\pi, -\frac{\pi}{2}\right]$, $y_0(x) \sim 2\pi i e^{-x}$. This is an example of *Stokes Phenomena*, where an analytic function $y_0(x)$ has different asymptotic representation for large $|x|$ as $\arg x$ varies. Indeed, a similar analysis as we cross $\arg x = 2\pi$ from below shows that

$$y_0(x) = \frac{1}{x}\int_0^\infty \frac{se^{-s}}{x-s}ds + 2\pi i e^{-x}$$

for $\arg x \in (2\pi, 4\pi)$. This means that largest sector over which the asymptotic relation $y_0(x) \sim \tilde{y}(x)$ is valid is $\arg x \in \left(-\frac{\pi}{2}, \frac{5}{2}\pi\right)$, of width 3π.

6 Borel Summability, Gevrey-Asymptotics and Least-term truncation

Recall that for a convergent power series representation of a function, there is a unique correspondence between the series and the function it represents. This section is motivated by the question whether or not an asymptotic series can be made to correspond uniquely to a function with additional assumptions.

Consider

$$\tilde{f}(x) = \sum_{j=0}^{\infty} c_j x^{-j} \tag{35}$$

be a formal power series and f a function asymptotic to it in the sense of Poincaré. This definition provides for large x estimates of $f(x)$ within $o(x^{-N})$, $N \in \mathbb{N}$, which are not sufficient to pin down a unique f associated with $\tilde{f}$. Simply widening the complex x-sector where $f \sim \tilde{f}$ is valid does not change this situation either, since we can always choose m large enough and suitable complex constant c so that $\exp\left[-cx^{1/m}\right]$ is smaller than any term in $\tilde{f}$ in a sector of width $m\pi$, which includes the sector for which we require $f \sim \tilde{f}$.

It is then reasonable to attempt to

1. Improve the quality of estimates of f from $\tilde{f}$ to within $O\left(\exp\left[-const.|x|\right]\right)$ and
2. Seek a sector wide enough for $f \sim \tilde{f}$ so as to rule out a contribution such as $\exp\left[-cx^{1/m}\right]$.

One important technique in this class is due to Gevrey.

Definition 1 *Consider a formal asymptotic series $\tilde{f}(x)$ of the form (35). This series is Gevrey of order $1/m$, if*

$$|c_k| \leq C_1 C_2^k (k!)^m \text{ for some } C_1, C_2 \tag{36}$$

We define the N-term trunction of (35) as $\tilde{f}^{[N]}(x)$. i.e.

$$\tilde{f}^{[N]}(x) = \sum_{k=0}^{N} c_k x^{-k}$$

Remark 1 *Note that if we change variables $x = y^m$, then $\tilde{f}(x(y))$ is Gevrey-1. This follows from using Stirling's formula $N! \sim \sqrt{2\pi N} e^{-N} N^N$ for large N and noting $\frac{(k!)^m}{(km)!} \sim \frac{(2\pi k)^{m/2-1/2}}{m^{km+1/2}}$ Hence, it is enough to concentrate on the case $m = 1$.*

Definition 2 *Let $\tilde{f}$ be Gevrey-1. A function f is Gevrey -1 asymptotic to $\tilde{f}$ as $x \to \infty$ in a sector S, if for some C_3 and C_4 and any $N \in \mathbb{N}$,*

$$|f(x) - \tilde{f}^{[N]}(x)| \leq C_3 C_2^{N+1} |x|^{-(N+1)} (N+1)! \quad \text{for} \quad |x| > C_4 \tag{37}$$

i.e. the error $f - \tilde{f}^{[N]}$ is of the same order as the first omitted term in $\tilde{f}$.

Lemma 3 *If $\tilde{f}$ is Gevrey-1 and f is Gevrey-1 asymptotic to $\tilde{f}$, then f is approximated by $\tilde{f}$ with exponential precision. Let $N = [|x/C_2|]$, the integer part of $|x/C_2|$; then for any $C > C_2$, we have from Stirling formula*

$$f(x) - \tilde{f}^{[N]}(x) = o\left(e^{-|x|/C}\right)$$

PROOF. From (37), with the choice of N as given above, we have

$$|f(x) - \tilde{f}^{[N]}(x)| \le C_3 N^{-(N+1)}(N+1)! = O(\sqrt{N}e^{-N})$$

by using Stirling's formula for $N!$. □

Remark 2 *Usually, the imprecision implied by (37) is larger than potential terms beyond Gevrey-1 series $\tilde{f}$, at least in some directions. This means we could still have two functions with the same Gevrey asymptotic series. However, if the estimate (37) holds in a sector $S_{>\pi}$, of width bigger than π, then (37) cannot hold for more than one function $f(x)$.*

On the other hand, with a sector $S_{>\pi}$ of validity of (37), $\hat{f}$ is Borel summable and f is precisely the Borel sum of $\tilde{f}$.

Theorem 4 *[see [1]] Let $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ be a Gevrey-1 series and assume the function f is analytic for large x in*

$$S_{>\pi} = \left\{ x : |\arg x| < \frac{\pi}{2} + \delta \right\}$$

for some $\delta > 0$ and Gevrey-1 asymptotic to $\tilde{f}$ in $S_{>\pi}$. Then

1. *f is unique.*
2. *$\tilde{f}$ is Borel summable and $f = \sum_{\mathcal{B}} \tilde{f}$.*
3. *$\mathcal{B}(\tilde{f})$ is analytic at $p = 0$ and in the sector $S_\delta = \{p : \arg p \in (-\delta, \delta)\}$*
4. *Conversely, if $\tilde{f}$ is Borel summable along any ray in the sector S_δ for $|\arg p| < \delta$, and uniformly bounded in any closed subsector of S_δ, then f is Gevrey-1 with respect to its asymptotic series $\tilde{f}$ in the sector $|\arg x| < \frac{\pi}{2} + \delta$.*

Remark 3 *Theorem 4 will be proved after we recall some properties of Laplace and inverse Laplace Transform in the following:*

Proposition 5 *Assume $\delta \ge 0$ and f is analytic in $e^{\pm i\delta}H$ defined as*

$$e^{\pm i\delta}H \equiv \left\{ x : |\arg x| < \frac{\pi}{2} + \delta \right\}$$

and assume f continuous on its boundary and that for some $K > 0$ and $x \in e^{\pm i\delta}H$, $|f(x)| \le K/(1+|x|^2)$, then
(i) $\mathcal{L}\mathcal{L}^{-1}f = f$ and in addition, $\|\mathcal{L}^{-1}f\|_\infty \le \frac{K}{2}$.
(ii) If $\delta > 0$, then $F = \mathcal{L}^{-1}f$ is analytic in the sector $\mathcal{S}_\delta = \{p \ne 0 : |\arg p| < \delta\}$, $\|F\|_{\infty, S_\delta} \le \frac{K}{2}$ and $F(p) \to 0$, as $p \to 0$ along any ray in S_δ.

PROOF. Note that using Fubini's theorem:

$$\int_0^\infty dpe^{-px}\int_{-\infty}^{\infty} e^{ips} if(is)ds = i\int_{-\infty}^{\infty} dsf(is)\left(\int_0^\infty dpe^{-px}e^{ips}dp\right)$$
$$= \int_{-\infty}^{\infty}\frac{ids}{x-is}f(s) = \int_{-i\infty}^{i\infty}\frac{f(z)dz}{x-z} = 2\pi i f(z)$$

Hence $\mathcal{L}\mathcal{L}^{-1}f = f$. The bounds on $\|\mathcal{L}^{-1}f\|_\infty$ simply follow by noting first that for $p\in\mathbb{R}^+$,

$$\left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{ips} if(is)ds\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K}{1+s^2}ds = \frac{K}{2}$$

while for other $p\in\mathcal{S}_\delta$, we rotate the contour of integration the s-plane so that ps is real and ranges from $(-\infty,\infty)$.

As far as (ii), we note that for $\delta'<\delta$,

$$\int_{-i\infty}^{i\infty} e^{ps}f(s)ds = \left(\int_{-i\infty}^{0}+\int_0^{i\infty}\right)e^{ps}f(s)ds = \left(\int_{-i\infty e^{-i\delta'}}^{0}+\int_0^{i\infty e^{-i\delta'}}\right)e^{ps}f(s)ds$$

The latter integrand is clearly L^1 in s and analytic for $p\in S_{\delta'}$; so the integral is analytic for $p\in S_{\delta'}$. Since this is true for any $\delta'<\delta$, it follows that F is analytic in S_δ.

Finally, we change variables and deform contour so that

$$F(p) = \frac{1}{2\pi i p}\int_C e^s f(s/p)ds$$

where contour C is consists of two straight lines, one connecting $i\infty e^{-i\delta'}$ to $s=1$ and the other from $s=1$ to $i\infty e^{i\delta'}$. It is clear on this contour that $|p^{-1}f(s/p)| \le \frac{K|p|}{|p|^2+|s|^2}$, whereas e^s is bounded by a constant. It follows easily that $|F(p)|\to 0$ as $p\to 0$.

□

Proposition 6 *Let F be analytic in the open sector $S_p = e^{i\phi}\mathbb{R}^+$, with $\phi\in(-\delta,\delta)$ be such that $|F(|p|e^{i\phi}) \le g(|p|)$ for some $g\in L^1[0,\varepsilon)$ and bounded outside the interval. Then $f = \mathcal{L}F$ is analytic in the sector $S_x = \left\{x : |\arg x| < \frac{\pi}{2}+\delta\right\}$ and $f(x)\to 0$ as $|x|\to\infty$, $\arg x\in\left(-\frac{\pi}{2}-\delta, \frac{\pi}{2}+\delta\right)$.*

PROOF. For any ϕ such that $|\phi|<\delta$, it is clear that

$$\left|\int_0^{\infty e^{i\phi}} e^{-px}F(p)dp\right| \le \int_0^\varepsilon e^{-\text{Re }(px)}g(|p|)d|p| + \int_\varepsilon^\infty e^{-|p||x|\cos(\phi+\arg x)}Md|p| \qquad (38)$$

where M is an upper-bound for $|g|$. For any $\arg x\in\left(-\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta\right)$, it is possible to choose $\phi\in(-\delta,\delta)$ so that $\phi+\arg x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Thus, the integral above exists. Since the integrand is analytic in x and L^1 in p on the ray of integration, it is clear that $f=\mathcal{L}F$ is analytic in S_x. Further, to show that $f(x)\to 0$ as $x\to\infty$ in S_x, we note that for given $\varepsilon'>0$, we can choose ε

sufficiently small, but independent of x, so that the first integral on the right of (38) $< \frac{\varepsilon'}{2}$. For the second integral, we choose $|x|$ large enough so as to ensure $< \frac{\varepsilon'}{2}$. $\square$

Proof of Theorem 4

Without loss of generality, we may assume $\tilde{f}^{[0]}$ and $\tilde{f}^{[1]} = 0$, *i.e* $c_0 = c_1 = 0$. Also, we can shift the origin of x sufficiently to the right so that it is possible to write $|f(x)| < \frac{K}{1+|x|^2}$ for all $x \in \mathcal{S}_{>\pi}$.

1. If both f_1 and f_2 is Gevrey-1 asymptotic to $\tilde{f}$, then by Lemma 3, for some constants C_1, C_2, we have

$$|f_1(x) - f_2(x)| \le C_1 e^{-C_2|x|}$$

in a sector $S_{>\pi}$ of opening more than π. From Proposition 5, $\mathcal{L}^{-1}(f_1 - f_2)(p)$ exists and is analytic for $\arg p \in (-\delta, \delta)$ and from the exponential bound, the integration path for $p \in (0, C_2)$

$$\int_{-i\infty}^{i\infty} (f_1(x) - f_2(x)) e^{px} dx$$

can be closed with a right semi-circle or radius $R \to \infty$, with no resulting contribution because of exponential decay of integrand. Hence $\mathcal{L}^{-1}(f_1 - f_2)(p) = 0$ for $p \in (0, C_2)$. From analyticity, $\mathcal{L}^{-1}(f_1 - f_2)(p) = 0$ for all p. Laplace transforming, we get $f_1 - f_2 = 0$.

2. & 3. By change of variables $g = f/C_1$ and $\tilde{g}_1 = \tilde{f}/C_1$ and $y = x/C_2$, $\tilde{g}$ and g satisfy the same relation involving y, as does f and $\tilde{f}$, except that corresponding $C_1 = 1$ and $C_2 = 1$. Thus, without loss of generality, we will assume $C_1 = 1 = C_2$ for f and $\tilde{f}$. From the bounds on c_k, we see that the series $F_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$. From Proposition 5, $F(p) = \mathcal{L}^{-1} f$ is analytic for $\arg p \in (-\delta, \delta)$, and is uniformly bounded if $|\arg p| \le \delta - \varepsilon$, for any small $\varepsilon > 0$. We now show that $F(p)$ is analytic for $|p| < 1$. Taking p real, $p \in [0, 1)$, we obtain

$$|F(p) - \tilde{F}^{[N-1]}(p)| \le \int_{-i\infty+N}^{i\infty+N} d|s| |f(s) - \tilde{f}^{[N-1]}(s)| \exp\left[\text{Re }(ps)\right]$$

$$\le N! e^{pN} \int_{-\infty}^{\infty} \frac{dt}{|t + iN|^N} = \frac{N! e^{pN}}{N^{N-1}} \int_{-\pi/2}^{\pi/2} \cos^{N-2} y dy < C N^{3/2} e^{(p-1)N} \text{where we use } t = N \tan y$$

Note that the last term above vanishes as $N \to \infty$, implying $F = F_1$ for $p \in [0, 1)$. From analytic continuation $F = F_1$ for all p. In particular, F is analytic for $|p| < 1$ or in the sector $|\arg p| < \delta$ (for any $|p|$).

4. Let $|\phi| < \delta$. We have, from integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N} \mathcal{L} \frac{d^N}{dp^N} F$$

On the other hand, F is analytic in $\mathcal{S}_a$ for some $a(\phi)$–neighborhood of the sector $\{p : |\arg p| < \phi\}$. Estimating derivatives through Cauchy's formula on a circle of radius a (see Fig. 3) so that either $|\arg p| < \phi$ or $|p| < 1$ for points on the circle, $|F^{(n)}(p)| \le N! [a(\phi)]^{-N} \|F(p)\|_{\infty; \mathcal{S}_a}$. Thus, with

$|\theta| \le \phi$ chosen so that $\gamma = \cos(\theta - \arg x) > 0$ and maximal, we have

$$\left| f(x) - \tilde{f}^{[N-1]}(x) \right| = \left| x^{-N} \int_0^{\infty e^{-i\theta}} F^{(N)}(p) e^{-px} dp \right|$$

$$\le N! a^{-N} |x|^{-N} \|F\|_{\infty;\mathcal{S}_a} \int_0^{\infty} e^{-s|x|\gamma} ds = N! a^{-N} \gamma^{-1} |x|^{-N-1} \|F\|_{\infty;\mathcal{S}_a}$$

Thus, *Gevrey-1* asymptotics follows, and the proof of Theorem 4 is complete.

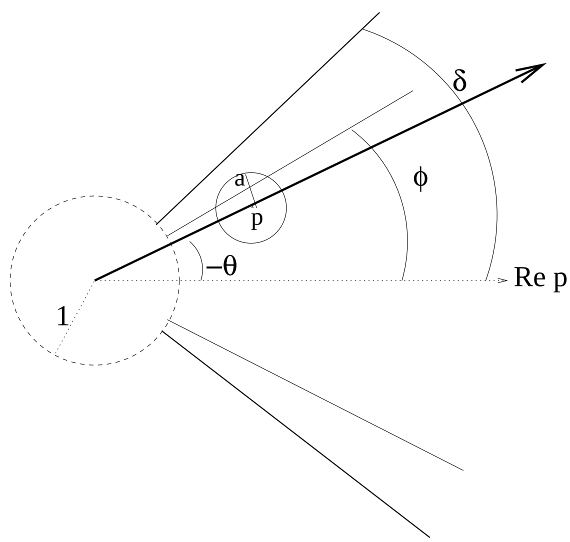

Figure 3: Radius $a(\phi)$ circle for estimation of $F^{(N)}(p)$ on ray $\arg p = -\theta$

7 Borel analysis for a nonlinear ODE

The steps carried out for the linear ODE $y' + y = \frac{1}{x^2}$ in the *Borel summation* process can be carried be out in practice only for a limited number of problems. Even if we know the full series representation for $\tilde{y}$, and that the Borel transform $\mathcal{B}\tilde{y}$ leads to a convergent series representation, it is nearly impossible to determine if analytic continuation on a complex ray satisfies exponential bounds to ensure Laplace transformability. Instead, for solutions to differential equations, we can apply the Borel Transform on the equation itself and study the analytic properties of the transform itself, assuming *a priori* that it exists.

We illustrate this in terms of a simple nonlinear version of (30):

$$y' + y = \frac{1}{x^2} + y^2 \tag{39}$$

We seek a solution y that behaves like x^{-2} as $x \to \infty$, for $\arg x \in \left(-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right)$ for some $\theta \in (-2\pi, 0)$. If indeed y has a series in inverse powers of x, starting with x^{-2}, and if $Y(p) := \mathcal{B}y$

is convergent, then it is clear from computation of the series' coefficients that $\mathcal{B}y' = -pY(p)$ and so, Borel transforming (39), $Y(p)$ must satisfy

$$-pY + Y = p + Y * Y \text{ implying } Y(p) = \frac{p}{1-p} + \frac{[Y * Y](p)}{1-p} := \mathcal{N}[Y](p) \tag{40}$$

where the symbol $\mathcal{N}$ is an operator, defined by the equality above.

We then show that the operator $\mathcal{N}$ is a contraction operator in the space of analytic functions in a domain $\mathcal{D}$ that are exponentially bounded at ∞, where

$$\mathcal{D} := \{p : |p| < 1 - \delta \text{ or } \arg p \in (-2\pi + \delta, -\delta)\} \tag{41}$$

It is convenient to define the norm

$$||Y||_\nu = M_0 \sup_{p\in\mathcal{D}} (1 + |p|^2) \exp\left[-\nu|p|\right] Y(p) \tag{42}$$

where $\nu > 4$ will be chosen sufficiently large, as described later, while the constant M_0 is defined by

$$M_0 \equiv \sup_{s\ge 0} \left\{ \frac{2(1+s^2)[\log(1+s^2) + s \arctan s]}{s(s^2+4)} \right\} = 3.76..$$

We consider the Banach space $\mathcal{A}$ of analytic functions in $\mathcal{D}$, continous in its closure, equipped with the norm $||.||_\nu$. From the exponential bound inherent in the definition of the norm, it is clear that for any fixed $\arg x \in \left(-\frac{\pi}{2}, \frac{5}{2}\pi\right)$, Laplace transform $\mathcal{L}_\theta Y(x)$ exists for large enough $|x|$ when δ is so small that $\theta \in (-2\pi + \delta, \delta)$ can be chosen so that $\arg x + \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. How large $|x|$ has to be depends on how close $\arg x$ is to the end points of the interval.

Lemma 7 *If $Y \in \mathcal{A}$, and $Y(p) \sim Cp^k$ for small p, $||Y||_\nu \le K\Gamma(k+1)\nu^{-k}$ for ν large enough and K independent of ν and k.*

PROOF. Note that for $|p| < \varepsilon < 1$, we have

$$\sup_{|p|<\varepsilon} (1 + |p|^2) e^{-\nu|p|} |Y(p)| \le C\nu^{-k} \sup_{|p|<\varepsilon} e^{-\nu|p|} (|p|\nu)^k \le C\nu^{-k} k^k e^{-k} \le \frac{K\Gamma(k+1)}{\nu^k}$$

Since for $|p| \ge \varepsilon$, $(1 + |p|^2)e^{-\nu|p|}|Y(p)|$ is exponentially small for ν large. Hence, the Lemma follows. □

Corollary 8

$$\left\| \frac{p}{1-p} \right\|_\nu \le \frac{K}{\nu} \tag{43}$$

for some constant K independent of ν

Lemma 9 *If Y_1, $Y_1 \in \mathcal{A}$, then $Y_1 * Y_2 \in \mathcal{A}$ and*

$$||Y_1 * Y_2||_\nu \le ||Y_1||_\nu ||Y_2||_\nu \tag{44}$$

PROOF. First, note that

$$[Y_1 * Y_2](p) = \int_0^p Y_1(s)Y_2(p-s)ds = p\int_0^1 Y_1(ps)Y_2(p(1-s))ds$$

and the latter integrand is clearly analytic in p and L^1 in s; therefore, integration over s preserves analyticity in p. Continuity on the boundary $\partial\mathcal{D}$ is also inherited from continuity of Y_1 and Y_2 on the boundary.

Also, for s on a straight line connecting 0 to $p \in \mathcal{D}$, we have $|p-s|+|s| = |p|$ and $|p-s| = |p|-|s|$, we obtain

$$\begin{aligned} |\left[Y_1 * Y_2\right](p)| \le |\int_0^p Y_1(p-s)Y_2(s)ds| &\le e^{\nu|p|}\|Y_1\|_\nu\|Y_2\|_\nu M_0^{-2}\int_0^{|p|}\frac{d\tilde{s}}{[1+(|p|-\tilde{s})^2][1+\tilde{s}^2]} \\ &\le \frac{e^{\nu|p|}}{M_0(1+|p|^2)}\|Y_1\|_\nu\|Y_2\|_\nu \end{aligned} \quad (45)$$

from which (44) follows. □

We now have the following result addressing the existence and uniqueness of the solution to the integral equation (40).

Lemma 10 *For ν large enough, the mapping $\mathcal{N}$ is a contraction mapping of the ball*

$$B_\nu := \left\{Y : Y \in \mathcal{A}, \|Y\|_\nu < \frac{2K}{\nu}\right\}$$

into itself and thus the integral equation (40) has a unique solution in B_ν.

PROOF. Since $1/|1-p|$ is bounded in $\mathcal{D}$, it follows that

$$\left|\frac{1}{1-p}[Y*Y](p)\right| \le C|Y*Y| \le C\frac{e^{\nu|p|}}{1+|p|^2}\|Y\|_\nu^2$$

Therefore, for $Y \in B_\nu$ we have

$$\|\mathcal{N}[Y]\|_\nu \le \frac{K}{\nu} + C\|Y\|_\nu^2 \le \frac{K}{\nu} + \frac{4CK^2}{\nu^2} \le \frac{2K}{\nu}$$

for sufficiently large ν. Thus, $\mathcal{N}$ maps B_ν back to itself. Now we note that

$$Y_1 * Y_1 - Y_2 * Y_2 = Y_1 * (Y_1 - Y_2) + Y_2 * (Y_1 - Y_2)$$

and so for Y_1, $Y_2 \in B_\nu$,

$$\left\|\frac{1}{1-p}(Y_1 * Y_1 - Y_2 * Y_2)\right\|_\nu = \frac{4CK}{\nu}\|Y_1 - Y_2\|$$

Therefore,

$$\|\mathcal{N}[Y_1] - \mathcal{N}[Y_2]\|_\nu \le \frac{4CK}{\nu}\|Y_1 - Y_2\|_\nu$$

and so $\mathcal{N}$ is a contraction mapping of B_ν into itself. The rest of the proof follows from the contraction mapping theorem. □

Remark 4 *Since $\delta > 0$ can be as small as we want, the integral equation (40) will have analytic solution $Y(p)$ for any $|p| < 1$ and $\arg p \in (-2\pi, 0)$.*

Lemma 11 *For fixed $\arg x \in (-\frac{\pi}{2}, \frac{5}{2}\pi)$, there exists a solution to (39) for large $|x|$ in the form*

$$y_0(x) \equiv \mathcal{L}_\theta Y = \int_0^{\infty e^{i\theta}} e^{-px} Y(p) dp,$$

where $\theta \in (-2\pi, 0)$ is chosen so that $\theta + \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and where Y is a solution to the nonlinear equation (39). Furthermore, for large $|x|$, it has the asymptotic expansion:

$$y_0(x) \sim x^{-2} + \sum_{k=3}^{\infty} a_k x^{-k}$$

PROOF. Since Lemma 10 guarantees Y and therefore $Y * Y$ to be analytic at the origin, it follows that

$$Y(p) = \sum_{k=0}^{\infty} A_k p^k$$

and therefore, using (28),

$$[Y * Y](p) = p \sum_{k=0}^{\infty} B_k p^k$$

for some constant B_k, related to A_0,...,A_k.

From the Taylor expansion of $p/(1-p)$ at the origin, it follows from the integral equation (40) that $A_0 = 0$. It is clear that $B_0 = A_0^2 = 0$ as well. Therefore, $A_1 = 1$ and

$$Y(p) = \sum_{k=1}^{\infty} A_k p^k$$

The expression for $y(x)$ is convergent for all sufficiently large $|x|$ since Y is exponentially bounded. From Watson's Lemma it follows that

$$y_0(x) \sim \sum_{k=1}^{\infty} A_k \frac{\Gamma(k+1)}{x^{k+1}} = x^{-2} + \sum_{k=3}^{\infty} a_k x^{-k} \equiv \tilde{y}(x)$$

where $a_k = A_{k-1}\Gamma(k)$.

The proof that $y_0(x)$ satisfies the differential equation (31) follows from noting that $\mathcal{L}_\theta[-pY] = y_0'$ and that $\mathcal{L}_\theta Y * Y = y_0^2$ as follows from using Fubini's theorem in the convolution, knowing *a priori* that the functions are integrable. □

8 Singularity of $Y(p)$ at $p = 1$ and Stokes Phenomena

Recall that we were considering

$$y' + y = \frac{1}{x^2} + y^2 \tag{46}$$

and found a Borel summable solution $y_0(x)$ for $|x|$ sufficiently large with $\arg x = \phi \in \left(-\frac{\pi}{2}, \frac{5}{2}\pi\right)$ in the form

$$y_0(x) = [\mathcal{L}_\theta Y](x) = \int_0^{\infty e^{i\theta}} Y(p)e^{-px}dp \tag{47}$$

where $Y(p)$ solves the integral equation

$$Y(p) = \frac{p}{1-p} + \frac{Y*Y}{1-p} \tag{48}$$

and $\theta \in (-2\pi, 0)$ is chosen to ensure $\arg(px) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If we choose instead θ suitably in $(0, 2\pi)$, and define

$$\hat{y}_0 = \int_0^{\infty e^{i\theta}} Y(p)e^{-px}dp, \tag{49}$$

then following essentially the same arguments as for y_0, it follows that $\hat{y}_0$ is a solution to (46) with asymptotic behavior $\hat{y}_0 \sim \tilde{y}(x)$ for $\arg x \in \left(-\frac{5}{2}\pi, \frac{\pi}{2}\right)$. Though both y_0 and $\hat{y}_0$ are Laplace transforms of the same function $Y(p)$ they are not analytic continuation of each other because of singularities of $Y(p)$ on the positive real axis at $p = 1$, $p = 2$, etc (see next section). Indeed, for $\arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it is seen by contour deformation that

$$y_0(x) - \hat{y}_0(x) = \int_C Y(p)e^{-px}dp \tag{50}$$

where the contour C is shown in Fig. 4.

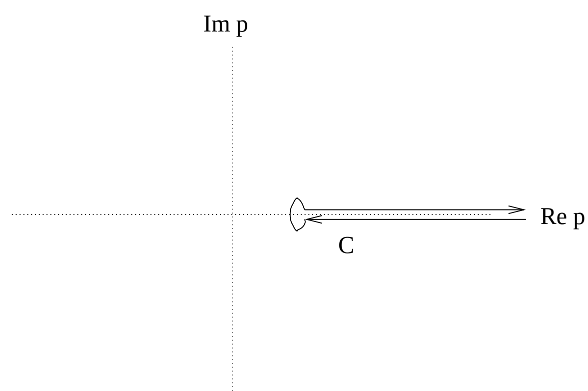

Figure 4: Contour C in (48)

The leading order asymptotic contribution from $\int_C$ for large x in this sector is of the type $Se^{-x}x^\gamma$ for some constants S (called Stokes constant) and γ. The precise value of γ depends on the nature of leading order singularity of $Y(p)$ at $p = 1$. If it is a simple pole, as for the linear problem, then $\gamma = 0$. We will determine this to be the case in the next section.

Relation (50) provides the analytic continuation of y_0 for $\arg x \in (-\frac{5}{2}\pi, 0)$. The consequence of this is that as $\arg x$ approaches $-\frac{\pi}{2}$ it is no longer correct to write $y_0(x) \sim \tilde{y}(x)$ when this *anti-Stokes line* is approached, since $\hat{y}_0(x) \sim \tilde{y}(x)$, but e^{-x} coming from the asymptotic behavior of $\int_C$ is no longer small compared to $\tilde{y}(x)$. This is referred to as the *Stokes phenomenon*, where a single analytic function (in this case y_0) has different asymptotics in different sectors of the complex plane. This is observed for the nonlinear problem as well as it is for the linear problem $y' + y = 1/x^2$. However, in the nonlinear case, the consequence is more catastrophic. As soon as e^{-x} becomes sizable, so do e^{-2x}, e^{-3x}, ..., etc, where such terms arise due to nonlinearity. As we shall see in a latter section, this has the cumulative effect of inducing singularities of $y_0(x)$ near the anti-Stokes line $\arg x = -\frac{\pi}{2}$ for large $|x|$. Similar arguments can be made for the analytically continued $\tilde{y}_0$ near the anti-Stokes line $\arg x = \frac{\pi}{2}$.

9 Analysis of singularity of $Y(p)$ at $p = 1$

We recall the Borel plane equation for $Y(p)$ satisfying:

$$(1-p)Y = p + Y * Y \tag{51}$$

It is convenient to define

$$H(p) = Y(p) \text{ for } |p| \leq 1-\varepsilon \text{ and } H(p) = 0 \text{ otherwise} \tag{52}$$

$$h(p) = Y(p) - H(p) \tag{53}$$

We want to look at the equation (51) in

$$\mathcal{D}_{1,\varepsilon} := \{p : |p-1| < \varepsilon, \arg(1-p) \in (-\pi, \pi)\}$$

Equation 51 becomes

$$(1-p)h(p) = p + H * H + 2H * h + h * h \tag{54}$$

Lemma 12 *For $p \in (1-\varepsilon, 1)$, for $\varepsilon < \frac{1}{4}$, $h * h(p) = 0$ and analytically extends to the zero analytic function in $\mathcal{D}_{1,\varepsilon}$.*

PROOF. First, consider

$$h * h = \int_{1-\varepsilon}^{p} h(s)h(p-s)ds = \int_0^{p-1+\varepsilon} h(s)h(p-s)ds = 0$$

since $\varepsilon - (1-p) < \varepsilon < 1 - \varepsilon$. □

Lemma 13 *For $p \in (1-\varepsilon, 1)$, with $\varepsilon < \frac{1}{4}$,*

$$H * H = \int_{\varepsilon-(1-p)}^{2\varepsilon} Y(s)Y(p-s) + \int_{2\varepsilon}^{1-\varepsilon} Y(s)Y(p-s)ds$$

and the above expression extends to an analytic function for any $p \in \mathcal{D}_{1,\varepsilon}$.

PROOF. Note that for $p \in (1-\varepsilon, 1)$,

$$(H*H)(p) = \int_0^{1-\varepsilon} H(s)H(p-s)ds = \int_{\varepsilon-(1-p)}^{p} H(s)H(p-s)ds = \int_{\varepsilon-(1-p)}^{1-\varepsilon} H(s)H(p-s)ds$$
$$= \int_{\varepsilon-(1-p)}^{1-\varepsilon} Y(s)Y(p-s)ds = \int_{\varepsilon-(1-p)}^{2\varepsilon} Y(s)Y(p-s)ds + \int_{2\varepsilon}^{1-\varepsilon} Y(s)Y(p-s)ds$$

Since $Y(p-s)$ is analytic for $p \in \mathcal{D}_{1,\varepsilon}$ for any s in the range of integration, the second part of the Lemma follows easily. □

Lemma 14 *For $p \in (1-\varepsilon, 1)$, for $\varepsilon < \frac{1}{4}$,*

$$(H*h)(p) = \int_{1-\varepsilon}^{p} H(p-s)h(s)ds = \int_{1-\varepsilon}^{p} Y(p-s)h(s)ds$$

This extends it to an analytic function for any $p \in \mathcal{D}_{1,\varepsilon}$.

PROOF. First for $p \in (1-\varepsilon, 1)$ we have

$$H*h(p) = \int_0^p H(s)h(p-s)ds = \int_0^{1-\varepsilon} H(s)h(p-s)ds = \int_{p-1+\varepsilon}^{p} H(p-s)h(s)ds$$
$$= \int_{1-\varepsilon}^{p} H(p-s)h(s)ds = \int_{1-\varepsilon}^{p} Y(p-s)h(s)ds$$

Since $Y(p-s)$ is analytic for $p \in \mathcal{D}_{1,\varepsilon}$ for any s in the range of integration and $h(s)$ is known to be integrable on any ray that avoids $s=1$, it follows that the above provides analytic continuation for $(H*h)(p)$ for any $p \in \mathcal{D}_{1,\varepsilon}$. □

Theorem 15 *For $p \in \mathcal{D}_{1,\varepsilon}$, $h(p)$, and therefore, $Y(p)$ has the ramified representation*

$$Y(p) = -\frac{A_1(1-p)}{1-p} - \log(1-p)A_1'(1-p) - A_2'(1-p)$$

where $A_1(z)$ and $A_2(z)$ are analytic for $|z| < \varepsilon$.

PROOF. For $p \in \mathcal{D}_{1,\varepsilon}$, is convenient to define $q(p) = \int_{1-\varepsilon}^{p} h(s)ds$. Then, it is clear that on integration by parts and using $Y(0)=0$, it follows that

$$(H*h)(p) = \int_{1-\varepsilon}^{p} H'(p-s)q(s)ds$$

Hence in this domain, (54) can be rewritten as

$$(1-p)q'(p) = p + H*H(p) + 2\int_{1-\varepsilon}^{1} H'(p-s)q(s)ds + 2\int_{1}^{p} H'(p-s)q(s)ds \qquad (55)$$

We define $z = 1 - p$ and $q(p) = Z(1-p)$. Then, replacing $1 - s = zt$ in the above integration, we obtain

$$-zZ'(z) = A(z) + 2z \int_0^1 H'(z(t-1))Z(zt)dt \tag{56}$$

where $A(z) = (1-z) + (H*H)(1-z) + 2\int_{1-\varepsilon}^1 H'(1-z-s)q(s)ds$. Dividing by z and integrating from $z = \varepsilon$ (where $Z(\varepsilon) = 0$) to z, we obtain

$$Z(z) = -A(0)(\log z - \log \varepsilon) - \int_0^z \frac{A(z') - A(0)}{z'}dz' - \int_\varepsilon^0 \frac{A(z') - A(0)}{z'}dz'$$
$$-2\int_\varepsilon^0 dz' \left[\int_0^1 H'(z'(t-1))Z(z't)dt\right] - 2\int_0^z dz' \left[\int_0^1 H'(z'(t-1))Z(z't)dt\right] := Z_0(z) + L[Z](z) \tag{57}$$

We now claim that $Z(z)$ is a ramified analytic function for $|z| < \varepsilon$ with the unique decomposition

$$Z(z) = A_1(z)\log z + A_2(z)$$

where A_1 and A_2 are analytic in $|z| < \varepsilon$. To show this first note that

$$\int_0^z dz' \left[\int_0^1 Y'(z'(t-1))[A_1(z't)\log(z't) + A_2(z't)]dt\right]$$
$$= z\int_0^1 d\tau \left[\int_0^1 Y'(z\tau(t-1))[A_1(z\tau t)[\log(z\tau t) + A_2(z\tau t)]dt\right] = z\tilde{A}_1(z)\log z + z\tilde{A}_2(z)$$

for some analytic functions $\tilde{A}_1$ and $\tilde{A}_2$ related to A_1 and A_2. Thus, the linear operator L preserves the ramified analytic structure of $Z(z)$. If we introduce the norm in this space

$$\|Z\|_R \equiv \|A_1\|_\infty + \|A_2\|_\infty$$

Then we find (since $|z| < \varepsilon$),

$$\|L[Z]\|_R \leq K\varepsilon \|Z\|_R$$

Thus, $Z(z)$ indeed has a unique solution in this ramified analytic space. Since the solution to (51) is unique, it follows that $Y(p)$ has the analytic structure

$$\int_{1-\varepsilon}^p Y(p')dp' = A_1 \log(1-p) + A_2$$

for p near 1. So, the ramified analytic structure of $Y(p)$ at $p = 1$ is given by the analytic representation:

$$Y(p) = -\frac{A_1(1-p)}{1-p} - A_1'(1-p)\log(1-p) - A_2'(1-p)$$

□

Remark 5 *The above relation proves that the leading order singularity of $Y(p)$ is indeed a simple pole with residue $S = A_1(0)$; however, unlike the linear problem, $p = 1$ is also a branch point as shown by the presence of logarithmic term. This is recognized in choice of contour $\int_C$ in Figure 4 by allowing a branch cut along the positive real axis from $p = 1$ to $p = +\infty$.*

Remark 6 *Having resolved the singularity at $p = 1$, we can repeat the analysis near $p = 2$ by considering $Y(p)$ known in the interval $(0, 1 + \varepsilon)$. The singularity at $p = 1$ is found to induce a singularity at $p = 2$. Inductively, we can prove that this is true for positive integer p. This gives rise to terms like e^{-2x}, e^{-3x}, ..., in the asymptotics of $\int_C Y(p)dp$ for large x.*

10 General solution to (39)

Having found one solution $y_0(x)$ in Lemma 11, we seek the representation of the general solution of (39).

Lemma 16 *The most general solution to the differential equation (39) that goes to zero as $|x| \to \infty$ for $\arg x = \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is of the form*

$$y(x) = y_0(x) + y_r(x)$$

where $y_r(x) \sim Ce^{-x}$ as $x \to \infty$ in the right half-plane for some constant C.

PROOF. Since y_0 and y are solutions to (39), y_r satisfies

$$y_r' + y_r = 2y_0 y_r + y_r^2 \tag{58}$$

Therefore,

$$y_r(x) = \int_{x_0}^{x} e^{-x+t}[2y_0(t)y_r(t) + y_r^2(t)]dt + \tilde{C}e^{-x} := \mathcal{M}[y_r](x) \tag{59}$$

for some constant $\tilde{C}$ and x_0 is a point on the complex ray $\arg x = \phi \in \left(-\frac{\pi}{2} \frac{\pi}{2}\right)$ and the path of integration is taken radially outwards from x_0 to x. We will consider the domain $|x| \geq |x_0|$, $\arg x = \phi = \arg x_0$. Define the ball

$$\mathcal{B}_\varepsilon \equiv \left\{f : f(x) \text{ continuous on } x = re^{i\phi}, |x_0| \leq r < \infty, \|f\|_\infty < \varepsilon\right\}$$

in the Banach space of continous function, equipped with sup norm. It is to be noted that for any $\varepsilon > 0$, this ball contains $y_r(x)$ that vanishes as $|x| \to \infty$ when $|x_0|$ is chosen sufficiently large. We now claim that $\mathcal{M}$ is a contraction mapping of $\mathcal{B}_\varepsilon$ onto itself for ε chosen sufficiently small and $|x_0|$ chosen sufficiently large.

To see this, we note that on the contour of integration, $t = se^{i\phi}$, where $|x_0| \leq s \leq r \equiv |x|$. So, using $|y_0(t)| \leq K|t|^{-2}$

$$|\mathcal{M}[y_r]| \leq |\tilde{C}|e^{-|x_0|\cos\phi} + \int_{|x_0|}^{r} e^{-(r-s)\cos\phi}\left[\frac{K}{s^2}|y_r(s)| + |y_r|^2\right] ds$$

$$\leq |\tilde{C}|e^{-|x_0|\cos\phi} + \frac{1}{\cos\phi}\left[\frac{K}{|x_0|^2}\|y_r\|_\infty + \|y_r\|_\infty^2\right] \leq \varepsilon$$

for sufficiently large $|x_0|$. Further,

$$\|\mathcal{M}[y_{r,1} - y_{r,2}]\| \leq \int_{|x_0|}^{r} e^{-(r-s)\cos\phi}\left[\frac{K}{s^2}|y_{r,1} - y_{r,2}| + |y_{r,1}(s) + y_{r,2}(s)||y_{r,1}(s) - y_{r,2}(s)|\right] ds$$

$$\leq \frac{1}{\cos\phi}\left(\frac{K}{|x_0|^2}+2\varepsilon\right)\|y_{r,1}-y_{r,2}\|_\infty$$

Clearly this is contractive for small enough ε and large enough $|x_0|$. Thus, for given $\tilde{C}$, there is a unique solution y_r to (59) that satisfies the condition $y_r \to 0$ as $\mathrm{Re}\, x \to +\infty$. Thus, the general solution to (58) that satisfies $y_r \to 0$ is a one-parameter family of solutions, characterized by constant $\tilde{C}$ in (59).

We will now show that this solution has the the asymptotic behavior given by $y_r \sim Ce^{-x}$ for some constant C, related to $\tilde{C}$.

It is convenient to define $z(x)$ so that

$$y_r(x) = Ce^{-x}\left[1+z(x)\right] \tag{60}$$

Then $z(x)$ satisfies

$$z' = 2y_0(x)(1+z(x)) + Ce^{-x}(2z(x)+z^2(x)) \tag{61}$$

Therefore,

$$z(x) = \int_{\infty e^{i\phi}}^{x} (2y_0(t)+2Ce^{-t})dt + \int_{\infty e^{i\phi}}^{x} \left[(2y_0(t)+2Ce^{-t})z(t) + Ce^{-t}[z(t)]^2\right] := \mathcal{N}[z](x) \tag{62}$$

The constant of integration implied in going from (61) to (62) is taken to be 0, without any loss of generality. This is because otherwise, this would merely correspond to a different choice of C in (60).

It is convenient to define

$$z_0(x) = \int_{\infty e^{i\phi}}^{x} \left[2y_0(t)+Ce^{-t}\right] dt$$

For $|x| > R$, with R large enough, it is clear that $\|z_0\|_\infty \leq \frac{K_0}{R}$. Also, note for $\|z\|_\infty \leq \frac{2K_0}{R}$, we have

$$\|\mathcal{N}[z]\|_\infty \leq \frac{K_1 K}{R^2}\|z_r\|_\infty + |C|\|z_r\|_\infty^2 \leq \frac{2K_0}{R}$$

for sufficiently large R. Furthermore,

$$\|\mathcal{N}[z_1]-\mathcal{N}[z_2]\|_\infty \leq \frac{KK_1}{R^2}\|z_1-z_2\|_\infty + \frac{4K_0 e^{-R\cos\phi}}{R}\|z_1-z_2\|_\infty$$

Thus, it is clear that unique solution, with $z(x) \to 0$ as $|x| \to \infty$ for $\phi = \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Given this unique solution $z(x)$ for given C, we define $\hat{y}(x) = Ce^{-x}(1+z(x))$ It is clear $\hat{y}$ satisfies

$$\hat{y} = \int_{\infty e^{i\phi}}^{x} e^{-x+t}[2y_0(t)\hat{y} + \hat{y}^2(t)]dt + Ce^{-x}$$

implying that

$$\hat{y} = \tilde{C}e^{-x} + \int_{x_0}^{x} e^{-x+t}[2y_0(t)\hat{y}(t) + \hat{y}^2(t)]dt$$

where

$$\tilde{C} = C + \int_{\infty e^{i\phi}}^{x_0} e^{-x+t}[2y_0(t)\hat{y} + \hat{y}(t)]dt$$

Thus $\hat{y}$ satisfies the same equation as y_r, which is known to have a unique solution for given $\tilde{C}$. Therefore, $y_r = \tilde{y}$ and since $z(x) = o(1)$ as $x \to \infty$ in the right half-plane, it follows that $y_r \sim Ce^{-x}[1 + o(1)]$. $\square$

Lemma 17 *For* $\arg x = \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$y_r(x) = \sum_{k=1}^{\infty} (Ce^{-x})^k y_k(x)$$

where $y_k(x) \sim \sum_{j=0}^{\infty} a_{j,k} x^{-j}$.

PROOF. We substitute the above into (62) and equate like powers of Ce^{-x} to obtain

$$y_k' + (1-k)y_k(x) = 2y_0 y_k + \sum_{m=1}^{k-1} y_m y_{k-m} \tag{63}$$

This gives rise to the integral equation:

$$y_k = e^{(k-1)x} \int_{\infty}^{x} e^{-(k-1)t} \left[2y_0 y_k + \sum_{m=1}^{k-1} y_m y_{k-m} \right] (t) dt \tag{64}$$

Since $y_r(x) \sim Ce^{-x}[1 + o(1)]$ as $x \to \infty$, it follows that an additive term of the form $C_k e^{(k-1)x}$ in going from (63) to (64) has to be zero.

We now show this is indeed the case. We decompose $y_k = a_{k,0} + z_k$, where $a_{k,0}$ for $k \geq 2$ is determined recursively from

$$(1-k)a_{k,0} = \sum_{m=1}^{k-1} a_{k-m,0} a_{m,0}$$

where $a_{1,0} = 1$ (It is clear from the recurrence relation that $|a_{k,0}| < A$ for some constant independent of k.) Then, z_k satisfies

$$z_k = e^{(k-1)x} \int_{\infty}^{x} e^{-(k-1)t} \left[2y_0(a_{k,0} + z_k) + \sum_{m=1}^{k-1} \{(a_{k-m,0} + z_{k-m})(a_{m,0} + z_m) - a_{k-m,0} a_{m,0}\} \right] (t) dt \tag{65}$$

It is easily proved that in the space of continuous functions with the norm

$$\|f\| := \sup_{|x_0| \leq |x|} < \infty |x||f(x)| < \infty$$

that the right side of (65) is a contraction mapping of a small ball in that function space onto itself, when $|x_0|$ is sufficiently large. Thus, $y_k = a_{k,0} + O(1/x)$. We can continue this process of peeling out higher order terms in y_k by formally determining

$$y_k = a_{k,0} + \frac{a_{k,1}}{x} + \ldots + \frac{a_{k,m}}{x^m} + y_R(x)$$

and demonstrating that $y_R(x) = O(x^{-(m+1)})$. $\square$

Remark 7 *The results above show that the general solution to the nonlinear equation (1) that goes to zero as* $\mathrm{Re}\, x \to +\infty$ *is of the form*

$$y(x) = \sum_{k=0}^{\infty} (Ce^{-x})^k y_k(x) \tag{66}$$

where

$$y_k(x) \sim \sum_{k=0}^{\infty} \frac{a_{k,m}}{x^m} \equiv \tilde{y}_k(x) \tag{67}$$

with $a_{0,0} = 0 = a_{0,1}$*. The resulting double asymptotic series in powers of* e^{-x} *and* $1/x$*:*

$$y(x) \sim \sum_{k=0}^{\infty} e^{-kx} \tilde{y}_k(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (Ce^{-x})^k \frac{a_{k,m}}{x^m} \tag{68}$$

is an example of a level one transseries. Note that each coefficient $a_{k,m}$ *of the formal asymptotic expansion can be determined from a formal algorithmic process of equating like powers of* $1/x$ *and* Ce^{-x}*. While* $\tilde{y}_k$ *is generally divergent, the representation (66) in terms of actual functions* $y_k(x)$ *is actually convergent. The function* y_k *is the Borel sum of* $\tilde{y}_k$*, after a suitable constant is subtracted from both* y_k *and* $\tilde{y}_k$*, i.e.*

$$y_k(x) = a_{k,0} + \mathcal{L}_\theta \mathcal{B}(\tilde{y}_k - a_{k,0}) \tag{69}$$

for constants $a_{k,0}$*, where* $y_k \sim a_{k,0}$ *as* $x \to \infty$ *in the right-half plane. This follows from Borel analysis on equation (63), similar to the one we have seen for* $y_0(x)$*. Borel analysis also shows that* $[\mathcal{B}(\tilde{y}_k - a_{k,0})](p)$ *has generally singularities at* $p = -1, -2, ..., -(k-1)$*, in addition to singularities at the positive integers. There is no singularity at* $p = 0$*, however. This means that the constant* C *appearing in the transseries representation (68) can only change at* $\arg x = 0$ *in the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*.*

11 Transseries and singularities of $y(x)$

Having established a general representation for solution to (39), we now consider the problem of determining singularities of $y(x)$. From the transseries representation (68), written out in the form:

$$y = \sum_{j=0}^{\infty} \frac{a_{0,j}}{x^j} + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (Ce^{-x})^k \frac{a_{k,j}}{x^j}$$

it follows that if we approach $\arg x = \pi/2$ from below, the terms $(Ce^{-x})^k$ are no longer smaller than all powers of x. The asymptotic series in the sense of Poincaré:

$$y \sim \tilde{y}_0 \equiv \sum_{j=0}^{\infty} \frac{a_{0,j}}{x^j}$$

cannot remain valid. Indeed, there exists a region near the *anti-Stokes line* $\arg x = \frac{\pi}{2}$ where $e^{-kx} \gg x^{-1}$ for any k. In that case, interchanging the order of k and j summation, the formal transseries results suggest that

$$y = \tilde{y}_0 + \sum_{j=0}^{\infty} \frac{1}{x^j} \sum_{k=1}^{\infty} \left[a_{k,j} (Ce^{-x})^k \right] = \sum_{j=0}^{\infty} \frac{F_j(\xi)}{x^j}$$

where $\xi = Ce^{-x}$. Since $\tilde{y}_0(x) = O(x^{-2})$, it follows that we may expect that in this region, the leading order behavior is given by $y(x) \sim F_0(Ce^{-x})$, with $F_0(\xi) \sim \xi$ as $\xi \to 0$, since $a_{1,0} = 1$. Instead of going through a complicated process of summing up $\sum_{k=1}^{\infty} a_{k,0} \xi^k$ to find $F_0(\xi)$, it is more convenient to go equation (1) and seek a two-scale solution[1] in the form

$$y(x) = q(\xi, x)$$

Then, (39) is equivalent to solving

$$-\xi q_\xi + q_x + q = \frac{1}{x^2} + q^2 \tag{70}$$

To the leading order, we seek a x dependent solution to (70). This gives rise to

$$-\xi q_\xi^0 + q^0 = {q^0}^2$$

This can be solved explicitly

$$q^0(\xi) = \frac{\xi}{\xi + C_1}$$

Since $a_{1,0} = 1$, it follows that for small ξ, it follows $q(\xi, x) \sim \xi$ as $\xi \to 0$. Therefore, it follows that $C_1 = 1$ and therefore,

$$q^0(\xi) = \frac{\xi}{\xi + 1} \tag{71}$$

There is an infinite set of poles of $q^0(Ce^{-x})$ at $x = x_s = i\pi + 2in\pi + \log C$ for postive integer n large enough to justify asymptotics for $x >> 1$.

To justify that the actual solution $q(x, \xi)$ has similar singularities nearby, we need to prove that if we decompose

$$q(\xi, x) = q^0(\xi) + r(\xi, x)$$

then $r(\xi, x)$ can be uniquely determined and that for large x, $r(\xi, x) = O(x^{-1})$. We note that $r(\xi, x)$ satisfies:

$$-\xi r_\xi + r_x + (1 - 2q^0) r = \frac{1}{x^2} + r^2$$

Borel transforming in x, we obtain:

$$-\xi R_\xi + (1 - 2q^0 - p) R = p + R * R$$

So, using observation $R(0, p) = Y(p)$ from the transseries,

$$R(\xi, p) = -\xi^{1-p} (\xi + 1)^{-2} \int_0^\xi {\xi'}^{-2+p} \left[(\xi' + 1)^2 [p + R * R](\xi', p) - [p + Y * Y] \right] d\xi' + \frac{Y}{(1 + \xi)^2}$$

[1] Costin and Costin[9] actually used bounds on $a_{k,j}$ to determine equivalent results. The two-scale method was worked out in collaboration with X. Liu

This can be shown to be a contraction mapping in a domain in the (ξ, p) variable that avoids $\xi = -1$ and is compact in ξ and restricted to $\arg p \in (-\delta, \delta)$ in the space of functions that are finite in the $\|.\|_\nu$ norm in this sector. Thus, the Laplace transform $[\mathcal{L}_\theta R(\xi, .)](x)$ exists for $\theta \in (-\delta, \delta)$ chosen so that $\arg x + \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $r(\xi, x) = O(x^{-1})$ for large x in this sector. Using relative smallness of r, it can be proved that indeed the actual singularities of $y(x)$ are poles and close (for large x) to the singularities of $q^0(Ce^{-x})$, i.e. near $x = x_s$, as described above.

12 General Theory for ODEs [10]

The differential system considered has the form

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n, \quad x \in \mathbb{C} \tag{72}$$

where
(i) $\mathbf{f}$ is *analytic* in a neighborhood $\mathcal{V}_x \times \mathcal{V}_\mathbf{y}$ of $(0, \mathbf{0})$, under the genericity conditions that:
(ii) the eigenvalues λ_j of the matrix $\hat{\Lambda} = -\left\{\frac{\partial f_i}{\partial y_j}(0, \mathbf{0})\right\}_{i,j=1,2,\ldots n}$ are linearly independent over $\mathbb{Z}$ (in particular $\lambda_j \neq 0$) and such that $\arg \lambda_j$ are all different.

By elementary changes of variables, the system (72) can be brought to the *normalized form*:

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{73}$$

where $\hat{\Lambda} = \text{diag}\{\lambda_j\}$, $\hat{A} = \text{diag}\{\alpha_j\}$ are constant matrices, $\mathbf{g}$ is analytic at $(0, \mathbf{0})$ and $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$ as $x \to \infty$ and $\mathbf{y} \to 0$.

Performing a further transformation of the type $\mathbf{y} \mapsto \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ (which takes out M terms of the formal asymptotic series solutions of the equation), makes

$$\mathbf{g}(|x|^{-1}, \mathbf{y}) = O(x^{-M-1}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|) \quad (x \to \infty;\ \mathbf{y} \to 0) \tag{74}$$

where

$$M \geq \max_j \text{Re }(\alpha_j)$$

and $O(a; b; c)$ means (at most) of the order of the largest among a, b, c.

Our analysis applies to solutions $\mathbf{y}(x)$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some arbitrary direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. A movable singularity of $\mathbf{y}(x)$ is a point $x \in \mathbb{C}$ with $x^{-1} \in \mathcal{V}_x$ where $\mathbf{y}(x)$ is not analytic. The point at infinity is an irregular singular point of rank 1; it is a fixed singular point of the system since, after the substitution $x = z^{-1}$ the r.h.s of the transformed system, $\frac{dy}{dz} = -z^{-2}\mathbf{f}(z, \mathbf{y})$ has, under the given assumptions, a pole at $z = 0$.

An n-parameter formal solution of (73) (under the assumptions mentioned) as a combination of powers and exponentials is found in the form

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k} \in (\mathbb{N} \cup \{0\})^n} \mathbf{C}^{\mathbf{k}} e^{-\boldsymbol{\lambda} \cdot \mathbf{k} x} x^{\boldsymbol{\alpha} \cdot \mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{75}$$

where $\tilde{\mathbf{s}}_{\mathbf{k}}$ are (usually factorially divergent) formal power series: $\tilde{\mathbf{s}}_{\mathbf{0}} = \tilde{\mathbf{y}}_{\mathbf{0}}$ and in general

$$\tilde{\mathbf{s}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^r} \tag{76}$$

that can be determined by formal substitution of (75) in (73); $\mathbf{C} \in \mathbb{C}^n$ is a vector of parameters[2](we use the notations $\mathbf{C}^{\mathbf{k}} = \prod_{j=1}^n C_j^{k_j}$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, $|\mathbf{k}| = k_1 + ... + k_n$).

Note the structure of (75): an infinite sum of (generically) divergent series multiplying exponentials. They are called *formal exponential power series* [21].

From the point of view of correspondence of these formal solutions to actual solutions it was recognized that not all expansions (75) should be considered meaningful; also they are defined relative to a sector (or a direction).

Given a direction d in the complex x-plane the *transseries* (on d), introduced by Écalle [15], are, in our context, those exponential series (75) which are formally *asymptotic* on d, i.e. the terms $\mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} x^{-r}$ (with $\mathbf{k} \in (\mathbb{N} \cup \{0\})^n$, $r \in \mathbb{N} \cup \{0\}$) form a well ordered set with respect to $\gg$ on d (see also [10]) (For example, this is the case when the terms of the formal expansion become (much) smaller when $\mathbf{k}$ becomes larger.)

We recall that the *antistokes lines* of (73) are the $2n$ directions of the x-plane $i\overline{\lambda_j}\,\mathbb{R}_+$, $-i\overline{\lambda_j}\,\mathbb{R}_+$, $j = 1, ..., n$, i.e. the directions along which some exponential $\mathrm{e}^{-\lambda_j x}$ of the general formal solution (75) is purely oscillatory.

In the context of differential systems with an irregular singular point, asymptoticity should be (generically) discussed relative to a direction towards the singular point; in fact, under the present assumptions (of non-degeneracy) asymptoticity can be defined on sectors.

Let d be a direction in the x-plane which is not an antistokes line. The solutions $\mathbf{y}(x)$ of (73) which satisfy

$$\mathbf{y}(x) \to 0 \quad (x \in d;\ |x| \to \infty) \tag{77}$$

are analytic for large x in a sector containing d, between two neighboring *anti-Stokes lines* and have the same asymptotic series

$$\mathbf{y}(x) \sim \tilde{\mathbf{y}}_{\mathbf{0}} \quad (x \in d;\ |x| \to \infty) \tag{78}$$

In the context of (73), a generalized Borel summation $\mathcal{LB}$ of transseries (75) is defined in [10].

The formal solutions (75) are determined by the equation (73) that they satisfy, except for the parameters $\mathbf{C}$. Then a correspondence between actual and formal solutions of the equation is an association between solutions and constants $\mathbf{C}$. This is done using a generalized Borel summation $\mathcal{LB}$.

The operator $\mathcal{LB}$ constructed in [10] can be applied to any transseries solution (75) of (73) (valid on its open sector S_{trans}, assumed non-empty) on any direction $d \subset S_{trans}$ and yields an actual solution $\mathbf{y} = \mathcal{LB}\tilde{\mathbf{y}}$ of (73), analytic in a domain S_{an}. Conversely, any solution $\mathbf{y}(x)$ satisfying (78) on a direction d is represented as $\mathcal{LB}\tilde{\mathbf{y}}(x)$, on d, for some unique $\tilde{\mathbf{y}}(x)$:

[2]In the general case when some assumptions made here do not hold, the general formal solution may additionally logs iterated exponentials, and powers [15]. The present paper only discusses equations in the setting explained at the beginning of the present section.

$$\mathbf{y}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \mathcal{LB}\tilde{\mathbf{y}}(x) \tag{79}$$

for some constants $\mathbf{C} \in \mathbb{C}^n$, where $M_j = \lfloor \mathrm{Re}\ \alpha_j \rfloor + 1$ ($\lfloor\cdot\rfloor$ is the integer part), and

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^{-\mathbf{k}\boldsymbol{\alpha}'+r}} \qquad (\boldsymbol{\alpha}' = \boldsymbol{\alpha} - \mathbf{M}) \tag{80}$$

(for technical reasons the Borel summation procedure is applied to the series

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = x^{\mathbf{k}\boldsymbol{\alpha}'} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{81}$$

rather than to $\tilde{\mathbf{s}}_{\mathbf{k}}(x)$ cf. (75),(76)).

12.1 Results on singularities [9]

The map $\tilde{\mathbf{y}} \mapsto \mathcal{LB}(\tilde{\mathbf{y}})$ depends on the direction d, and (typically) is discontinuous at the finitely many *Stokes lines*, see [10], Theorem 4.

For linear equations only the directions $\overline{\lambda_j}\,\mathbb{R}_+$, $j = 1, ..., n$ are *Stokes lines*, but for nonlinear equations there are also other *Stokes lines*, recognized first by Écalle. $\mathcal{LB}$ is only discontinuous because of the jump discontinuity of the vector of "constants" $\mathbf{C}$ across *Stokes directions* (Stokes' phenomenon); between Stokes lines $\mathcal{LB}$ does not vary with d.

The function series in (79) is uniformly *convergent* and the functions $\mathbf{y}_{\mathbf{k}}$ are analytic on domains S_{an} (for some $\delta > 0$, $R = R(\mathbf{y}(x), \delta) > 0$.

Theorem 18 *(i) There exists $\delta_1 > 0$ so that if $\xi = C_1 x^{\alpha_1} e^{-\lambda_1 x}$ satisfies restriction $|\xi| < \delta_1$ then the power series*

$$\mathbf{F}_m(\xi) = \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;m}, \quad m = 0, 1, 2, ... \tag{82}$$

converge. Furthermore

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m} \mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{S}_{\delta_1},\ x \to \infty) \tag{83}$$

uniformly in a domain $\mathcal{S}_{\delta_1}$ near anti-Stokes line associated with direction $\pm i\bar{\lambda}_1\mathbb{R}^+$, where ξ is restricted to a compact set not containing singularities of $\mathbf{F}_0(\xi)$. Further, the asymptotic representation (83) is differentiable. The functions $\mathbf{F}_m$ are uniquely defined by (83), the requirement of analyticity at $\xi = 0$, and $\mathbf{F}_0'(0) = \mathbf{e}_1$.

Remark 8 *A direct calculation shows that the functions $\mathbf{F}_m$ are solutions of the system of equations*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_0 = \xi^{-1}\left(\hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0,\mathbf{F}_0)\right) \tag{84}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_m + \hat{N}\mathbf{F}_m = \alpha_1\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_{m-1} + \mathbf{R}_{m-1} \qquad \textit{for } m \geq 1 \tag{85}$$

where $\hat{N}$ *is the matrix*

$$\xi^{-1}(\partial_{\mathbf{y}}\mathbf{g}(0,\mathbf{F}_0) - \hat{\Lambda}) \tag{86}$$

and the function $\mathbf{R}_{m-1}(\xi)$ *depends only on the* $\mathbf{F}_k$ *with* $k < m$*:*

$$\xi\mathbf{R}_{m-1} = -\left[(m-1)I + \hat{A}\right]\mathbf{F}_{m-1} - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z;\sum_{j=0}^{m-1} z^j\mathbf{F}_j\right)\Bigg|_{z=0} \tag{87}$$

13 Example: Painlevé's equation P_{I}

We write this equation in the form[3]

$$u'' = u^2 - z \tag{88}$$

We seek a solution which for large z has the leading order asymptotic behavior $u(z) \sim z^{1/2}$. Substituting $u(z) = z^{1/2}(1 + v(z))$, $v(z)$ satisfies

$$-\frac{1}{4z^2} - \frac{v}{4z^2} + \frac{1}{z}v' + v'' - 2\sqrt{z}v - \sqrt{z}v^2 = 0 \tag{89}$$

A dominant balance argument gives rise to $v(z) \sim -\frac{1}{8}z^{-5/2}$ for large z. Continuing in a similar way, it is not difficult to argue that

$$v(z) \sim \sum_{j=1}^{\infty}\frac{b_j}{z^{5j/2}} \tag{90}$$

Exponential correction to this asymptotic series comes from looking at WKB solutions to the associated linearized homogeneous equation:

$$v_H'' - 2\sqrt{z}v_H + \frac{1}{z}v_H' = 0 \tag{91}$$

for which $v_H \sim z^{-5/8}\exp\left[\pm 4\sqrt{2}z^{5/4}/5\right]$. This formal expansion and the form of the asymptotic series (91) suggests that an appropriate change of variable that will convert Painlevé equation close to the normal form in [10] is:

$$u(z) = z^{1/2}[1 + y(z^{5/4})] \tag{92}$$

[3] Usually it is presented as $u'' = 6u^2 + z$. Clearly, a scaling of dependent and independent variable and a switch of sign of z will result in the form given in (88)

Substituting this into (88), we get the following differential equation for $y(x)$:

$$y'' - \frac{32}{25}y + \frac{1}{x}y' = \frac{16}{25}y^2 - \frac{4}{25}\frac{y}{x^2} - \frac{4}{25}x^{-2} \tag{93}$$

This can be put in the normal form of [10] by defining $\mathbf{y} = (y, y')$. The matrix Λ in this case has eigenvalues $\pm\frac{4}{5}\sqrt{2}$ and we have $\alpha_1 = \alpha_2 = -\frac{1}{2}$. From the results in [10], the transseries for $y(x)$ that vanishes as $|x| \to \infty$ along any ray in the right half x-plane is given by

$$y(x) \sim \tilde{y}_0(x) + \sum_{k=1}^{\infty} \xi^k \tilde{y}_k(x), \text{where} \xi = Cx^{-1/2} \exp\left[-\frac{4}{5}\sqrt{2}x\right] \tag{94}$$

and

$$\tilde{y}_0 = \sum_{j=2}^{\infty} \frac{a_{0,j}}{x^j}, \tilde{y}_k = \sum_{j=0}^{\infty} \frac{a_{k,j}}{x^j} \text{with} a_{1,0} = 1 \tag{95}$$

Furthermore, the transseries are Borel summable. There are singularities of such $y(x)$, straddling the *anti-Stokes line* $\arg x = \pm\frac{\pi}{2}$. Their form is given by looking for expansion in the form

$$y(x) = \sum_{j=0}^{\infty} \frac{F_j(\xi)}{x^j} \tag{96}$$

and doing asymptotics for large x. The equation for $F_0(\xi)$ is given by

$$\xi^2 F_0'' + \xi F_0' = F_0 + \frac{F_0^2}{2}, \text{ with condition } F_0(\xi) \sim \xi(1 + o(1)) \tag{97}$$

Such solution is given by

$$F_0(\xi) = \frac{\xi}{\left(\frac{\xi}{12} - 1\right)^2} \tag{98}$$

which has a double pole at $\xi = 12$. This corresponds to an infinite array of singularities x_s determined by the transcendental relation:

$$x_s^{-1/2} \exp\left[-\frac{4}{5}\sqrt{2}x_s\right] = 12 \tag{99}$$

For large x_s, the theory [9] predicts location of double poles for such solutions to Painlevé I solution in terms of the *Stokes constant* C.

14 Application to PDEs

The methology and rigorous results extend to PDEs as well. We illustrate the methods developed on the the following modified Harry-Dym problem [20][4] that arises in viscous fingering:

$$H_t - H^3 H_{zzz} + \frac{1}{2}H^3 = 0 \text{ with } H(z, 0) = z^{-1/2} \tag{100}$$

[4]The equation, as it appears in the reference, uses the variable $\xi = z + t$, instead of z.

A formal asymptotic expansion in powers of t, (justified rigorously as well, [12]), results in

$$H(z,t) \sim z^{-1/2} \sum_{n=0}^{\infty} t^n P_n \left(\frac{1}{z^{9/2}}, \frac{1}{z} \right) = z^{-1/2} \sum_{n=0}^{\infty} P_n \left(\frac{t}{z^{9/2}}, \frac{t}{z} \right) \tag{101}$$

where P_n is a completely determined homogeneous polynomial of order n and the asymptotics is valid for z with $\arg z \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$, $|z| \gg t^{2/9}$. For $z = O(t^{2/9})$ we introduce rescaled variables

$$\eta = \frac{z}{t^{2/9}}; \tau = t^{7/9}; H(z(\eta,t),t) = t^{1/9} G(\eta,\tau), \tag{102}$$

Then, by Corollary 37 in [12][5], for $|\eta|$ sufficiently large, with $\arg \eta \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$, the series

$$G(\eta,\tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta) \tag{103}$$

is convergent for small τ. Substituting (101) into (100), the G_k satisfy

$$\frac{1}{9} G_0 + \frac{2}{9} \eta G_0' + G_0^3 G_0''' = 0; G_0^3 \mathcal{L}_k G_k = R_k \text{ for } k \geq 1 \tag{104}$$

where the operator $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k u = u''' + \frac{2}{9G_0^3} \eta u' - \left(\frac{\beta}{G_0^3} + \frac{3G_0'''}{G_0} \right) u \text{ where } \beta = \frac{7k-1}{9} \tag{105}$$

and R_k on the right hand side of (104) is completely determined by $\{G_j\}_{j<k}$. Matching with (101) requires algebraic decay of G_k for large $|\eta|$ in the sector $\arg \eta \in \left(-\frac{4\pi}{9}, \frac{4\pi}{9}\right)$.

Using transasymptotic matching as outlined in previous section, after a normalizing change of variable, it can be proved that for some $\hat{\delta} \in \left(0, \frac{2}{9}\pi\right)$ and large η, excluding an exponentially small region around the singularities of G_0, the following asymptotic series holds uniformly in the sector $\arg \eta \in \left[-\frac{4\pi}{9} - \hat{\delta}, \frac{4\pi}{9} - \hat{\delta}\right]$

$$G_0 \sim \eta^{-1/2} U(\zeta) + O(\eta^{-5}) \tag{106}$$

where

$$\zeta = -\log C + \frac{9}{8} \log \eta + \frac{i4\sqrt{2}}{27} \eta^{9/4} \tag{107}$$

with the principal branch of the log and where C is a specific Stokes constant, and $U(\zeta)$ is determined from

$$\zeta = \log 4 - 2 - i\pi - 2\sqrt{U} - \ln \left(\frac{1-\sqrt{U}}{1+\sqrt{U}} \right) \tag{108}$$

The function U has a singularity at $\zeta = \zeta_s = \log 4 - 2 - i\pi$, corresponding to a string of singularities at $\eta = \eta_s$, where

$$\frac{i4\sqrt{2}}{27} \eta_s^{9/4} + \frac{9}{8} \log \eta_s = -2 + \log 4 - (2\hat{n} - 1)i\pi + \log C \tag{109}$$

[5] A different scaled variable, $\zeta = \eta^{3/2}$, was used in that paper.

and where $\hat{n} \in \mathbb{N}$ has to be large for η_s to be large. For large $\hat{n}$, $\arg \eta_s \sim -\frac{4\pi}{9}$, i.e. η_s approaches the anti-Stokes line. A similar quasi-periodic array of singularities exists near $\arg \eta = \frac{4\pi}{9}$. It is known that for large $|\eta_s|$, the singularities $\hat{\eta}_s$ of G_0 are exponentially close to η_s and of the same type as those of U. Further, G_0 cannot be zero except at $\hat{\eta}_s$. The main result in [13] is the following:

Theorem 19 ([13]) *For any singularity η_s of U with $|\eta_s|$ large, there is an annular domain around and close to $|\eta_s|$ so that the series (103) is convergent for small enough τ. In particular, there exists an actual branch-point singularity of $G(\eta, \tau)$ within a small neighborhood of η_s.*

It is to be noted that a singularity η_s in the η variable corresponds to a moving singularity in the z variable at the location $z = z_s(t) \equiv t^{2/9}\eta_s$. A crucial and non-trivial part of the proof relies on control of inversion $\mathcal{L}_k^{-1}$ for large k, which is explained in detail in [13].

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