INTEGRAL FORMULATION OF 3-D NAVIER-STOKES AND LONGER TIME EXISTENCE OF SMOOTH SOLUTIONS.

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Abstract. We extend Borel summability methods to the analysis of the 3-D Navier-Stokes initial value problem,

$$v_t - \nu\Delta v = -\mathcal{P}\left[v \cdot \nabla v\right] + f\ ,\ v(x,0) = v_0(x),\ x \in \mathbb{T}^3 \quad (*)$$

where $\mathcal{P}$ is the Hodge projection to divergence-free vector fields. We assume that the Fourier transform norms $\|\hat{f}\|_{l^1(\mathbb{Z}^3)}$ and $\|\hat{v}_0\|_{l^1(\mathbb{Z}^3)}$ are finite. We prove that the integral equation obtained from (*) by Borel transform and Écalle acceleration, $\hat{U}(k,q)$, is exponentially bounded for q in a sector centered on $\mathbb{R}^+$, where q is the inverse Laplace dual to $1/t^n$ for $n \geq 1$.

This implies in particular local existence of a classical solution to (*) for $t \in (0,T)$, where T depends on $\|\hat{v}_0\|_{l^1}$ and $\|\hat{f}\|_{l^1}$. Global existence of the solution to NS follows if $\|\hat{U}(\cdot,q)\|_{l^1}$ has subexponential bounds as $q \to \infty$.

If $f = 0$, then the converse is also true: if NS has global solution, then there exists $n \geq 1$ for which $\|\hat{U}(\cdot,q)\|$ necessarily decays. More generally, if the exponential growth rate in q of $\hat{U}$ is α, then a classical solution to NS exists for $t \in \left(0, \alpha^{-1/n}\right)$.

We show that α can be better estimated based on the values of $\hat{U}$ on a finite interval $[0, q_0]$. We also show how the integral equation can be solved numerically with controlled errors.

Preliminary numerical calculations of the integral equation over a modest $[0,10]$, q-interval for $n = 2$ corresponding to Kida ([21]) initial conditions, though far from being optimized or rigorously controlled, suggest that this approach gives an existence time for 3-D Navier-Stokes that substantially exceeds classical estimate.

1. Introduction

We consider the 3-D Navier-Stokes (NS) initial value problem

$$v_t - \nu\Delta v = -\mathcal{P}[v \cdot \nabla v] + f(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{T}^3[0,2\pi], \quad t \in \mathbb{R}^+ \tag{1.1}$$

where v is the fluid velocity and $\mathcal{P} = I - \nabla\Delta^{-1}(\nabla\cdot)$ is the Hodge projection operator to the space of divergence-free vector fields. For simplicity we assume that the forcing f is time-independent.

In Fourier space, (1.1) can be written as

$$\hat{v}_t + \nu|k|^2\hat{v} = -ik_j P_k\left[\hat{v}_j \hat{*} \hat{v}\right] + \hat{f}\ ,\ \hat{v}(k,0) = \hat{v}_0, \tag{1.2}$$

where $\hat{v}(k,t) = \mathcal{F}\left[v(\cdot,t)\right](k)$ is the Fourier transform of the velocity, $\hat{*}$ denotes Fourier convolution, a repeated index j indicates summation over $j = 1,2,3$ and

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$P_k = \mathcal{F}(\mathcal{P})$ is the Fourier space representation of the Hodge projection operator on the space of divergence-free vector fields, given explicitly by

$$P_k \equiv 1 - \frac{k(k\cdot)}{|k|^2}. \tag{1.3}$$

We assume that $\hat{v}_0$ and $\hat{f} \in l^1(\mathbb{Z}^3)$ and, without loss of generality, that the average velocity and force in the periodic box are zero, and hence $\hat{v}(0,t) = 0 = \hat{f}(0)$.

Global existence of smooth solutions to the 3-D Navier-Stokes problem remains a formidable open mathematical problem, even for zero forcing, despite extensive research in this area. The problem is important not only in mathematics but it has wider impact, particularly if singular solutions exist. It is known [4] that the singularities can only occur if ∇v blows up. This means that near a potential blow-up time, the relevance of NS to model fluid flow becomes questionable, since the linear approximation in the constitutive stress-strain relationship, the assumption of incompressibility and even the continuum hypothesis implicit in derivation of NS become doubtful. In some physical problems (such as inviscid Burger's equation) the singularity of an idealized approximation is mollified by inclusion of regularizing effects. It may be expected that if 3-D NS solutions exhibited blow up, then actual fluid flow, on very small time and space scales, has to involve parameters other than those considered in NS. This could profoundly affect our understanding of small scale in turbulence. In fact, some 75 years back, Leray [23], [24], [25] was motivated to study weak solutions of 3-D NS, conjecturing that turbulence was related to blow-up of smooth solutions.

The typical method used in the mathematical analysis of NS, and of more general PDEs, is the so-called energy method. For NS, the energy method involves *a priori* estimates on the Sobolev $\mathbb{H}^m$ norms of v. It is known that if $\|v(\cdot,t)\|_{\mathbb{H}^1}$ is bounded, then so are all the higher order energy norms $\|v(\cdot,t)\|_{\mathbb{H}^m}$ if they are bounded initially. The condition on v has been further weakened [4] to $\int_0^t \|\nabla \times v(\cdot,t)\|_{L^\infty} dt < \infty$. Prodi [29] and Serrin [30] have found a family of other controlling norms for classical solutions [22]. In particular, no singularity is possible if $\|v(\cdot,t)\|_{L^\infty}$ is bounded. The L^3 norm is also controlling, as has been recently shown in [32]. For classical solutions, global existence proofs exist only for small initial data and forcing or for large viscosity (*i.e.* when the non-dimensional Reynolds number is small). On a sufficiently small initial interval the solution is classical and unique. Global weak solutions (possibly non-unique) are only known to exist [23], [24], [25] in a space of functions for which ∇v can blow-up on a small set in space-time(1). However, when $f = 0$ (no forcing), a time T_c may be estimated in terms of the $\|v_0\|_{H^1}$ beyond which any weak Leray solution becomes smooth again. Such an estimate, which also follows directly from Leray's observation on the cumulative dissipation being bounded, is worked out in the Appendix. (2)

Classical energy methods have so far failed to give global existence because of failure to obtain conservation laws involving any of the controlling norms [33].

(1)The 1-D Hausdorff measure of the set of blow-up points in space-time is known to be zero [6]

(2)We are grateful to Alexey Cheskidov for pointing out the fact that classical estimates are easily obtainable.

Numerical solutions to (1.1) are physically revealing but do not shed enough light into the existence issue. Indeed, the numerical errors in Galerkin/finite-difference/finite-element approximations depend on derivatives of v that are not known to exist *a priori* beyond an initial time interval.

In [11], we proved Borel summability of the solution of the 3-D Navier Stokes system with analytic initial data and forcing. We now allow for rough initial conditions and forcing (the only requirement is that their Fourier coefficients are in l^1) and still obtain Laplace transform representations of the solutions.(3)

We show that sufficiently strong Écalle acceleration has the important consequence that the Borel solution decays as the accelerated variable q goes to $+\infty$ **iff** the associated NS solution exists globally in time. We use this to establish a number of properties of the solution in the time domain, in particular extending the time of provable existence of solutions with specific data.(4).

The method of analysis is rather general and it is clear that it can be applied to a broad class of dissipative nonlinear PDEs.

In our formulation, the velocity $v(x,t)$ is obtained as a Laplace transform:

$$v(x,t) = v_0(x) + \int_0^\infty U(x,q) e^{-q/t^n} dq, \qquad n \geq 1 \tag{1.4}$$

where U satisfies an integral equation (IE) which always has a unique acceptable smooth solution.

Remark 1.1. For general initial data and forcing, U is in $L_1(\mathbb{R}^+, e^{-\alpha q} dq)$, as defined in (2.8). If $n > 1$, then U is analytic in q in an open sector. For $n = 1$, the solution is q-analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$ (5) iff $\hat{v}(x,0)$ and $\hat{f}(x)$ are analytic in x.

In Fourier space, (1.4) implies

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k,q) e^{-q/t^n} dq. \tag{1.5}$$

Notation. Variables in the Fourier domain are marked with a hat $\hat{}$, Laplace convolution is denoted by $*$, Fourier-convolution by $\hat{*}$, while $\overset{*}{*}$ denotes Fourier followed by Laplace convolution (their order is unimportant).

As seen in §4, $\hat{U}$ satisfies the following IE:

$$\hat{U}(k,q) = -ik_j \int_0^q \mathcal{G}(q,q';k) \hat{H}_j(k,q') dq' + \hat{U}^{(0)}(k,q) =: \mathcal{N}\left[\hat{U}\right](k,q), \tag{1.6}$$

where

$$\hat{H}_j(k,q) = P_k \left[\hat{v}_{0,j} \hat{*} \hat{U} + \hat{U}_j \hat{*} \hat{v}_0 + \hat{U}_j \overset{*}{*} \hat{U} \right](k,q). \tag{1.7}$$

The kernel $\mathcal{G}$, the inhomogeneous term $\hat{U}^{(0)}(k,q)$ and their essential properties are given in (4.22) and (4.24) in §4.

Note 1.2. The solutions of (1.6), needed on $\mathbb{R}^+$, are very regular, see Note 1.1. The existence time of $\hat{v}$ is determined by the behavior of $\hat{U}$ for large q. In this formulation, global existence of $\hat{v}$ is equivalent to subexponential behavior of $\hat{U}$.

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(3) Of course, this does not imply Borel summability anymore, since now the solutions may not even be differentiable at $t = 0$.

(4) Some of the present results have been announced mostly without proofs in [12], [13]

(5) This together with the L^1 estimate proves Borel summability of the small t series.

The IE formulation was first introduced in [11] in a narrower context, and provides a new approach towards solving IVPs.

2. Main results

We define

$$L_1(\mathbb{R}^+, e^{-\alpha q}dq) = \left\{ g : \mathbb{R}^+ \mapsto \mathbb{C} \;\Big|\; \int_0^\infty e^{-\alpha q}|g(q)|dq < \infty \right\}. \tag{2.8}$$

Assumptions 1. *In the following, unless otherwise specified, we assume that* $\hat{v}_0$ *and* $\hat{f}$ *are in* $l^1(\mathbb{Z}^3)$, $\hat{v}_0(0) = 0 = \hat{f}(0)$, $n \geq 1$, $\nu > 0$ *and* α *in (2.8) is large enough (see Proposition 5.11).*

Theorem 2.1. *(i) Eq. (1.6) has a unique solution* $\hat{U}(\cdot, q) \in L_1(\mathbb{R}^+, e^{-\alpha q}dq)$*. For* $n > 1$ *this solution is analytic in an open sector, cf. Note 1.1. We let* $U(x, q) = \mathcal{F}^{-1}\left[\hat{U}(\cdot, q)\right](x)$*.*

(ii) With this $\hat{U}$, $\hat{v}$ *in (1.5) (*$v(x,t)$ *in (1.4) respectively) is a classical*[6] *solution of (1.2) ((1.1), resp.) for* $t \in \left(0, \alpha^{-1/n}\right)$*.*

(iii) Conversely, any classical solution of (1.1), $v(x,t)$, $t \in (0, T_0)$ *has a Laplace representation of the form (1.4) with* U *as in (i) and with*

$$\hat{U}(k,q) := \mathcal{L}^{-1}\left[\mathcal{F}[v(\cdot, \tau^{-1/n})](k) - \mathcal{F}[v_0](k)\right](q)$$

a solution of (1.6) in $L_1(\mathbb{R}^+, e^{-\alpha q}dq)$, $\alpha > T_0^{-n}$*.*

The proof is given at the end of §5.

Remark 2.1. Proposition 5.11 below provides (relatively rough) estimates on α. Theorem 3.1 gives sharper bounds in terms of the values of $\hat{U}$ on a finite interval $[0, q_0]$. Smaller bounds on α entail smooth solutions of (1.1) over a longer time.

We have the following result which, in a sense, is a converse of Theorem (2.1).

Theorem 2.2. *For* $f = 0$*, if (1.1) has a global classical solution, then for all sufficiently large* n, $U(x,q) = O(e^{-C_n q^{1/(n+1)}})$ *as* $q \to +\infty$*, for some* $C_n > 0$*.*

The proof is given in §7.

Corollary 2.2. *Theorems 2.1 and 2.2 imply that global existence is equivalent to an asymptotic problem:* $\hat{v}$ *exists for all time iff* $\hat{U}$ *decays in* q *for some* $n \in \mathbb{Z}^+$*.*

The existence interval $\left(0, \alpha^{-1/n}\right)$ guaranteed by Theorem 2.1 is suboptimal. It does not take into account the fact that the initial data v_0 and forcing f are real valued. (Blow up of Navier-Stokes solution for complex initial conditions is known to occur [31]). Also, the estimate ignores possible cancellations in the integrals.

In the following we address the issue of sharpening the estimates, in principle arbitrarily well, based on more detailed knowledge of the solution of the IE on an interval $[0, q_0]$. This knowledge may come, for instance, from computer assisted estimates or from rigorous bounds based on optimal truncation of asymptotic series. In this context, the fact that the radius of convergence of the series in p [7] does

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[6] That is, $C^2(\mathbb{T}^3)$ in x (by Lemma 12.1 below) and therefore, from general properties of the Navier-Stokes system, C^1 in time.

[7] Note that $q = p$ when $n = 1$.

not depend, in a number of cases, on the size of initial data and forcing [13] is most relevant. Furthermore, the calculation of the IE solution in the interval $[0, q_0]$ can account for cancellation ignored in the estimates of Theorem 2.1 and in fact uses the full structure of NS. If this information shows that the solution is sufficiently small for q near the right end of the interval, then α can be shown to be small. This in turn results in longer times of guaranteed existence of the NS solution possibly global existence for $f = 0$ if this time exceeds T_c, the time after which it is known that any weak solution becomes classical.

3. Sharpening the estimates; rigorous numerical analysis

Let $\hat{U}(k, q)$ be the solution of (1.6), provided by Theorem 2.1. Define

$$\hat{U}^{(a)}(k,q) = \begin{cases} \hat{U}(k,q) & \text{for } q \in (0, q_0] \subset \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}, \tag{3.9}$$

$$\hat{U}^{(s)}(k,q) = -ik_j \int_0^{\min\{q,2q_0\}} \mathcal{G}(q,q';k) \hat{H}_j^{(a)}(k,q')\, dq' + \hat{U}^{(0)}(k,q),$$

$$\hat{H}_j^{(a)}(k,q) = P_k \left[\hat{v}_{0,j} \hat{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*} \hat{v}_0 + \hat{U}_j^{(a)} \hat{\underset{*}{*}} \hat{U}^{(a)} \right](k,q).$$

Using (3.9) we introduce the following functionals of $\hat{U}^{(a)}(k,q)$, $\hat{v}_0$ and $\hat{f}$:

$$b := \alpha^{1/2+1/(2n)} \int_{q_0}^{\infty} e^{-\alpha q} \|\hat{U}^{(s)}(\cdot, q)\|_{l^1}\, dq, \tag{3.10}$$

$$\epsilon_1 = \Gamma\left(\frac{1}{2} + \frac{1}{2n}\right) \left[B_1 + \int_0^{q_0} e^{-\alpha q'} B_2(q')\, dq' \right], \tag{3.11}$$

$$\epsilon = \Gamma\left(\frac{1}{2} + \frac{1}{2n}\right) B_3, \tag{3.12}$$

where

$$B_1 = 4 \sup_{k \in \mathbb{Z}^3} \left\{ |k| B_0(k) \right\} \|\hat{v}_0\|_{l^1}, \qquad B_0(k) = \sup_{q_0 \le q' \le q} \left\{ (q-q')^{1/2-1/(2n)} |\mathcal{G}(q,q';k)| \right\},$$

$$B_2(q) = 4 \sup_{k \in \mathbb{Z}^3} \left\{ |k| B_0(k) \right\} \|\hat{U}^{(a)}(\cdot, q)\|_{l^1}, \qquad B_3 = 2 \sup_{k \in \mathbb{Z}^3} \left\{ |k| B_0(k) \right\}.$$

Theorem 3.1. *The exponential growth rate α of $\hat{U}$ is estimated in terms of the restriction of $\hat{U}$ to $[0, q_0]$ as follows.*

$$\text{If} \quad \alpha^{1/2+1/(2n)} > \epsilon_1 + 2\sqrt{\epsilon b} \quad \text{then} \quad \int_0^{\infty} \|\hat{U}(\cdot, q)\|_{l^1} e^{-\alpha q} dq < \infty. \tag{3.13}$$

The proof of Theorem 3.1 is given in §8.

Remark 3.1. In §8.1, it is shown that for a given *global classical* solution to (1.1), in adapted variables, the quantity $\epsilon_1 + 2\sqrt{\epsilon b}$ is small for large q_0.

Remark 3.2. In the proof it is also seen that if $\|\hat{U}^{(a)}(\cdot,q)\|_{l^1}$ is small enough in a sufficiently large subinterval $[q_d,q_0]$, then the right side of (3.13) is small, implying a large existence time $\left(0,\alpha^{-1/n}\right)$ of a classical solution v. The guaranteed existence time is larger if q_0 is larger. If for $f=0$, the estimated $\alpha^{-1/n}$ exceeds T_c, the time for Leray's weak solution to become classical again (see Appendix), then global existence of a classical solution v follows.

Since the improved estimates in Theorem 3.1 rely on the values of $\hat{U}$ on a sufficiently large initial interval, we analyze the properties of a discretized scheme for numerical computation of $\hat{U}$ with *controlled errors*.

Definition 3.3. *We introduce the following norm on functions defined on a δ-grid in q*

$$\|\hat{W}\|^{(\alpha,\delta)} = \sup_{m_s\le m\in\mathbb{Z}^+} m^{1-1/n}\delta^{1-1/n}(1+m^2\delta^2)e^{-\alpha m\delta}\|\hat{W}(\cdot,m\delta)\|_{l^1}.$$

Theorem 3.2. *Consider a discretized integral equation consistent with (1.6) (cf. definition 9.2) based on Galerkin truncation to $[-N,N]^3$ Fourier modes and uniform discretization in q,*

$$\hat{U}_\delta^{(N)} = \mathcal{N}_\delta^{(N)}\left[\hat{U}_\delta^{(N)}\right],$$

see (9.77) below. Then, the error $\hat{U}-\hat{U}_\delta^{(N)}$ at the points $q=m\delta$, satisfies

$$\|\hat{U}(\cdot,m\delta)-\hat{U}_\delta^{(N)}(\cdot,m\delta)\|_{l^1} \le \left[2\|T_{E,N}\|^{(\alpha,\delta)}+2\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}+\|(I-\mathcal{P}_N)\hat{U}\|^{(\alpha,\delta)}\right]\frac{e^{\alpha m\delta}}{m^{1-1/n}\delta^{1-1/n}(1+m^2\delta^2)} \tag{3.14}$$

for $m\ge m_s\in\mathbb{Z}^+$, where $m_s\delta =: q_m>0$ is independent of δ. In (3.14), $T_{E,N}$ is the truncation error due to Galerkin projection $\mathcal{P}_N$ and $T_{E,\delta}^{(N)}$ is the truncation error due to the δ-discretization in q for a given N. We have $\|T_{E,N}\|^{(\alpha,\delta)}, \|(I-\mathcal{P}_N)\hat{U}\|^{(\alpha,\delta)}\to 0$ as $N\to\infty$ for any δ and $\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}\to 0$ as $\delta\to 0$, uniformly in N.

Remark 3.4. For small q, independent of δ, an asymptotic expansion of $\hat{U}$ exists, and solving the equation numerically for $q\in[0,q_m]$ can be avoided. For this reason we start with $q=q_m$.

4. Integral Equation (1.6) and its properties

We define $\hat{u}$ through the decomposition

$$\hat{v}(k,t)=\hat{v}_0(k)+\hat{u}(k,t). \tag{4.15}$$

Then, (1.2) implies

$$\hat{u}_t+\nu|k|^2\hat{u} = -ik_jP_k\left[\hat{v}_{0,j}\hat{*}\hat{u}+\hat{u}_j\hat{*}\hat{v}_0+\hat{u}_j\hat{*}\hat{u}\right]+\hat{v}_1(k) =: -ik_j\hat{h}_j(k,t)+\hat{v}_1(k), \tag{4.16}$$

where $\hat{v}_1$ is given by (5.59). Using $\hat{v}(k,0)=\hat{v}_0$, we have $\hat{u}(k,0)=0$ and we obtain from (4.16),

$$\hat{u}(k,t) = -ik_j\int_0^t e^{-\nu|k|^2(t-s)}\hat{h}_j(k,s)ds+\hat{v}_1(k)\left(\frac{1-e^{-\nu|k|^2t}}{\nu|k|^2}\right). \tag{4.17}$$

We look for $\hat{u}$ in the form of a Laplace transform

$$\hat{u}(k,t) = \int_0^\infty \hat{U}(k,q)e^{-q/t^n}dq; \quad n \geq 1 \tag{4.18}$$

We apply the inverse Laplace transform of (4.17) with respect to $\tau = 1/t^n$ (justified at the end of the proof of Lemma 4.6, with more details in the Appendix) to obtain (1.6). The inverse Laplace transform of f is given, as usual, by

$$\left[\mathcal{L}^{-1}f\right](p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{ps}ds, \tag{4.19}$$

where c is chosen so that f is analytic and has suitable asymptotic decay for Re $s \geq c$.

For $n = 1$ the kernel $\mathcal{G}$ is given by, see [11],

$$\mathcal{G}(q,q';k) = \frac{\pi z'}{z}\left(J_1(z)Y_1(z') - J_1(z')Y_1(z)\right) \tag{4.20}$$
$$\text{where } z = 2|k|\sqrt{\nu q}\ ,\ z' = 2|k|\sqrt{\nu q'}, \quad (n=1)$$

J_1 and Y_1 are Bessel functions of order 1, and

$$\hat{U}^{(0)}(k,q) = 2\hat{v}_1(k)\frac{J_1(z)}{z}\ , \text{ where } z = 2|k|\sqrt{\nu q} \tag{4.21}$$

For $n \geq 2$ the kernel has the form (derived in the Appendix, see (12.92))

(4.22)
$$\mathcal{G}(q,q';k) = \int_{(q'/q)^{1/n}}^1 \left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tau^{-1/n}\exp\left[-\nu|k|^2\tau^{-1/n}(1-s) + (q-q's^{-n})\tau\right]d\tau\right\}ds$$
$$= \frac{\gamma^{1/n}}{\nu^{1/2}|k|q^{1-1/(2n)}}\int_1^{\gamma^{-1/n}}(1-s^{-n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2}\mu^{1/2}F(\mu)\,ds,$$

where

$$\gamma = \frac{q'}{q}, \qquad \mu = \nu|k|^2q^{1/n}(1-s\gamma^{1/n})(1-s^{-n})^{1/n},$$

$$F(\mu) = \frac{1}{2\pi i}\int_C \zeta^{-1/n}e^{\zeta-\mu\zeta^{-1/n}}\,d\zeta, \tag{4.23}$$

and C is a contour starting at $\infty e^{-i\pi}$ and ending at $\infty e^{i\pi}$ turning around the origin counterclockwise. The function $\hat{U}^{(0)}(k,q)$ in (1.6) is defined by

$$\hat{U}^{(0)}(k,q) = \frac{\hat{v}_1(k)}{\nu|k|^2}\mathcal{L}^{-1}\left\{1-\exp\left[-\nu|k|^2\tau^{-1/n}\right]\right\}(q) = \frac{\hat{v}_1(k)}{\nu|k|^2q}G(\nu|k|^2q^{1/n}), \tag{4.24}$$

where

$$G(\tilde{\mu}) = -\frac{1}{2\pi i}\int_C e^{\zeta-\tilde{\mu}\zeta^{-1/n}}\,d\zeta. \tag{4.25}$$

4.1. Properties of F, G, $\mathcal{G}$, $\hat{U}^{(0)}$ and the relation between the IE and NS.

Lemma 4.1. *The functions F, G in (4.23) and (4.25) are entire and $G'(\mu) = F(\mu)$. Furthermore $F(0) = \frac{1}{\Gamma(1/n)}$, $G(0) = 0$ and, for $n \ge 2$, their asymptotic behavior for large positive μ is given by*

(4.26)

$$F(\mu) \sim \begin{cases} \sqrt{\dfrac{2}{\pi(n+1)}} n^{\frac{3}{2(n+1)}} \mu^{\frac{n-2}{2(n+1)}} \operatorname{Im}\left\{\exp\left[\dfrac{3i\pi}{2(n+1)}\right] e^{-z}\right\} & \text{if } \arg\mu = 0 \\ -i\sqrt{\dfrac{1}{2\pi(n+1)}} n^{\frac{3}{2(n+1)}} \mu^{\frac{n-2}{2(n+1)}} \exp\left[\dfrac{3i\pi}{2(n+1)}\right] e^{-z} & \text{if } \arg\mu \in (0, \frac{n+3}{2n}\pi) \\ i\sqrt{\dfrac{1}{2\pi(n+1)}} n^{\frac{3}{2(n+1)}} \mu^{\frac{n-2}{2(n+1)}} \exp\left[\dfrac{-3i\pi}{2(n+1)}\right] e^{-\hat{z}} & \text{if } \arg\mu \in (-\frac{n+3}{2n}\pi, 0) \end{cases}$$

(4.27)

$$G(\mu) \sim \begin{cases} -\sqrt{\dfrac{2}{\pi(n+1)}} n^{\frac{1}{2(n+1)}} \mu^{\frac{n}{2(n+1)}} \operatorname{Im}\left\{\exp\left[\dfrac{i\pi}{2(n+1)}\right] e^{-z}\right\} & \text{if } \arg\mu = 0 \\ i\sqrt{\dfrac{1}{2\pi(n+1)}} n^{\frac{1}{2(n+1)}} \mu^{\frac{n}{2(n+1)}} \exp\left[\dfrac{i\pi}{2(n+1)}\right] e^{-z} & \text{if } \arg\mu \in (0, \frac{n+3}{2n}\pi) \\ -i\sqrt{\dfrac{1}{2\pi(n+1)}} n^{\frac{1}{2(n+1)}} \mu^{\frac{n}{2(n+1)}} \exp\left[\dfrac{-i\pi}{2(n+1)}\right] e^{-\hat{z}} & \text{if } \arg\mu \in (-\frac{n+3}{2n}\pi, 0) \end{cases}$$

where

(4.28)

$$z = \xi_0 \mu^{n/(n+1)} e^{i\pi/(n+1)} \ , \ \xi_0 = n^{-n/(n+1)} \ (n+1); \ \hat{z} = \xi_0 \mu^{n/(n+1)} e^{-i\pi/(n+1)}.$$

Proof. These results follow from standard steepest descent analysis and from the ordinary differential equation that F and G satisfy, see §12.1.1. ∎

Remark 4.2. *We see that $F(\mu)$ and $G(\mu)$ are exponentially small for large μ when $\arg\mu \in \left(-\frac{(n-1)\pi}{2n}, \frac{(n-1)\pi}{2n}\right)$, that is, when $\arg q \in \left(-\frac{(n-1)\pi}{2}, \frac{(n-1)\pi}{2}\right)$.*

Definition 4.3. *For $\delta > 0$ and $n \ge 2$ we define the sector*

$$\mathcal{S}_\delta := \left\{ q : \arg q \in \left(-\frac{(n-1)\pi}{2} + \delta, \frac{(n-1)\pi}{2} - \delta \right) \right\}.$$

Lemma 4.4. *For $n \ge 2$, $q, q' \in e^{i\phi}\mathbb{R}^+ \subset \mathcal{S}_\delta$, with $0 < |q'| \le |q| < \infty$ and $k \in \mathbb{Z}^3$ we have*

$$|\mathcal{G}(q, q'; k)| \le \frac{C_2 |q - q'|^{\frac{1}{2n} - \frac{1}{2}}}{\nu^{1/2} |k| |q|^{1/2}},$$

where C_2 only depends on δ. For $n = 1$, the same inequality holds for $q, q' \in \mathbb{R}^+$ with $0 < q' \le q$.

Proof. The case $n = 1$ follows from the behavior of J_1 and Y_1, see [11][8]. For $n \ge 2$, it follows from Lemma 4.1 that $|\mu^{1/2} F(\mu)|$ is bounded, with a bound dependent on

[8] In that paper the viscosity ν was scaled to 1.

δ. Below, C is a generic constant, possibly δ and n dependent. From (4.22) we get

$$(4.29)\quad |\mathcal{G}(q,q';k)| \le \frac{C\gamma^{1/n}}{\nu^{1/2}|k||q|^{1-1/(2n)}}\left[\int_1^{\frac{1}{2}(1+\gamma^{-1/n})} + \int_{\frac{1}{2}(1+\gamma^{-1/n})}^{\gamma^{-1/n}}\right]$$

$$\times(1-s^{-n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2}\,ds =: \frac{C\gamma^{1/n}}{\nu^{1/2}|k||q|^{1-1/(2n)}}\Big(I_1+I_2\Big);\ \text{ where } \gamma=\frac{q'}{q}$$

For $s\in\left(1,\frac{1}{2}(1+\gamma^{-1/n})\right]$ we have

(4.30)

$$(1-s^{-n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2} \le \left(\frac{s-1}{1-s^{-n}}\right)^{1-1/(2n)}(s-1)^{1/(2n)-1}\left(\frac{1}{2}-\frac{1}{2}\gamma^{1/n}\right)^{-1/2}$$
$$\le C\Big[1+(s-1)^{1-1/(2n)}\Big](s-1)^{1/(2n)-1}\left(\frac{1}{2}-\frac{1}{2}\gamma^{1/n}\right)^{-1/2}$$
$$\le C(1-\gamma^{1/n})^{-1/2}\Big[1+(s-1)^{1/(2n)-1}\Big],$$

and for $s\in[\frac{1}{2}(1+\gamma^{-1/n}),\gamma^{-1/n})$,

$$(1-s^{-n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2} \le C\Big[1+(s-1)^{1/(2n)-1}\Big](1-s\gamma^{1/n})^{-1/2}$$
$$\le C(1-\gamma^{1/n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2}.$$

Thus

$$I_1 \le C(1-\gamma^{1/n})^{-1/2}\int_1^{\frac{1}{2}(1+\gamma^{-1/n})}\Big[1+(s-1)^{1/(2n)-1}\Big]\,ds$$
$$\le C\gamma^{-1/n}(1-\gamma^{1/n})^{-1/2}\Big[(1-\gamma^{1/n})+(1-\gamma^{1/n})^{1/(2n)}\Big]$$
$$\le C\gamma^{-1/n}(1-\gamma^{1/n})^{1/(2n)-1/2},$$
$$I_2 \le C(1-\gamma^{1/n})^{1/(2n)-1}\int_{\frac{1}{2}(1+\gamma^{-1/n})}^{\gamma^{-1/n}}(1-s\gamma^{1/n})^{-1/2}\,ds$$
$$\le C\gamma^{-1/n}(1-\gamma^{1/n})^{1/(2n)-1/2}.$$

∎

Lemma 4.5. *(i) For $n\ge 2$ and $0\ne q\in\mathcal{S}_\delta$, we have for $\alpha\ge 1$,*

$$\|\hat{U}^{(0)}(\cdot,q)\|_{l^1} \le c_1\|\hat{v}_1\|_{l^1}|q|^{-1+1/n}\exp\left[-c_2\nu^{n/(n+1)}|q|^{1/(n+1)}\right],$$

$$\|k\hat{U}^{(0)}(\cdot,q)\|_{l^1} \le c_1\||k|\hat{v}_1\|_{l^1}|q|^{-1+1/n}\exp\left[-c_2\nu^{n/(n+1)}|q|^{1/(n+1)}\right],$$

where c_1 and c_2 depend on δ and n. Thus, we have

$$(4.31)\qquad \int_0^\infty e^{-\alpha|q|}\|\hat{U}^{(0)}(\cdot,q)\|_{l^1}\,d|q| \le c_1\|\hat{v}_1\|_{l^1}\alpha^{-1/n}\Gamma\left(\frac{1}{n}\right).$$

With $c_1 = 1$ and $q \in \mathbb{R}^+$, the bound in (4.31) holds for $n = 1$ as well. For $n \geq 2$, noting that $\hat{v}_1(0) = 0 = \hat{f}(0)$, we have

(4.32)
$$\int_0^\infty \|\hat{U}^{(0)}(\cdot, q)\|_{l^1} d|q| \leq C_G \left\| \frac{\hat{v}_1}{\nu |k|^2} \right\|_{l^1} \leq C_G \left\{ \|\hat{v}_0\|_{l^1} \left(1 + \frac{2}{\nu} \|\hat{v}_0\|_{l^1}\right) + \frac{1}{\nu} \left\| \frac{\hat{f}}{|k|^2} \right\|_{l^1} \right\},$$

where
$$C_G = \sup_{\phi \in \left[-\frac{n-1}{2n}\pi + \frac{\delta}{n}, \frac{n-1}{2n}\pi - \frac{\delta}{n}\right]} n \int_0^\infty s^{-1} |G(s e^{i\phi})| ds.$$

(ii) If moreover $|k|^{j+2}\hat{v}_0, |k|^j \hat{f} \in l^1$ $(j = 0, 1)$, then

$$\sup |q|^{1-1/n} (1 + |q|^2) e^{-\alpha |q|} \|k^j \hat{U}^{(0)}(\cdot, q)\|_{l^1}$$
$$\leq 2c_1 \||k|^j \hat{v}_1\|_{l^1} \leq 2c_1 \left[\nu \||k|^{j+2}\hat{v}_0\|_{l^1} + 2\||k|^j \hat{v}_0\|_{l^1} \||k|\hat{v}_0\|_{l^1} + \||k|^j \hat{f}\|_{l^1}\right]$$

where the sup is taken over $\mathbb{R}^+$ if $n = 1$ and over $\mathcal{S}_\delta$ if $n > 1$.

Proof. The result follows from (4.24) and (5.59) using the asymptotics of G, cf. (4.27) and the behavior $G(\tilde{\mu}) \sim C\tilde{\mu}$ near $\tilde{\mu} = 0$. For $n = 1$, the bound (4.31) follows from the fact that $\left|2z^{-1} J_1(z)\right| \leq 1$. ∎

The following lemma proves that a suitable solution to the integral equation (1.6) gives rise to a solution of NS.

Lemma 4.6. *For any solution $\hat{U}$ of (1.6) such that $\|\hat{U}(\cdot, q)\|_{l^1} \in L_1(\mathbb{R}^+, e^{-\alpha q} dq)$, the Laplace transform*
$$\hat{v}(k, t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k, q) e^{-q/t^n} dq$$
solves (1.2) for $t \in \left(0, \alpha^{-1/n}\right)$. For $n = 1$, $\hat{v}(k, t)$ is analytic in t for $\mathrm{Re}\frac{1}{t} > \alpha$.

It will turn out, cf. Lemma 12.1 in the appendix, that $|k|^2 \hat{v}(\cdot, t) \in l^1$ for $t \in \left(0, \alpha^{-1/n}\right)$. Therefore, $v(x, t) = \mathcal{F}^{-1}[\hat{v}(\cdot, t)](x)$ is the classical solution of (1.1).

Proof. From (4.24), we obtain
$$\int_0^\infty e^{-qt^{-n}} \hat{U}^{(0)}(k, q) dq = \hat{v}_1(k) \int_0^\infty e^{-qt^{-n}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1 - e^{-\nu |k|^2 \tau^{-1/n}}}{\nu |k|^2} e^{q\tau} d\tau dq$$
$$= \hat{v}_1(k) \left(\frac{1 - e^{-\nu |k|^2 t}}{\nu |k|^2}\right).$$

Furthermore , we may rewrite (4.22) as

(4.33) $\mathcal{G}(q, q'; k)$
$$= \frac{1}{2\pi i} \int_0^1 \int_{c-i\infty}^{c+i\infty} \tau^{-1/n} \left\{\exp\left[-\nu |k|^2 \tau^{-1/n}(1 - s) + (q - q'/s^n)\tau\right] d\tau\right\} ds$$

since the integral with respect to τ is identically zero when $s \in \left(0, (q'/q)^{1/n}\right)$ (the τ contour can be pushed to $+\infty$), we can replace the lower limit in the outer integral in (4.33) by $(q'/q)^{1/n}$. Note that $\|\hat{H}_j(\cdot, q)\|_{l^1} \in L_1\left(e^{-\alpha |q|} d|q|\right)$, since

$$\|F * G\|_\alpha \leq \|F\|_\alpha \|G\|_\alpha \tag{4.34}$$

(see[10] and also Lemma 5.5 below). Changing variable $q'/s^n \to q'$ and applying Fubini's theorem we get
(4.35)

$$-ik_j \int_0^q \hat{H}_j(k,q')\mathcal{G}(q,q';k)dq' = \int_0^1 s^n \left\{ \int_0^q \left[-ik_j \hat{H}_j \right](k, q' s^n) \mathcal{Q}(q-q', s; k) dq' \right\} ds$$

where for $q > 0$ we have

$$\mathcal{Q}(q,s;k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\left[-\nu |k|^2 \tau^{-1/n} (1-s) + q\tau) \right] \tau^{-1/n} d\tau. \tag{4.36}$$

Laplace transforming (4.35) with respect to q, again by Fubini we have

$$\int_0^\infty e^{-qt^{-n}} \left\{ \int_0^1 \int_0^q \left\{ -ik_j \hat{H}_j \right\} (k, q's^n) \mathcal{Q}(q-q'; s, k) s^n dq' ds \right\} \, dq \tag{4.37}$$

$$= -ik_j \int_0^1 \, ds \; g(t,s;k) \hat{h}_j(k, st),$$

where $\hat{h}_j(k,t) = \mathcal{L}\left[\hat{H}_j(k,\cdot) \right](t^{-n})$, $g(t,s;k) = \mathcal{L}\left[\mathcal{Q}(\cdot,s;k) \right](t^{-n})$. By assumption, $\|\hat{U}(\cdot,q)\|_{l^1} \in L_1\left(\mathbb{R}^+, e^{-\alpha q} dq\right)$ and $\hat{v}_0(k) \in l^1$. From (1.7) and (4.34) it follows that $\hat{H}_j$ is Laplace transformable in q and

$$\hat{h}_j(k,t) = P_k \left\{ \hat{v}_{0,j} \hat{*} \hat{u} + \hat{u}_j \hat{*} \hat{v}_0 + \hat{u}_j \hat{*} \hat{u} \right\} (k,t),$$

while

$$g(t,s;k) = t \exp\left[-\nu |k|^2 t (1-s) \right].$$

This leads to

$$\hat{u}(k,t) = t \int_0^1 e^{-\nu |k|^2 t(1-s)} \left[-ik_j \hat{h}_j \right] (k, st) ds + \hat{v}_1(k) \left(\frac{1 - e^{-\nu |k|^2 t}}{\nu |k|^2} \right)$$

$$= \int_0^t e^{-\nu |k|^2 (t-\tau)} \left[-ik_j \hat{h}_j \right] (k,\tau) d\tau + \hat{v}_1(k) \left(\frac{1 - e^{-\nu |k|^2 t}}{\nu |k|^2} \right)$$

and thus

$$\hat{u}_t + \nu |k|^2 \hat{u} = -ik_j \hat{h}_j(k,t) \; + \; \hat{v}_1 \;, \text{ with } \hat{u}(k,0) = 0.$$

Therefore, using expression (4.16) for $\hat{h}_j$, we see that $\hat{v}(k,t) = \hat{u}(k,t) + \hat{v}_0(k)$ (1.2), with $\hat{v}(k,0) = \hat{v}_0(k)$. Analyticity in t of this solution in region $\mathrm{Re}\frac{1}{t} > \alpha$ follows from the representation (1.5). It is clear that $|k|^2 \hat{v}(\cdot,t), \hat{f} \in l^1$, ensures that $\mathcal{F}^{-1}\left[\hat{v}(\cdot,t) \right](x)$ is a classical solution to (1.1). ∎

5. Existence of a solution to (1.6)

First, we prove some preliminary lemmas.

Lemma 5.1. *By standard Fourier theory, if $\hat{v}, \hat{w} \in l^1\left(\mathbb{Z}^3\right)$, then so is $\hat{v} \hat{*} \hat{w}$, and $\|\hat{v} \hat{*} \hat{w}\|_{l^1} \le \|\hat{v}\|_{l^1} \|\hat{w}\|_{l^1}$.* ∎

Lemma 5.2.

$$\|P_k \left[\hat{w}_j \hat{*} \hat{v} \right] \|_{l^1} \le 2 \|\hat{w}_j\|_{l^1} \|\hat{v}\|_{l^1}.$$

Proof. It is easily seen from the representation of P_k in (1.3) that

$$|P_k \hat{g}(k)| \leq 2|\hat{g}(k)|. \tag{5.38}$$

The rest follows from Lemma (5.1). ∎

Lemma 5.3. *Let $C_2 = C_2(\delta, n)$ be given by*

$$C_2 = 2 \sup_{\substack{q,q' \in e^{i\phi}\mathbb{R}^+ \subset \mathcal{S}_\delta,\, 0 \leq |q'| \leq |q| \\ k \in \mathbb{Z}^3}} \nu^{1/2} |k| |q|^{1/2} |q - q'|^{1/2 - 1/(2n)} |\mathcal{G}(q, q'; k)| \quad \text{for } n \geq 2,$$

$$C_2 = 2 \sup_{\substack{q,q' \in \mathbb{R}^+,\, 0 \leq q' \leq q \\ k \in \mathbb{Z}^3}} \nu^{1/2} |k| q^{1/2} |\mathcal{G}(q, q'; k)| \quad \text{for } n = 1.$$

Then, for $n \geq 2$, we have

$$\|\mathcal{N}[\hat{U}](\cdot, q)\|_{l^1} \leq \frac{C_2}{\nu^{1/2}|q|^{1/2}} \int_0^{|q|} (|q| - s)^{-1/2 + 1/(2n)} \left\{ \|\hat{U}(\cdot, se^{i\phi})\|_{l^1} \right. \\ \left. * \|\hat{U}(\cdot, se^{i\phi})\|_{l^1} + 2\|\hat{v}_0\|_{l^1} \|\hat{U}(\cdot, se^{i\phi})\|_{l^1} \right\} ds + \|\hat{U}^{(0)}(\cdot, q)\|_{l^1}, \tag{5.39}$$

$$\|\mathcal{N}[\hat{U}^{[1]}](\cdot, q) - \mathcal{N}[\hat{U}^{[2]}](\cdot, q)\|_{l^1} \\ \leq \frac{C_2}{\nu^{1/2}|q|^{1/2}} \int_0^{|q|} (|q| - s)^{-1/2 + 1/(2n)} \left\{ \left(\|\hat{U}^{[1]}(\cdot, se^{i\phi})\|_{l^1} + \|\hat{U}^{[2]}(\cdot, se^{i\phi})\|_{l^1} \right) \right. \\ \left. * \|\hat{U}^{[1]}(\cdot, se^{i\phi}) - \hat{U}^{[2]}(\cdot, se^{i\phi})\|_{l^1} + 2\|\hat{v}_0\|_{l^1} \|\hat{U}^{[1]}(\cdot, se^{i\phi}) - \hat{U}^{[2]}(\cdot, se^{i\phi})\|_{l^1} \right\} ds. \tag{5.40}$$

For $n = 1$, (5.39) and (5.40) hold for $q \in \mathbb{R}^+$, i.e. when $\phi = 0$.

Proof. From Lemma 5.2, we have, for any q

$$\|P_k \left\{ \hat{U}_j \hat{*} \hat{U} \right\} (k, q)\|_{l^1} \leq 2\|\hat{U}(\cdot, q)\|_{l^1} * \|\hat{U}(\cdot, q)\|_{l^1},$$

and similarly

$$\|P_k \left\{ \hat{v}_{0,j} \hat{*} \hat{U}(\cdot, q) + \hat{U}_j(\cdot, q) \hat{*} \hat{v}_0 \right\} \|_{l^1} \leq 4\|\hat{v}_0\|_{l^1} \|\hat{U}(\cdot, q)\|_{l^1},$$

and (5.39) follows.

The second part of the lemma follows by noting that

$$\hat{U}_j^{[1]} \hat{*} \hat{U}^{[1]} - \hat{U}_j^{[2]} \hat{*} \hat{U}^{[2]} = \hat{U}_j^{[1]} \hat{*} \left(\hat{U}^{[1]} - \hat{U}^{[2]} \right) + \left(\hat{U}_j^{[1]} - \hat{U}_j^{[2]} \right) \hat{*} \hat{U}^{[2]}. \tag{5.41}$$

Applying Lemma 5.2 to (5.41), we obtain

$$\|P_k \left\{ \hat{U}_j^{[1]} \hat{*} \hat{U}^{[1]}(\cdot, q) - \hat{U}_j^{[2]} \hat{*} \hat{U}^{[2]}(\cdot, q) \right\} \|_{l^1} \leq 2\|\hat{U}^{[1]}(\cdot, q)\|_{l^1} * \|\hat{U}^{[1]}(\cdot, q) - \hat{U}^{[2]}(\cdot, q)\|_{l^1} \\ + 2\|\hat{U}^{[2]}(\cdot, q)\|_{l^1} * \|\hat{U}^{[1]}(\cdot, q) - \hat{U}^{[2]}(\cdot, q)\|_{l^1},$$

from which (5.40) follows easily. ∎

It is convenient to define a number of different q-norms, $q \in e^{i\phi}\mathbb{R}^+ \cup \{0\} \subset \mathcal{S}_\delta$.

Definition 5.4. *(i) For $\alpha > 0$, $n \geq 2$, we let $\mathcal{A}^{(\alpha)}$ be the set of analytic functions in $\mathcal{S}_\delta$ with the norm*

$$\|\hat{f}\|^{(\alpha)} = \sup_{q \in \mathcal{S}_\delta} |q|^{1-1/n}(1+|q|^2)e^{-\alpha|q|}\|\hat{f}(\cdot,q)\|_{l^1} < \infty, \tag{5.42}$$

while for $n = 1$, $\mathcal{A}^{(\alpha)}$ will denote the set of continuous functions on $[0,\infty)$ with norm $\|\cdot\|^{(\alpha)}$.

(ii) Let $\alpha > 0$, $n \geq 2$, $\delta > 0$. We define a Banach space $\mathcal{A}_1^{\alpha;\phi}$ of functions along the ray $|q|e^{i\phi} \in \mathcal{S}_\delta$ with the norm

$$\|\hat{f}\|_1^{\alpha;\phi} = \int_0^\infty e^{-\alpha|q|}\|\hat{f}(\cdot,|q|e^{i\phi})\|_{l^1} d|q| < \infty. \tag{5.43}$$

We agree to omit the superscript ϕ when $\phi = 0$ (which is always the case if $n = 1$).

Lemma 5.5. *We have the following Banach algebra properties:*

$$\|\hat{f} \,\hat{*}\, \hat{g}\|_1^{\alpha;\phi} \leq \|\hat{f}\|_1^{\alpha;\phi}\|\hat{g}\|_1^{\alpha;\phi}, \tag{5.44}$$

$$\|\hat{f} \,\hat{*}\, \hat{g}\|^{(\alpha)} \leq M_0\|\hat{f}\|^{(\alpha)}\|\hat{g}\|^{(\alpha)}, \tag{5.45}$$

where

$$M_0 = 2^{4-1/n}\int_0^\infty \frac{ds}{s^{1-1/n}(1+s^2)}.$$

Proof. In the following, we take $u(s) = \|\hat{f}(\cdot, se^{i\phi})\|_{l^1}$ and $v(s) = \|\hat{g}(\cdot, se^{i\phi})\|_{l^1}$. For (5.44) we note that for any $L > 0$,

$$\int_0^L e^{-\alpha|q|}\int_0^{|q|} u(s)v(|q|-s)ds\; d|q| \tag{5.46}$$
$$= \int_0^L \int_0^{|q|} e^{-\alpha s}e^{-\alpha(|q|-s)}u(s)v(|q|-s)ds\; d|q| \leq \int_0^L e^{-\alpha s}u(s)ds\int_0^L e^{-\alpha\tau}v(\tau)d\tau.$$

From (5.42), we note that

$$\int_0^{|q|} u(s)v(|q|-s)ds \leq \|\hat{f}\|^{(\alpha)}\|\hat{g}\|^{(\alpha)}e^{\alpha|q|}\int_0^{|q|}\frac{ds}{s^{1-1/n}(|q|-s)^{1-1/n}[1+s^2][1+(|q|-s)^2]}.$$

Finally,

$$\int_0^{|q|}\frac{ds}{s^{1-1/n}(|q|-s)^{1-1/n}[1+s^2][1+(|q|-s)^2]}$$
$$= 2\int_0^{|q|/2}\frac{ds}{s^{1-1/n}(|q|-s)^{1-1/n}[1+s^2][1+(|q|-s)^2]}$$
$$\leq \frac{2^{2-1/n}}{|q|^{1-1/n}(1+|q|^2/4)}\int_0^{|q|/2}\frac{ds}{s^{1-1/n}[1+s^2]} \leq \frac{2^{4-1/n}}{|q|^{1-1/n}(1+|q|^2)}\int_0^\infty\frac{ds}{s^{1-1/n}[1+s^2]},$$

where we used $\sup \frac{1+|q|^2}{1+|q|^2/4} = 4$. ∎

Lemma 5.6. *Let C_2 be as in Lemma 5.3 and $\alpha \geq 1$. The operator $\mathcal{N}$ in (1.6) is well defined on:*

(i) $\mathcal{A}_1^{\alpha;\phi}$*, where it satisfies the following inequalities*

(5.47)
$$\|\mathcal{N}[\hat{U}]\|_1^{\alpha;\phi} \le C_2 \nu^{-1/2} \Gamma\left(\frac{1}{2n}\right) \alpha^{-1/(2n)} \left\{ \left(\|\hat{U}\|_1^{\alpha;\phi} \right)^2 + 2\|\hat{v}_0\|_{l^1} \|\hat{U}\|_1^{\alpha;\phi} \right\} + \|\hat{U}^{(0)}\|_1^{\alpha;\phi},$$

$$(5.48)\quad \|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_1^{\alpha;\phi} \le C_2 \nu^{-1/2} \Gamma\left(\frac{1}{2n}\right) \alpha^{-1/(2n)}$$
$$\times \left\{ \left(\|\hat{U}^{[1]}\|_1^{\alpha;\phi} + \|\hat{U}^{[2]}\|_1^{\alpha;\phi} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{\alpha;\phi} + 2\|\hat{v}_0\|_{l^1} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{\alpha;\phi} \right\}.$$

(ii) $\mathcal{A}^{(\alpha)}$*, where it satisfies the inequalities:*

(5.49)
$$\|\mathcal{N}[\hat{U}]\|^{(\alpha)} \le C_2 C_3 \nu^{-1/2} \alpha^{-1/(2n)} \left\{ M_0 \left(\|\hat{U}\|^{(\alpha)} \right)^2 + 2\|\hat{v}_0\|_{l^1} \|\hat{U}\|^{(\alpha)} \right\} + \|\hat{U}^{(0)}\|^{(\alpha)},$$

$$(5.50)\quad \|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|^{(\alpha)} \le C_2 C_3 \nu^{-1/2} \alpha^{-1/(2n)}$$
$$\times \left\{ M_0 \left(\|\hat{U}^{[1]}\|^{(\alpha)} + \|\hat{U}^{[2]}\|^{(\alpha)} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|^{(\alpha)} + 2\|\hat{v}_0\|_{l^1} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|^{(\alpha)} \right\},$$

where C_3 *is defined in (5.52) and depends on* n *alone.*

Proof. (i) For any $0 < L \le \infty$ and $u \ge 0$ we have

$$\int_0^L e^{-\alpha|q|} |q|^{-1/2} \left(\int_0^{|q|} (|q| - s)^{-1/2+1/(2n)} u(se^{i\phi}) ds \right) d|q|$$
$$= \int_0^L u(se^{i\phi}) e^{-\alpha s} \left(\int_s^L |q|^{-1/2} (|q| - s)^{-1/2+1/(2n)} e^{-\alpha(|q|-s)} d|q| \right) ds$$
$$\le \int_0^L e^{-\alpha s} u(se^{i\phi}) \left\{ \int_0^L {s'}^{-1/2+1/(2n)} (s' + s)^{-1/2} e^{-\alpha s'} ds' \right\} ds.$$

Using (5.39) it follows that

$$\int_0^\infty e^{-\alpha|q|} \|\mathcal{N}[\hat{U}](\cdot, |q|e^{i\phi})\|_{l^1} d|q|$$
$$\le C_2 \nu^{-1/2} \Gamma\left(\frac{1}{2n}\right) \alpha^{-1/(2n)} \left(\left[\|\hat{U}\|_1^{\alpha;\phi} \right]^2 + 2\|v_0\|_{l^1} \|\hat{U}\|_1^{\alpha;\phi} \right) + \|\hat{U}^{(0)}\|_1^{\alpha;\phi}.$$

From (5.40), it now follows that

$$\int_0^\infty \|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_{l^1} e^{-\alpha|q|} d|q|$$
$$\le C_2 \nu^{-1/2} \Gamma\left(\frac{1}{2n}\right) \alpha^{-1/(2n)} \left\{ \left(\|\hat{U}^{[1]}\|_1^{\alpha;\phi} + \|\hat{U}^{[2]}\|_1^{\alpha;\phi} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{\alpha;\phi} \right.$$
$$\left. + 2\|\hat{v}_0\|_{l^1} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{\alpha;\phi} \right\}.$$

(ii) We first note that

$$
\begin{aligned}
&|q|^{1/2-1/n}\int_0^{|q|} e^{-\alpha(|q|-s)}(|q|-s)^{-1/2+1/(2n)}s^{-1+1/n}(1+s^2)^{-1}ds \\
&\quad = |q|^{1/(2n)}\int_0^1 e^{-\alpha|q|(1-t)}t^{-1+1/n}(1-t)^{-1/2+1/(2n)}(1+t^2|q|^2)^{-1}dt \\
&\quad = |q|^{1/(2n)}\left\{\int_0^{1/2} e^{-\alpha|q|(1-t)}\frac{t^{-1+1/n}(1-t)^{-1/2+1/(2n)}}{(1+t^2|q|^2)}dt+\right. \\
&\qquad\qquad \left.\int_{1/2}^1 e^{-\alpha|q|(1-t)}\frac{t^{-1+1/n}(1-t)^{-1/2+1/(2n)}}{(1+t^2|q|^2)}dt\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text{(5.51)}\qquad &\le |q|^{1/(2n)}e^{-\alpha|q|/2}\int_0^{1/2} t^{-1+1/n}(1-t)^{-1/2+1/(2n)}dt \\
&\quad + \frac{2^{1-1/n}|q|^{1/(2n)}}{1+|q|^2/4}\int_{1/2}^1 e^{-\alpha|q|(1-t)}(1-t)^{-1/2+1/(2n)}dt.
\end{aligned}
$$

The first term on the right of (5.51) is bounded by $n2^{1/2-3/(2n)}|q|^{1/(2n)}e^{-\alpha|q|/2}$. For the second term we separate two cases. Let first $\alpha|q|\le 1$. It is then clear that

$$
\begin{aligned}
&|q|^{1/(2n)}\int_{1/2}^1 e^{-\alpha|q|(1-t)}(1-t)^{-1/2+1/(2n)}dt \\
&\qquad\qquad \le |q|^{1/(2n)}\int_{1/2}^1 (1-t)^{-1/2+1/(2n)}dt \le \frac{2n}{(n+1)\alpha^{1/(2n)}}.
\end{aligned}
$$

Now, if $\alpha|q|>1$, we have

$$
\begin{aligned}
&|q|^{1/(2n)}\int_{1/2}^1 e^{-\alpha|q|(1-t)}(1-t)^{-1/2+1/(2n)}dt = |q|^{1/(2n)}\int_0^{1/2} e^{-\alpha|q|t}t^{-1/2+1/(2n)}dt \\
&\qquad \le |q|^{1/(2n)}\Gamma\left(\frac12+\frac1{2n}\right)[\alpha|q|]^{-1/2-1/(2n)} \le \alpha^{-1/(2n)}\Gamma\left(\frac12+\frac1{2n}\right).
\end{aligned}
$$

Combining these results we get

$$
|q|^{1/(2n)}\int_{1/2}^1 e^{-\alpha|q|(1-t)}(1-t)^{-1/2+1/(2n)}dt \le \alpha^{-1/(2n)}C_1,
$$

where

$$
C_1 = \max\left\{\Gamma\left(\frac12+\frac1{2n}\right),\frac{2n}{n+1}\right\}.
$$

Therefore,

(5.52)

$$
\begin{aligned}
&\sup_{|q|>0}\left\{|q|^{1-1/n}(1+|q|^2)e^{-\alpha|q|}|q|^{-1/2}\int_0^{|q|} e^{\alpha s}(|q|-s)^{-1/2+1/(2n)}s^{-1+1/n}(1+s^2)^{-1}ds\right\} \\
&\qquad\qquad \le (C_0+2^{3-1/n}C_1)\alpha^{-1/(2n)} \equiv C_3\alpha^{-1/(2n)},
\end{aligned}
$$

where

$$
C_0 = n2^{1/2-1/n}\left[\sup_{\gamma>0}\gamma^{1/(2n)}e^{-\gamma}+4\sup_{\gamma>0}\gamma^{2+1/(2n)}e^{-\gamma}\right].
$$

From (5.39) and the definition of $\|\cdot\|^{(\alpha)}$, it follows that

$$\|\mathcal{N}[\hat{U}]\|^{(\alpha)} \leq C_2 C_3 \nu^{-1/2} \alpha^{-1/(2n)} \left[M_0 \left(\|\hat{U}\|^{(\alpha)} \right)^2 + 2\|\hat{v}_0\|_{l^1} \|\hat{U}\|^{(\alpha)} \right] + \|\hat{U}^{(0)}\|^{(\alpha)}.$$

Inequality (5.50) follows similarly. ∎

Lemma 5.7. *The integral equation (1.6) has a unique solution in:*
(i) the ball of radius $2\|\hat{U}^{(0)}\|_1^{\alpha;\phi}$ in $\mathcal{A}_1^{\alpha;\phi}$, if α is large enough so that

$$C_2 \nu^{-1/2} \Gamma\left(\frac{1}{2n}\right) \alpha^{-1/(2n)} \left(4\|\hat{v}_0\|_{l^1} + 4\|\hat{U}^{(0)}\|_1^{\alpha;\phi} \right) < 1. \tag{5.53}$$

Here C_2 is the same as in Lemma 5.3 and depends on δ and n for $n \geq 2$. For $n = 1$ we have $\phi = 0$.
(ii) the ball of radius $2\|\hat{U}^{(0)}\|^{(\alpha)}$ in $\mathcal{A}^{(\alpha)}$ if α is large enough so that

$$C_2 C_3 \nu^{-1/2} \alpha^{-1/(2n)} \left(4\|\hat{v}_0\|_{l^1} + 4M_0 \|\hat{U}^{(0)}\|^{(\alpha)} \right) < 1, \tag{5.54}$$

where C_2 (defined in Lemma 5.3) and C_3 (defined in (5.52)) depend on δ and n for $n \geq 2$.

Proof. The estimates in Lemma 5.6 imply that $\mathcal{N}$ maps a ball of size $2\|\hat{U}^{(0)}\|_1^{\alpha;\phi}$ in $\mathcal{A}_1^{\alpha;\phi}$ back to itself and that $\mathcal{N}$ is contractive in that ball when α satisfies (5.53). In $\mathcal{A}^{(\alpha)}$, the estimates of Lemma 5.6 imply that $\mathcal{N}$ maps a ball of size $2\|\hat{U}^{(0)}\|^{(\alpha)}$ to itself and that $\mathcal{N}$ is contractive in that ball when α satisfies (5.54). ∎

Remark 5.8. *If α satisfies both (5.53) and (5.54), then it follows from Lemma 4.6 and the uniqueness of classical solution of 1.2 that the solutions $\hat{U}$ in $\mathcal{A}_1^{\alpha;\phi}$ and $\mathcal{A}^{(\alpha)}$ are one and the same.*

Lemma 5.9. *The q-derivatives of $\hat{U}(k,q)$ in $\mathcal{A}^{(\alpha)}$ for $q > 0$ are estimated by:*

$$\left\| \frac{\partial^m}{\partial q^m} \hat{U}(\cdot, q) \right\|_{l^1} \leq C_m \|\hat{v}_1\|_{l^1} \frac{q^{-1+1/n} \omega^{-m}}{1+q^2} e^{\alpha q + \omega \alpha}, \tag{5.55}$$

where $\omega = q/2$ for $q \leq 2$, $\omega = 1$ for $q > 2$.

Proof. For $q \leq 2$, we use Cauchy's integral formula on a circle of radius $q/2$ around q and Lemma 4.5 to bound $\hat{U}$ for $|q| > 0$, $\arg q \in \left[-(n-1)\frac{\pi}{2} + \delta, (n-1)\frac{\pi}{2} - \delta\right]$ (we may pick for instance $\delta = \frac{\pi}{4}$ to obtain specific values of constants here). For $q > 2$, the argument is similar, now on a circle of radius 1. ∎

In the following we need bounds on $\|k\hat{U}\|^{(\alpha)}$. We rewrite (1.6) using the divergence-free condition (note that $k\hat{U}$ is a tensor of rank 2) as

(5.56)

$$k\hat{U}(k,q) = -ik \int_0^q \mathcal{G}(q,q';k) P_k \left\{ \hat{U}_j \hat{*} [k_j \hat{U}] + \hat{v}_{0,j} \hat{*} [k_j \hat{U}] \right\} (k,q') dq' + \hat{U}^{(0,1)}(k,p)$$
$$:= \tilde{\mathcal{N}}\left[k\hat{U}\right]$$

$$\text{where } \hat{U}^{(0,1)}(k,p) := -ik \int_0^q \mathcal{G}(q,q';k) P_k \left[\hat{U}_j \hat{*} [k_j \hat{v}_0] \right] (k,q') dq' + k\hat{U}^{(0)}(k,j).$$

We now think of $\hat{U}$ in (5.56) as known; then $\tilde{\mathcal{N}}$ becomes linear in $k\hat{U}$.

Lemma 5.10. *If $|k|^3\hat{v}_0 \in l^1$ and α is large enough so that (5.54) is satisfied, then*

$$\||k|\hat{U}\|^{(\alpha)} \leq 4c_1 \left(\nu \||k|^3\hat{v}_0\|_{l^1} + 2\||k|\hat{v}_0\|_{l^1}^2 + \||k|\hat{f}\|_{l^1}\right) + \||k|\hat{v}_0\|_{l^1}.$$

Proof. From (5.56), we obtain

$$\begin{aligned}&\||k|\hat{U}\|^{(\alpha)} = \|\tilde{\mathcal{N}}[|k|\hat{U}]\|^{(\alpha)} \\ &\leq C_2 C_3 \nu^{-1/2}\alpha^{-1/(2n)}\left\{M_0\|\hat{U}\|^{(\alpha)}\||k|\hat{U}\|^{(\alpha)} + \|\hat{v}_0\|_{l^1}\||k|\hat{U}\|^{(\alpha)}\right\} + \|\hat{U}^{(0,1)}\|^{(\alpha)}.\end{aligned}$$

Lemma 5.7, which applies when α satisfies (5.54), implies that $\|\hat{U}\|^{(\alpha)} \leq 2\|\hat{U}^{(0)}\|^{(\alpha)}$ and thus

$$\begin{aligned}\||k|\hat{U}\|^{(\alpha)} &\leq C_2 C_3 \nu^{-1/2}\alpha^{-1/(2n)}\||k|\hat{U}\|^{(\alpha)}\left\{2M_0\|\hat{U}^{(0)}\|^{(\alpha)} + \|\hat{v}_0\|_{l^1}\right\} + \|\hat{U}^{(0,1)}\|^{(\alpha)} \\ &\leq \frac{1}{2}\||k|\hat{U}\|^{(\alpha)} + \|\hat{U}^{(0,1)}\|^{(\alpha)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\||k|\hat{U}\|^{(\alpha)} \\ &\leq 2\|\hat{U}^{(0,1)}\|^{(\alpha)} \leq 2\||k|\hat{U}^{(0)}\|^{(\alpha)} + 4M_0 C_2 C_3 \nu^{-1/2}\alpha^{-1/(2n)}\||k|\hat{v}_0\|_{l^1}\|\hat{U}^{(0)}\|^{(\alpha)}.\end{aligned}$$

Lemma follows from (5.54) and bounds on $\hat{U}^{(0)}$ given in Lemma 4.5. ∎

Proposition 5.11. *Assume $\hat{f}(0) = 0 = \hat{v}_0(0)$ and we define $\left\|\frac{\hat{f}}{|k|^2}\right\|_{l_1} = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|\hat{f}|^2}{|k|^2}$. If for $n \geq 2$, α satisfies the condition:*

(5.57)

$$C_2\nu^{-1/2}\Gamma\left(\frac{1}{2n}\right)\alpha^{-1/(2n)}\left\{4\|\hat{v}_0\|_{l^1} + 4C_G\left[\|\hat{v}_0\|_{l^1}\left(1 + \frac{2}{\nu}\|\hat{v}_0\|_{l^1}\right) + \frac{1}{\nu}\left\|\frac{\hat{f}}{|k|^2}\right\|_{l^1}\right]\right\} < 1,$$

with constants C_2 and C_G defined in Lemmas 5.3 and 4.5, then the integral equation (1.6) has a unique solution in a ball of size $2\|\hat{U}^{(0)}\|_1^{\alpha;\phi}$ in $\mathcal{A}_1^{\alpha;\phi}$. If in addition $|k|^2\hat{v}_0 \in l^1$, then for $n \geq 1$ and $\alpha = \alpha_1$ is such that

$$C_2\nu^{-1/2}\Gamma\left(\frac{1}{2n}\right)\alpha_1^{-1/(2n)}\left\{4\|\hat{v}_0\|_{l^1} + 4c_1\Gamma\left(\frac{1}{n}\right)\alpha_1^{-1/n}\|\hat{v}_1\|_{l^1}\right\} < 1, \tag{5.58}$$

where

$$\hat{v}_1(k) = \left(-\nu|k|^2\hat{v}_0 - ik_j P_k\left[\hat{v}_{0,j}\hat{*}\hat{v}_0\right]\right) + \hat{f}(k) \tag{5.59}$$

with c_1 defined in Lemma 4.5, then the integral equation (1.6) has a unique solution in a ball of size $2\|\hat{U}^{(0)}\|_1^{\alpha_1;\phi}$ in $\mathcal{A}_1^{\alpha_1;\phi}$.

Proof. The proof follows from Lemma 4.5 since (5.57) and (5.58) imply (5.53), and thus Lemma 5.7 applies. ∎

Proof of Theorem 2.1

Proposition 5.11 gives a unique solution to (1.6) in some small ball in the Banach space $\mathcal{A}_1^{\alpha;\phi}$ for sufficiently large α. From Lemma 4.6, we see that $\hat{U}$ generates via (1.5) a solution $\hat{v}$ to (1.2) for $t \in \left[0, \alpha^{-1/n}\right)$. Classical arguments (presented for completeness in Lemma 12.1 in the Appendix), show that $|k|^2\hat{v}(\cdot,t) \in l^1$ and hence $\mathcal{F}^{-1}\left[\hat{v}(\cdot,t)\right](x)$ is a smooth solution to (1.1) for $t \in \left(0, \alpha^{-1/n}\right)$. Analyticity in t for

$\mathrm{Re}\frac{1}{t^n} > \alpha$ follows from the Laplace representation. For optimal analyticity region in t, we choose $n = 1$.

It is well known that (1.1) has locally a unique classical solution [34], [15], [9]. Thus, given $\hat{v}_0, \hat{f} \in l^1$, all solutions obtained via the integral equation coincide. Furthermore, $\hat{v}(k,t) - \hat{v}_0$ is inverse-Laplace transformable in $1/t^n$ and the inverse Laplace transform satisfies (1.6). Therefore, no restriction on the size of ball in spaces $\mathcal{A}_1^{\alpha;\phi}$, $\mathcal{A}^{(\alpha)}$ is necessary for uniqueness of the solution of (1.6).

Remark 5.12. The arguments in the proof of Theorem 2.1 show that $\|\hat{v}(\cdot,t)\|_{l^1} < \infty$ over an interval of time implies that the solution is classical. This is not a new result. Standard Fourier arguments show that, in this case, we have $\|v(\cdot,t)\|_{L^\infty} < \infty$, *i.e.* one of the Prodi-Serrin criteria for existence of classical solutions [29], [30] is satisfied.

6. Error bounds in a Galerkin approximation involving $[-N,N]^3$ Fourier modes

Definition 6.1. *We define the operator $\mathcal{N}^{(N)}$ (associated to $\mathcal{N}$) by*

$$\mathcal{N}^{(N)}\left[\hat{U}\right](k,q) \tag{6.60}$$
$$= -ik_j \int_0^q \mathcal{G}(q,q';k)\mathcal{P}_N P_k \left[\hat{U}_j \hat{*} \hat{U} + \hat{v}_{0,j} \hat{*} \hat{U} + \hat{U}_j \hat{*} \hat{v}_0\right](k,q')dq' + \mathcal{P}_N \hat{U}^{(0)}(k,q),$$

where $\mathcal{P}_N$, the Galerkin projection to $[-N,N]^3$ Fourier modes, is given by

$$\left[\mathcal{P}_N \hat{U}\right](k,q) = \hat{U}(k,q) \text{ for } k \in [-N,N]^3 \ , \ \left[\mathcal{P}_N \hat{U}\right](k,q) = \ 0 \text{ otherwise.}$$

Lemma 6.2. *The integral equation*

$$\hat{U}^{(N)} = \mathcal{N}^{(N)}\left[\hat{U}^{(N)}\right]$$

has a unique solution in $\mathcal{A}_1^{\alpha}$(9) *as well as in $\mathcal{A}^{(\alpha)}$, if α satisfies the conditions in Theorem 2.1.*

Proof. The proof is very similar to that of Theorem 2.1 part 1, noting that the Galerkin projection $\mathcal{P}_N$ does not increase l^1 norms and $\mathcal{N}^{(N)}$ and $\mathcal{N}$ have similar properties. ∎

Lemma 6.3. *Assume that α is large enough so that*

$$C_2 C_3 \nu^{-1/2} \alpha^{-1/(2n)} \left(4\|\hat{v}_0\|_{l^1} + 4M_0 \|\hat{U}^{(0)}\|^{(\alpha)}\right) \le \frac{1}{2}, \tag{6.61}$$

and that $|k|^3 \hat{v}_0,\ |k|\hat{f} \in l^1$. Define the Galerkin truncation error:

$$T_{E,N} = \mathcal{P}_N \hat{U} - \mathcal{N}^{(N)}\left[\mathcal{P}_N \hat{U}\right] = \mathcal{P}_N \mathcal{N}\left[\hat{U}\right] - \mathcal{N}^{(N)}\left[\mathcal{P}_N \hat{U}\right] \tag{6.62}$$
$$= -ik_j \int_0^q \mathcal{G}(q,q';k)\mathcal{P}_N P_k \Big[\hat{v}_{0,j} \hat{*} (I-\mathcal{P}_N)\hat{U} + (I-\mathcal{P}_N)\hat{U}_j \hat{*} \hat{v}_0$$
$$+ (I-\mathcal{P}_N)\hat{U}_j \hat{*} \mathcal{P}_N \hat{U} + \mathcal{P}_N \hat{U}_j \hat{*} (I-\mathcal{P}_N)\hat{U} + (I-\mathcal{P}_N)\hat{U}_j \hat{*} (I-\mathcal{P}_N)\hat{U}\Big](k,q')dq'.$$

(9)Recall this means $\mathcal{A}_1^{\alpha;\phi}$ with $\phi = 0$

Then,

$$\|\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} \le \|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)} + 2\|T_{E,N}\|^{(\alpha)},$$

where

$$\begin{aligned}
&\|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)} + 2\|T_{E,N}\|^{(\alpha)} \\
&\qquad \le \frac{1}{N}\left[2c_1\left(\nu\||k|^3\hat{v}_0\|_{l^1} + 2\||k|\hat{v}_0\|_{l^1}^2 + \||k|\hat{f}\|_{l^1}\right) + \||k|\hat{v}_0\|_{l^1}\right] \\
&\qquad\quad \times\left\{1 + 4c\|\hat{v}_0\|_{l^1} + 12c\left(\nu\||k|^2\hat{v}_0\|_{l^1} + 2\||k|\hat{v}_0\|_{l^1}\|\hat{v}_0\|_{l^1} + \|\hat{f}\|_{l^1}\right)\right\}.
\end{aligned}$$

Proof. Clearly,

$$\|\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} \le \|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)} + \|\mathcal{P}_N\hat{U} - \hat{U}^{(N)}\|^{(\alpha)}.$$

By (6.61), (6.62) and contractivity of $\mathcal{N}^{(N)}$,

$$\begin{aligned}
\|\mathcal{P}_N\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} &\le \|\mathcal{N}^{(N)}[\mathcal{P}_N\hat{U}] - \mathcal{N}^{(N)}[\hat{U}^{(N)}]\|^{(\alpha)} + \|T_{E,N}\|^{(\alpha)} \\
&\le \frac{1}{2}\|\mathcal{P}_N\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} + \|T_{E,N}\|^{(\alpha)},
\end{aligned}$$

so

$$\|\hat{U} - \hat{U}^{(N)}\|^{(\alpha)} \le \|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)} + 2\|T_{E,N}\|^{(\alpha)}.$$

Now estimates similar to (5.49) imply that

$$\begin{aligned}
\|T_{E,N}\|^{(\alpha)} &\le c\|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)}\left[2\|\hat{v}_0\|_{l^1} + 2\|\mathcal{P}_N\hat{U}\|^{(\alpha)} + \|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)}\right] \\
&\le c\|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)}\left[2\|\hat{v}_0\|_{l^1} + 6\|\hat{U}^{(0)}\|^{(\alpha)}\right],
\end{aligned}$$

and Lemma 5.10 implies that

$$\begin{aligned}
\|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha)} &\le \frac{1}{N}\|k\hat{U}\|^{(\alpha)} \\
&\le \frac{1}{N}\left[2c_1\left(\nu\||k|^3\hat{v}_0\|_{l^1} + 2\||k|\hat{v}_0\|_{l^1}^2 + \||k|\hat{f}\|_{l^1}\right) + \||k|\hat{v}_0\|_{l^1}\right].
\end{aligned}$$

Hence the lemma follows. ∎

7. The exponential rate α and the singularities of v

We have already established that at most subexponential growth of $\|\hat{U}(\cdot, q)\|_{l^1}$ implies global existence of a classical solution to (1.1).

We now look for a converse: suppose (1.1) has a global solution, is it true that $\hat{U}(\cdot, q)$ always is subexponential in q? The answer is no. For $n = 1$, any complex singularity t_s in the right-half complex t-plane of $v(x, t)$ produces exponential growth of $\hat{U}$ with rate $\mathrm{Re}(1/t_s)$ (oscillatory with a frequency $\mathrm{Im}(1/t_s)$).

However, if $f = 0$, we will see that for any *given global classical* solution of (1.1), there is a $c > 0$ so that for any t_s we have $|\arg t_s| > c$. This means that for sufficiently large n, the function $v(x, \tau^{-1/n})$ has no singularity in the right-half τ plane. Then the inverse Laplace transform

$$U(x, q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{v(x, \tau^{-1/n}) - v_0(x)\right\}e^{q\tau}d\tau$$

can be shown to decay for q near $\mathbb{R}^+$.

We now seek to find conditions for which there are no singularities of $v(x,\tau^{-1/n})$ in $\{\tau : \mathrm{Re}\,\tau \geq 0,\ \tau \notin \mathbb{R}^+ \cup \{0\}\}$.

Lemma 7.1. *(Special case of* [17]*) If $f = 0$ and $v(\cdot,t_0) \in H^1\left(\mathbb{T}^3\left[0,2\pi\right]\right)$, then $v(x,t)$ is analytic in x and t in the domain $|\mathrm{Im}\ x_j| < c\nu|t-t_0|$, $0 < |t-t_0| < C$ for $\arg(t-t_0) \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right]$, where c and C are positive constants (C depends on $\|v_0\|_{H^1}$ and ν and bounded away from 0 when $\|v_0\|_{H^1}$ is bounded).*

See page 71 of [16].

Lemma 7.2. *(i) Assume $k\hat{v}_0, \hat{f} \in l^1$ and α is large enough so that (5.53) holds for $n = 1$. The classical solution of (1.1) has no singularity in $\mathrm{Re}\frac{1}{t} > \alpha$, $x \in \mathbb{T}^3$.*

(ii) Furthermore, for $f = 0$ (no forcing), no singularity can exist for $\arg(t - T_{c,a}) \in (-\tilde{\delta},\tilde{\delta})$ for any $0 < \tilde{\delta} < \frac{\pi}{2}$ and any $x \in \mathbb{T}^3$. ($T_{c,a}$ is estimated in terms of $\|v_0\|_{H^1}$, ν, and $\tilde{\delta}$ in Theorem 12.1 in the Appendix using standard arguments.)

Proof. (i) The assumption implies $v_0 \in H^1(\mathbb{T}^3)$. Since it is well known (see for instance [16], [34], [9], [15]) that a classical solution to (1.1) is unique, it follows that this solution equals the one given in Theorem 2.1 in the form (1.4). From standard properties of Laplace transforms this solution is analytic for $\mathrm{Re}\frac{1}{t} > \alpha$, where α is given in Theorem 2.1.

(ii) We know that under these assumptions $\|v(\cdot,t)\|_{H^1} \to 0$ as $t \to \infty$. There is then a critical time $T_{c,a}$ so that standard contraction mapping arguments show that $v(\cdot,t)$ is analytic for $t - T_{c,a} \in \tilde{S}_{\tilde{\delta}}$ as seen in Theorem 12.1 in the Appendix. ∎

Corollary 7.3. *If $f = 0$, for any v_0 there exists a $c > 0$ so that any singularity t_s of the solution v of (1.1) is either a positive real time singularity, or else $|\arg t_s| > c$.*

Proof. If there exists a classical solution on $\mathbb{R}^+$ then $\|v(\cdot,t)\|_{H^1}$ is uniformly bounded and by the proof of Lemma 7.2 (ii) there is a $T_{c,a}$ (as given in Theorem 12.1 in the Appendix) such that $v(\cdot,t)$ is analytic for $\arg(t-T_{c,a}) \in \left[-\frac{\pi}{4},\frac{\pi}{4}\right]$. Let now $M_1 = \max_{t\in[0,T_{c,a}+\epsilon]} \|v(\cdot,t)\|_{H^1}$. Then by Lemma 7.1, for any $t' \in [0, T_{c,a}+\epsilon]$ there exists a $c_2 = c_2(M_1)$ such that v is analytic in the region $|t-t'| < c_2,\ |\arg(t-t')| \leq \pi/4$. Thus v is analytic in (see Fig.1)

$$\left\{t : |t-t'| < c_2, |\arg(t-t')| \leq \frac{\pi}{4}, 0 \leq t' \leq T_{c,a}+\epsilon\right\} \bigcup \left\{t : |\arg(t-T_{c,a})| \leq \frac{\pi}{4}\right\}.$$

Thus, if t_s is a singular point of v, then $\tan|\arg(t_s)| > c$ where

$$c = \frac{c_2/\sqrt{2}}{T_{c,a}+c_2/\sqrt{2}} = \frac{c_2}{\sqrt{2}\,T_{c,a}+c_2}.$$

∎

Proof of Theorem 2.2

By definition,

$$\hat{U}(k,q) = \frac{1}{2\pi i}\int_C \hat{u}(k,\tau^{-1/n})e^{q\tau}\,d\tau = \frac{1}{2\pi i}\int_C \left[\hat{v}(k,\tau^{-1/n}) - \hat{v}_0(k)\right]e^{q\tau}\,d\tau,$$

where the Bromwich contour C lies to the right of all singularities of $\hat{u}(k,\tau^{-1/n})$ in the complex τ-plane. By Corollary 7.3, $\hat{u}(k,t)$ has no singularities in the sector

$$S_{t,\phi} := \left\{t : |\arg t\,| \leq \phi := \tan^{-1} c\right\},$$

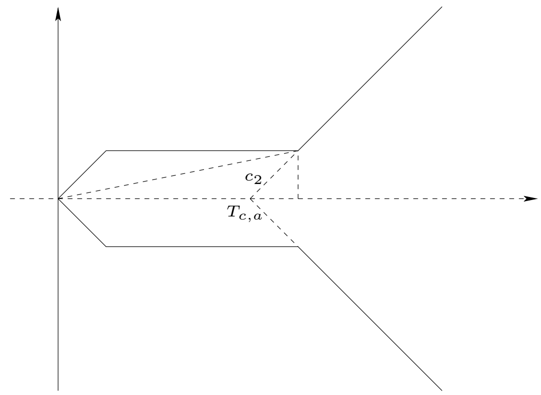

FIGURE 1. The region of analyticity of v.

so $\hat{u}(k, \tau^{-1/n})$ has no singularities in the sector

$$S_{\tau,\phi} := \Big\{\tau : |\arg \tau| < n\phi\Big\}.$$

Clearly, if $n\phi \in (\frac{\pi}{2}, \pi)$, then $\hat{u}(k, \tau^{-1/n})$ is analytic in a sector of width between π and 2π, and in particular the Bromwich contour C can be chosen to be the imaginary axis. With the suitable decay of $\hat{u}(k, \tau^{-1/n})$ at $\tau = \infty$:

$$\hat{u}(k,t) = \hat{v}(k,t) - \hat{v}_0(k) = O(t) \qquad \text{as } t \to 0, \qquad \text{which means that}$$
$$\hat{u}(k, \tau^{-1/n}) = O(\tau^{-1/n}) \qquad \text{as } \tau \to \infty,$$

Jordan's lemma applies and C can be deformed to the edges of the sector $S_{\tau,\phi}$, i.e.

$$\hat{U}(k,q) = \frac{1}{2\pi i}\Big\{\int_{\infty e^{-in\phi}}^{0} + \int_0^{\infty e^{in\phi}}\Big\}\Big[\hat{v}(k,\tau^{-1/n}) - \hat{v}_0(k)\Big] e^{q\tau}\, d\tau$$
$$= \frac{1}{2\pi i}\Big\{\int_{\infty e^{-in\phi}}^{0} + \int_0^{\infty e^{in\phi}}\Big\}\hat{v}(k,\tau^{-1/n}) e^{q\tau}\, d\tau$$

(carefully note that the integral of $\hat{v}_0(k)$ over the contour is 0). Further, as shown in Theorem 12.1 in the Appendix, there is a sector $\tilde{S}_{\tilde{\delta}}$ in the right-half t-plane (with $\phi < \tilde{\delta} < \frac{\pi}{2}$) so that

$$\|\hat{v}(\cdot,t)\|_{l^1} \le Ce^{-\frac{3}{4}\nu \operatorname{Re} t} \qquad \text{as } t \to \infty \text{ in } \tilde{S}_{\tilde{\delta}}.$$

So

$$\|\hat{v}(\cdot, \tau^{-1/n})\|_{l^1} \le Ce^{-\frac{3}{4}\nu \operatorname{Re}(\tau^{-1/n})} \qquad \text{as } \tau \to 0 \text{ along } e^{\pm in\phi}(0,\infty),$$

and the boundedness of $\|\hat{v}(\cdot, \tau^{-1/n})\|_{l^1}$ for large $|\tau|$ implies that

$$\|\hat{v}(\cdot, \tau^{-1/n})\|_{l^1} \le Ce^{-\frac{3}{4}\nu \operatorname{Re}(\tau^{-1/n})} \qquad \text{for all } \tau \in e^{\pm in\phi}(0,\infty).$$

It follows that

$$\|\hat{U}(\cdot,q)\|_{l^1} \le C\int_0^\infty e^{-\frac{3}{4}\nu \operatorname{Re}(\tau^{-1/n}) + q\operatorname{Re}\tau}\, d|\tau| \le C\int_0^\infty e^{-\frac{3}{4}\nu r^{-1/n}\cos\phi + qr\cos n\phi}\, dr,$$

and a standard application of the Laplace method (with the change of variable $r = q^{-n/(n+1)}s$) shows that

$$\|\hat{U}(\cdot,q)\|_{l^1} \le C_1 e^{-C_2 q^{1/(n+1)}} \qquad \text{as } q \to +\infty.$$

8. Estimates of α based the solution of (1.6) in $[0, q_0]$

Define $\hat{U}^{(a)}$ as in (3.9) and $\hat{U}^{(b)} = \hat{U} - \hat{U}^{(a)}$. Using (1.6), it is convenient to write an integral equation for $\hat{U}^{(b)}$ for $q > q_0$:

$$\hat{U}^{(b)}(k,q) = -ik_j \int_{q_0}^{q} \mathcal{G}(q,q';k)\hat{H}_j^{(b)}(k,q')\,dq' + \hat{U}^{(s)}(k,q), \tag{8.63}$$

where

$$\hat{U}^{(s)}(k,q) = -ik_j \int_0^{\min\{q,2q_0\}} \mathcal{G}(q,q';k)\hat{H}_j^{(a)}(k,q')\,dq' + \hat{U}^{(0)}(k,q), \tag{8.64}$$

and

$$\hat{H}_j^{(a)}(k,q) = P_k\Big[\hat{v}_{0,j}\hat{*}\hat{U}^{(a)} + \hat{U}_j^{(a)}\hat{*}\hat{v}_0 + \hat{U}_j^{(a)}{\ast\atop\ast}\hat{U}^{(a)}\Big](k,q), \tag{8.65}$$

(8.66)

$$\hat{H}_j^{(b)}(k,q) = P_k\Big[\hat{v}_{0,j}\hat{*}\hat{U}^{(b)} + \hat{U}_j^{(b)}\hat{*}\hat{v}_0 + \hat{U}_j^{(a)}{\ast\atop\ast}\hat{U}^{(b)} + \hat{U}_j^{(b)}{\ast\atop\ast}\hat{U}^{(a)} + \hat{U}_j^{(b)}{\ast\atop\ast}\hat{U}^{(b)}\Big](k,q).$$

Also, we define $\hat{R}^{(b)}(k,q) = -ik_j\hat{H}_j^{(b)}(k,q)$. It is to be noted that the support of $\hat{H}^{(a)}$ is $[0,2q_0]$. Thus, if $\hat{U}^{(a)}$ is known (computationally or otherwise), then $\hat{H}^{(a)}$ and therefore $\hat{U}^{(s)}$ are known for all q.

Proof of Theorem 3.1: Note that

$$|\hat{R}^{(b)}(k,q)| \le 2|k|\Big[2|\hat{v}_0|\hat{*}|\hat{U}^{(b)}| + 2|\hat{U}^{(a)}|{\ast\atop\ast}|\hat{U}^{(b)}| + |\hat{U}^{(b)}|{\ast\atop\ast}|\hat{U}^{(b)}|\Big](k,q),$$

where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^3$. By Lemma 4.4 we can define a best constant

$$B_0(k) = \sup_{q_0 \le q' \le q}\Big\{(q-q')^{1/2-1/(2n)}|\mathcal{G}(q,q';k)|\Big\} \tag{8.67}$$

and conclude that

$$|\mathcal{G}(q,q';k)\hat{R}^{(b)}(k,q')| \le 2|k|B_0(k)(q-q')^{1/(2n)-1/2}\Big[2|\hat{v}_0|\hat{*}|\hat{U}^{(b)}| + 2|\hat{U}^{(a)}|{\ast\atop\ast}|\hat{U}^{(b)}| + |\hat{U}^{(b)}|{\ast\atop\ast}|\hat{U}^{(b)}|\Big](k,q').$$

It follows from Lemma 5.1 that

$$\|\mathcal{G}(q,q';\cdot)\hat{R}^{(b)}(\cdot,q')\|_{l^1} \le \psi(q-q')\Big[B_1 u + B_2 * u + B_3 u * u\Big](q'),$$

where $\psi(q) = q^{1/(2n)-1/2}$ and

$$u(q) = \|\hat{U}^{(b)}(\cdot,q)\|_{l^1}, \qquad B_1 = 4\sup_{k\in\mathbb{Z}^3}\big\{|k|B_0(k)\big\}\|\hat{v}_0\|_{l^1},$$
$$B_2(q) = 4\sup_{k\in\mathbb{Z}^3}\big\{|k|B_0(k)\big\}\|\hat{U}^{(a)}(\cdot,q)\|_{l^1}, \qquad B_3 = 2\sup_{k\in\mathbb{Z}^3}\big\{|k|B_0(k)\big\}.$$

Taking the l^1-norm in k on both sides of (8.63), multiplying the equation by $e^{-\alpha q}$ for some $\alpha \geq \alpha_0 \geq 0$ and integrating over the interval $[q_0, M]$, we obtain

$$L_{q_0,M} \leq \int_{q_0}^{M} e^{-\alpha q} \int_{q_0}^{q} \psi(q-q') \Big[B_1 u + B_2 * u + B_3 u * u\Big](q')\, dq'\, dq + \int_{q_0}^{M} e^{-\alpha q} u^{(s)}(q)\, dq$$

$$\leq \int_{q_0}^{M} \Big[B_1 u + B_2 * u + B_3 u * u\Big](q') \int_{q'}^{M} e^{-\alpha q} \psi(q-q')\, dq\, dq' + \int_{q_0}^{M} e^{-\alpha q} u^{(s)}(q)\, dq$$

$$\leq \int_{0}^{\infty} e^{-\alpha q} \psi(q)\, dq \int_{q_0}^{M} e^{-\alpha q'} \Big[B_1 u + B_2 * u + B_3 u * u\Big](q')\, dq' + \int_{q_0}^{M} e^{-\alpha q} u^{(s)}(q)\, dq,$$

where

$$L_{q_0,M} := \int_{q_0}^{M} e^{-\alpha q} u(q)\, dq, \qquad u^{(s)}(q) = \|\hat{U}^{(s)}(\cdot, q)\|_{l^1}. \tag{8.68}$$

If we use the fact that

$$\int_{q_0}^{M} e^{-\alpha q'} u * v(q')\, dq' = \int_{q_0}^{M} e^{-\alpha q'} \int_{q_0}^{q'} u(s) v(q'-s)\, ds\, dq'$$
$$= \int_{q_0}^{M} u(s) \int_{s}^{M} e^{-\alpha q'} v(q'-s)\, dq'\, ds$$

for any function v on $[0, M]$ (recall that $u = 0$ on $[0, q_0]$), then

$$L_{q_0,M} \leq \int_{0}^{\infty} e^{-\alpha q} \psi(q)\, dq \left\{ \left[B_1 + \int_{0}^{q_0} e^{-\alpha q'} B_2(q')\, dq' \right] L_{q_0,M} + B_3 L_{q_0,M}^2 \right\}$$
$$+ b\alpha^{-1/2-1/(2n)} \leq \alpha^{-1/2-1/(2n)} \Big[\epsilon_1 L_{q_0,M} + \epsilon L_{q_0,M}^2 \Big] + b\alpha^{-1/2-1/(2n)}, \tag{8.69}$$

where

$$b = \alpha^{1/2+1/(2n)} \int_{q_0}^{\infty} e^{-\alpha q} u^{(s)}(q)\, dq, \tag{8.70}$$

$$\epsilon_1 = \Gamma\left(\frac{1}{2} + \frac{1}{2n}\right) \left[B_1 + \int_{0}^{q_0} e^{-\alpha_0 q'} B_2(q')\, dq' \right], \qquad \epsilon = \Gamma\left(\frac{1}{2} + \frac{1}{2n}\right) B_3. \tag{8.71}$$

For

$$\epsilon_1 < \alpha^{1/2+1/(2n)} \qquad \text{and} \qquad \left(\epsilon_1 - \alpha^{1/2+1/(2n)}\right)^2 > 4\epsilon b,$$

this leads to an estimate for $L_{q_0,M}$ independent of M:

$$L_{q_0,M} \leq \frac{1}{2\epsilon} \left[\alpha^{1/2+1/(2n)} - \epsilon_1 - \sqrt{\left(\epsilon_1 - \alpha^{1/2+1/(2n)}\right)^2 - 4\epsilon b} \right]. \tag{8.72}$$

So $\|\hat{U}(\cdot, q)\|_{l^1} \in L^1(e^{-\alpha q}\, dq)$ and the solution to (1.1) exists for $t \in (0, \alpha^{-1/n})$, if α is sufficiently large so that

$$\alpha \geq \alpha_0, \qquad \alpha^{1/2+1/(2n)} > \epsilon_1 + 2\sqrt{\epsilon b}.$$

Alternatively, we may choose $\alpha_0 = \alpha$, in which case α has to be large enough to satisfy:

$$\alpha^{1/2+1/(2n)} > \epsilon_1 + 2\sqrt{\epsilon b}.$$

This completes the proof of Theorem 3.1

8.1. **Further estimates on ϵ_1, b and ϵ.** By Lemma 4.4,

$$c_g = \sup_{\substack{k\in\mathbb{Z}^3\\ q_0\le q'\le q}} \left\{ |k|\, q^{1/2}(q-q')^{1/2-1/(2n)} |\mathcal{G}(q,q';k)| \right\} < \infty, \tag{8.73}$$

and by (8.68), (8.64), Lemma 4.4, Lemma 4.5 and the compact support of $\hat{H}^{(a)}$,

$$c_s = \sup_{q_0\le q} \left\{ q^{1/2-1/(2n)} u^{(s)}(q) \right\} < \infty. \tag{8.74}$$

It follows that

$$b \le c_s \Gamma\left(\frac{1}{2}+\frac{1}{2n}, \alpha_0 q_0\right), \qquad \epsilon \le 2\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right) c_g q_0^{-1/2}, \tag{8.75}$$

where

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\, dt$$

is the incomplete Gamma function, and condition (3.13) is satisfied if

$$\alpha > \alpha_0, \qquad \alpha^{1/2+1/(2n)} > \epsilon_1 + 2\left[2\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2n}, \alpha_0 q_0\right) c_g c_s \right]^{1/2} q_0^{-1/4}. \tag{8.76}$$

If on a large subinterval $[q_d, q_0]$, $\hat{U}^{(a)}(\cdot, q)$, and therefore $\hat{H}^{(a)}$, decays, cf. the exponential decay in Theorem 2.2, then the estimated c_s is small. Also, ϵ_1 in (8.71) is small for large q_0, ultimately since $B_0(k)$ in (8.67) is small. It is then clear that α in (8.76) can be chosen small as well.

9. Control of numerical errors in $[0, q_0]$ in a discretized scheme

The errors in a numerical discretization scheme for 3-D Navier-Stokes cannot be readily controlled since these depend on derivatives of the classical solution; and these are not known to exist beyond some initial time interval. In contrast to physical space approaches, the q derivatives of the solution $\hat{U}$ to (1.6), are *a priori* bounded on any interval $[q_m, q_0] \subset \mathbb{R}^+$ for $q_m > 0$, by Lemma 5.9. Further, if q_m is chosen appropriately small, then the small t expansion of NS solution provides an accurate representation for $\hat{U}$ and therefore of $\hat{H}_j$ in $[0, q_m]$ to any desired accuracy. Calculating the numerical solution to (1.6) with rigorous error control is relevant in more than one way.

In §8, we have shown that control of $\hat{U}$ on a finite q-interval provides sharper estimates on the exponent α, and therefore an improved classical existence time estimate for v. If this estimate exceeds T_c, the time beyond which Leray's weak solution becomes classical again (see the Appendix for a bound on T_c) then, of course, global existence of v follows.

Furthermore, a numerical scheme to calculate (1.6), which is analyzed in this section is interesting in its own right. It provides, through Laplace transform, an alternative calculation method for Navier-Stokes dynamics. Evidently, this method is not numerically efficient to determine $v(x,t)$ for fixed time t; nonetheless it may be advantageous in finding long time averages involving v and ∇v needed for turbulent flow. These can sometimes be expressed as functionals of $\hat{U}$.

Definition 9.1. *We introduce a discrete operator* $\mathcal{N}_\delta^{(N)}$ *by*

$$\left\{\mathcal{N}_\delta^{(N)}[\hat{V}]\right\}(k, m\delta) = -ik_j \sum_{m'=m_s}^{m-1} w^{(1)}(m, m'; k, \delta)\mathcal{P}_N \hat{H}_{j,\delta}^{(N)}(k, m'\delta) + \hat{U}^{(0,N)}(k, m\delta) - ik_j w^{(1,1)}(m, k, \delta)\mathcal{P}_N \hat{H}_{j,\delta}^{(N)}(k, m\delta), \tag{9.77}$$

where $k \in [-N, N]^3 \setminus \{0\}$, $\mathbb{N} \ni m \geq m_s$, $q_m = m_s\delta$ *(*q_m *is independent of* δ*) and*

$$\hat{U}^{(0,N)}(k, m\delta) = \hat{U}^{(0)}(k, m\delta) - ik_j \int_0^{q_m} \mathcal{G}(m\delta, q'; k)\mathcal{P}_N \hat{H}_j^{(N)}(k, q')dq' \tag{9.78}$$

is considered known, while for $m' \geq m_s$,

$$\begin{aligned}\hat{H}_{j,\delta}^{(N)}(k, m'\delta) = &\sum_{k' \in [-N,N]^3 \setminus \{0,k\}} P_k\left[\hat{v}_{0,j}(k')\hat{V}(k - k', m'\delta) + \hat{v}_0(k')\hat{V}_j(k - k', m'\delta)\right] \\ &+ \sum_{\substack{k' \in [-N,N]^3 \setminus \{0,k\} \\ m''=m_s..,m'-m_s}} P_k\left[\hat{V}_j(k', m''\delta)\hat{V}(k - k', (m' - m'')\delta)\right] w^{(2)}(m', m''; k, \delta) \\ &+ 2\sum_{l=0}^{m_s-1} w^{(2,l)}(m', k, \delta)P_k\left[\hat{E}^{(l)}(k)\hat{*}\hat{V}(k, (m' - l)\delta)\right].\end{aligned} \tag{9.79}$$

In (9.79), $\hat{E}^{(l)}(k)$ involves $\hat{v}_0(k)$–this representation encapsulates the singular contribution of $\hat{U}(\cdot, q')$ and $\hat{U}(\cdot, q - q')$ when q' and $q - q'$ are small respectively. The precise form of these functions and of the weights $w^{(1)}(m, m'; k, \delta)$, $w^{(1,1)}(m, k, \delta)$, $w^{(2)}(m', m''; k, \delta)$ and $w^{(2,l)}(m', k, \delta)$ generally depend on the particular discretization scheme employed to calculate $\mathcal{N}_\delta^{(N)}[\hat{U}]$. Also, note that in (9.79), the nonlinear terms in the summation are absent when $m_s \leq m' < 2m_s$. To simplify the discussion, we do not specify the weights, but only require that they ensure consistency, namely that in the formal limit $\delta \to 0$, the discrete operator $\mathcal{N}_\delta^{(N)}$ becomes $\mathcal{N}^{(N)}$. Based on behavior of the kernel $\mathcal{G}$, consistency implies that

$$|k||w^{(1)}(m, m'; k, \delta)| \leq \frac{C_1\delta^{1/(2n)}}{m^{1/2}(m - m')^{1/2-1/(2n)}} \tag{9.80}$$

$$|kw^{(1,1)}| \leq C_{1,1}\delta^{1/2+1/(2n)}(m\delta)^{-1/2}\,,\; |w^{(2)}| \leq C_2\delta\,,\; |w^{(2,l)}| \leq C_3\delta^{1/n}(l+1)^{-1+1/n}.$$

Consider the solution

$$\hat{U}_\delta^{(N)}(k, m\delta) = \left\{\mathcal{N}_\delta^{(N)}\left[\hat{U}_\delta^{(N)}\right]\right\}(k, m\delta) \text{ for } m_s \leq m,\; k \in [-N, N]^3, \tag{9.81}$$

where as noted before, $q_m = m_s\delta$ is small enough so that the known asymptotic series of $\hat{U}$ at $q = 0$ can be used to accurately calculate $\hat{U}^{(N)}$ and $\hat{H}_j^{(N)}$ for $q < q_m$, and thus of $\hat{U}^{(0,N)}$ and $\hat{E}^{(l)}$ in (9.78) and (9.79).

Definition 9.2. *We let*

$$T_{E,\delta}^{(N)} = \mathcal{N}^{(N)}\hat{U}^{(N)} - \mathcal{N}_\delta^{(N)}\hat{U}^{(N)}$$

be the truncation error due to q-discretization for a fixed number of Fourier modes, $[-N, N]^3$*. The discretization is consistent (in the numerical analysis sense) if* $T_{E,\delta}^{(N)}$

scales with some positive power of δ and involves a finite number of derivatives of $\hat{U}$.

Definition 9.3. *We define $\|\cdot\|^{(\alpha,\delta)}$, the discrete analog of $\|\cdot\|^{(\alpha)}$, as follows:*

$$\|\hat{f}\|^{(\alpha,\delta)} = \sup_{m\geq m_s} m^{1-1/n}\delta^{1-1/n}(1+m^2\delta^2)e^{-\alpha m\delta}\|\hat{f}(\cdot,m\delta)\|_{l^1}.$$

Remark 9.4. More specific bounds on the truncation error depend on the specific numerical scheme. It is however standard for numerical quadratures to choose the weights $w^{(j)}$ so that q-integration is exact on $q\in[q_m,q_0]$ for a polynomial of some order l. For a general $\hat{V}(\cdot,q)$, the interpolation errors involve $l+1$ q-derivatives. Lemma 5.9 guarantees that the derivatives of $\hat{U}$ are exponentially bounded for large q. It follows that $\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}\to 0$ as $\delta\to 0$.

Remark 9.5. *In the rest of this section, with slight abuse of notation, we write $*$ for the discrete summation convolution in q-space (i.e. sum over m') and $\overset{*}{*}$ for the discrete double, Fourier-Laplace, convolution. Since the rest of the paper deals with discrete systems, this should not cause confusion.*

Lemma 9.6. *For $m\geq m_s$, $\hat{H}_{j,\delta}^{(N)}(\cdot,m\delta)$ satisfies the following estimate:*

$$\begin{aligned}(9.82)\quad &\|\hat{H}_{j,\delta}^{(N)}(\cdot,m\delta)\|_{l^1}\\ &\leq C\frac{e^{\alpha m\delta}}{(1+m^2\delta^2)m^{1-1/n}\delta^{1-1/n}}\|\hat{U}_\delta^{(N)}\|^{(\alpha,\delta)}\left\{\|\hat{U}_\delta^{(N)}\|^{(\alpha,\delta)}+\|\hat{v}_0\|_{l^1}+C_E\right\}\end{aligned}$$

for some known constant C_E.

Proof. Using the properties of discrete convolution we see that

$$\begin{aligned}&\|P_k\left\{\hat{v}_{0,j}\overset{*}{*}\hat{U}_\delta^{(N)}+\hat{U}_{\delta,j}^{(N)}\overset{*}{*}\hat{v}_0+\hat{U}_{\delta,j}^{(N)}\overset{*}{*}\hat{U}_\delta^{(N)}\right\}\|_{l^1}\\ &\leq C\Bigg\{\|\hat{v}_0\|_{l^1}\|\hat{U}_\delta^{(N)}(\cdot,m\delta)\|_{l^1}+\delta\sum_{m'=m_s}^{m-m_s}\|\hat{U}_\delta^{(N)}(\cdot,m'\delta)\|_{l^1}\|\hat{U}_\delta^{(N)}(\cdot,(m-m')\delta)\|_{l^1}\\ &\qquad+\delta^{1/n}\sum_{m'=0}^{m_s-1}(m'+1)^{-1+1/n}\|\hat{E}^{(m')}\|_{l^1}\|\hat{U}_\delta^{(N)}(\cdot,(m-m')\delta)\|_{l^1}\Bigg\}\\ &\leq C\frac{e^{\alpha m\delta}}{m^{1-1/n}\delta^{1-1/n}(1+m^2\delta^2)}\left\{(C_E+\|\hat{v}_0\|_{l^1})\,\|\hat{U}_\delta^{(N)}\|^{(\alpha,\delta)}+\left(\|\hat{U}_\delta^{(N)}\|^{(\alpha,\delta)}\right)^2\right\},\end{aligned}$$

where, by a standard integral estimate,

$$\delta^{1-1/n}m^{1-1/n}(1+m^2\delta^2)\sum_{m'=1}^{m-1}\frac{\delta}{[\delta m'\delta(m-m')]^{1-1/n}(1+\delta^2m'^2)(1+\delta^2(m-m')^2)}<C,$$

$$\delta^{1-1/n}m^{1-1/n}(1+m^2\delta^2)\sum_{m'=0}^{m_s-1}\frac{\delta}{[\delta(m'+1)\delta(m-m')]^{1-1/n}(1+\delta^2(m-m')^2)}<C,$$

for C independent of m, m' and δ. In the above estimates we have used

$$\|\hat{E}^{(m')}\|_{l^1}\leq C_E e^{\alpha_0 m'\delta}\qquad(\alpha_0\leq\alpha)$$

which can be obtained from the definition of $\hat{E}^{(m')}$. ∎

Define $\hat{H}_{j,\delta}^{(N,1)}$ and $\hat{H}_{j,\delta}^{(N,2)}$ by substituting $\hat{U}_{\delta}^{(N)} = \hat{U}_{\delta}^{(N,1)}$ and $\hat{U}_{\delta}^{(N,2)}$, respectively, in $\hat{H}_{j,\delta}^{(N)}$.

Lemma 9.7. *For $m \geq m_s$, we have*

$$\begin{aligned}\|\hat{H}_{j,\delta}^{(N,1)}(\cdot, m\delta) - \hat{H}_{j,\delta}^{(N,2)}(\cdot, m\delta)\|_{l^1} \\ \leq C \frac{e^{\alpha m \delta}}{(1+m^2\delta^2) m^{1-1/n} \delta^{1-1/n}} \|\hat{U}_{\delta}^{(N,1)} - \hat{U}_{\delta}^{(N,2)}\|^{(\alpha,\delta)} \\ \times \left\{ \|\hat{U}_{\delta}^{(N,1)}\|^{(\alpha,\delta)} + \|\hat{U}_{\delta}^{(N,2)}\|^{(\alpha,\delta)} + \|\hat{v}_0\|_{l^1} + C_E \right\}.\end{aligned}$$

Proof. The proof is similar to that of Lemma 9.6 ∎

Lemma 9.8. *(i) For C_4 defined in (9.85), assume α is large enough so that*

$$2 C_4 \alpha^{-1/2-1/(2n)} \left([C_E + \|\hat{v}_0\|_{l^1}] + 2 \|\hat{U}^{(0,N)}\|^{(\alpha,\delta)} \right) < 1. \tag{9.83}$$

Then, for any $\alpha^{-1} \geq \delta_0 \geq \delta > 0$, $\mathcal{N}_{\delta}^{(N)}$ is contractive and there is a unique solution to $\hat{U}_{\delta}^{(N)} = \mathcal{N}_{\delta}^{(N)} \left[\hat{U}_{\delta}^{(N)} \right]$, which satisfies the bounds

$$\|\hat{U}_{\delta}^{(N)}(\cdot, m\delta)\|_{l^1} \leq \frac{2 e^{\alpha m \delta}}{m^{1-1/n} \delta^{1-1/n} (1+m^2\delta^2)} \|\hat{U}^{(0,N)}\|^{(\alpha,\delta)}.$$

(ii) If α is such that

$$2 C_4 \alpha^{-1/2-1/(2n)} \left([C_E + \|\hat{v}_0\|_{l^1}] + 2 \|\hat{U}^{(0,N)}\|^{(\alpha,\delta)} \right) \leq \frac{1}{2}, \tag{9.84}$$

then

$$\|\hat{U}_{\delta}^{(N)}(\cdot, m\delta) - \hat{U}^{(N)}(\cdot, m\delta)\|_{l^1} \leq \frac{2 e^{\alpha m \delta}}{m^{1-1/n} \delta^{1-1/n} (1+m^2\delta^2)} \|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}.$$

Proof. (i) We have

$$\begin{aligned}\|\mathcal{N}_{\delta}^{(N)}[\hat{U}_{\delta}^{(N)}](\cdot, m\delta)\|_{l^1} \leq \|\hat{U}^{(0,N)}(\cdot, m\delta)\|_{l^1} \\ + C \sum_{m'=m_s}^{m-1} \frac{\delta^{1/(2n)}}{m^{1/2} (m-m')^{1/2-1/(2n)}} \|\hat{H}_{\delta}^{(N)}(\cdot, m'\delta)\|_{l^1} + C \delta^{1/(2n)} m^{-1/2} \|\hat{H}_{\delta}^{(N)}(\cdot, m\delta)\|_{l^1} \\ \leq \frac{e^{\alpha m \delta}}{(1+m^2\delta^2) m^{1-1/n} \delta^{1-1/n}} \left\{ \|\hat{U}^{(0,N)}\|^{(\alpha,\delta)} \right. \\ \left. + C_4 \alpha^{-1/2-1/(2n)} \|\hat{U}_{\delta}^{(N)}\|^{(\alpha,\delta)} \left(\|\hat{U}_{\delta}^{(N)}\|^{(\alpha,\delta)} + \|\hat{v}_0\|_{l^1} + C_E \right) \right\},\end{aligned} \tag{9.85}$$

where, by a standard integral estimate,

$$\begin{aligned}\delta^{1-1/n} m^{1-1/n} (1+m^2\delta^2) \sum_{m'=m_s}^{m-1} \frac{\delta e^{\alpha(m'-m)\delta}}{[\delta m]^{1/2} [\delta(m-m')]^{1/2-1/(2n)} [\delta m']^{1-1/n} (1+m'^2\delta^2)} \\ \leq C \alpha^{-1/2-1/(2n)},\end{aligned}$$

and

$$\frac{\delta^{1/2+1/(2n)}}{[\delta m]^{1/2}} \leq C \delta^{1/2+1/(2n)} \leq C \alpha^{-1/2-1/(2n)}.$$

Thus $\hat{U}_\delta^{(N)} = \mathcal{N}_\delta^{(N)}\left[\hat{U}_\delta^{(N)}\right]$ has a unique solution such that

$$\|\hat{U}_\delta^{(N)}\|^{(\alpha,\delta)} \le 2\|\hat{U}^{(0,N)}\|^{(\alpha,\delta)}.$$

Hence the first part of the lemma follows.

(ii) Under the assumption,

$$\begin{aligned}\|\hat{U}^{(N)} - \hat{U}_\delta^{(N)}\|^{(\alpha,\delta)} &\le \|\mathcal{N}_\delta^{(N)}[\hat{U}^{(N)}] - \mathcal{N}_\delta^{(N)}[\hat{U}_\delta^{(N)}]\|^{(\alpha,\delta)} + \|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)} \\ &\le \frac{1}{2}\|\hat{U}^{(N)} - \hat{U}_\delta^{(N)}\|^{(\alpha,\delta)} + \|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}.\end{aligned}$$

So

$$\|\hat{U}^{(N)} - \hat{U}_\delta^{(N)}\|^{(\alpha,\delta)} \le 2\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)}$$

and the second part of the lemma follows. ∎

Proof of Theorem 3.2: Note that

$$\hat{U}_\delta^{(N)} - \hat{U} = \hat{U}_\delta^{(N)} - \hat{U}^{(N)} + \hat{U}^{(N)} - \hat{U}.$$

From Lemmas 6.3 and 9.8, it follows that

$$\|\hat{U}_\delta^{(N)} - \hat{U}\|^{(\alpha,\delta)} \le 2\|T_{E,N}\|^{(\alpha,\delta)} + 2\|T_{E,\delta}^{(N)}\|^{(\alpha,\delta)} + \|(I - \mathcal{P}_N)\hat{U}\|^{(\alpha,\delta)},$$

which tends to zero as $N \to \infty$, $\delta \to 0$, by Lemmas 6.3 and 9.8.

10. Numerical Method

In this section we describe a numerical scheme for calculating the solution $\hat{U}_\delta^{(N)}$ over a fixed interval. The procedure can be further optimized in a number of ways, such as adapting the quadrature scheme to the features of the kernel.

10.1. **Outline of the Algorithm.** The main algorithm is summarized as follows:

```
initialization;
startup routine;
for each time step
  advance the solution using second order Runge-Kutta integration;
end
estimate the error and output the results.
```

10.2. **Startup Routine.** One difficulty in numerically solving (1.6) is that the equation is singular at $q = 0$. To overcome it, we first compute $\hat{u}$ for small t by solving (4.16) using Taylor expansion:

$$\hat{u}(k,t) = \sum_{m=1}^{\infty} \hat{c}_m(k)\, t^m,$$

where

$$\hat{c}_1 = \hat{v}_1,$$

$$\hat{c}_{m+1} = \frac{1}{m+1}\left[-\nu|k|^2\hat{c}_m - ik_jP_k\left(\hat{v}_{0,j}\hat{*}\hat{c}_m + \hat{c}_{m,j}\hat{*}\hat{v}_0 + \sum_{\ell=1}^{m-1}\hat{c}_{\ell,j}\hat{*}\hat{c}_{m-\ell}\right)\right],\ m \ge 1.$$

Then $\hat{U}$ is computed for small q by

$$\hat{U}(k,q) = \sum_{m=1}^{m_0} \hat{d}_m(k)\, q^{m/n-1},$$

where[(10)]

$$\hat{d}_m = \frac{\hat{c}_m}{\Gamma(m/n)}.$$

10.3. **Second Order Runge-Kutta Integration.** After computing the solution on $[0, q_m]$ for some $q_m > 0$ by using Taylor expansions, we solve the integral equation (1.6) on $[q_m, q_0]$ using second order Runge-Kutta (predictor-corrector) method. Since this numerical scheme is preliminary and far from being optimized, we do not include the details here.

What is worth mentioning is the evaluation of the functions $F(\mu)$ and $G(\mu)$. As shown in earlier sections, both F and G are entire functions and have power series expansions at $\mu = 0$. For small μ, these expansions converge very rapidly (super-factorially) and provide an efficient way to evaluate F and G. For large μ, however, the alternating nature of the expansions raises the issue of catastrophic cancellation, and it is no longer appropriate to use them for numerical computation. In this regime we use the asymptotic expansions of F and G, which we derive below.

While the complete asymptotics of F and G can be derived using Laplace's method, a faster and easier way is to use the differential equations they satisfy. For example, recall that for $n = 2$,

$$F(\mu) = \frac{1}{2\pi i}(I_1 - \bar{I}_1) = \frac{1}{\pi}\mathrm{Im} I_1,$$

where

$$I_1 = i\int_0^\infty r^{-1/2} e^{-r - i\mu r^{-1/2}}\, dr.$$

It is easy to check that I_1 satisfies the third-order ODE (the same equation satisfied by F)

$$\mu I_1''' + I_1'' - 2I_1 = 0,$$

and it has the leading order asymptotics

$$I_1 \sim 2\sqrt{\frac{\pi}{3}}\, e^{-z},$$

where

$$z = 3 \cdot 2^{-2/3} \mu^{2/3} e^{i\pi/3}.$$

If we make the change of dependent variable

$$I_1 = 2\sqrt{\frac{\pi}{3}}\, e^{-z} J_1(z),$$

then J_1 must have the form

$$J_1(z) = 1 + \sum_{m=1}^{\infty} a_m z^{-m},$$

---

(10) Note that

$$\int_0^\infty \hat{d}_m q^{m/n-1} e^{-q/t^n}\, dq = \hat{d}_m t^m \int_0^\infty q^{m/n-1} e^{-q}\, dq = \Gamma\left(\frac{m}{n}\right) \cdot \hat{d}_m t^m,$$

so $\Gamma(m/n) \cdot \hat{d}_m = \hat{c}_m$.

and it solves the ODE

$$J_1''' - 3J_1'' + \left(3 + \frac{1}{4z^2}\right)J_1' - \frac{1}{4z^2}\,J_1 = 0.$$

It follows that

$$F(\mu) \sim \frac{2}{\sqrt{3\pi}}\mathrm{Im}\left\{e^{-z}\left(1 + \sum_{m=1}^{\infty} a_m z^{-m}\right)\right\},$$

where a_1, a_2, etc. are determined by the recurrence

$$a_0 = 1, \qquad a_1 = -\frac{1}{12},$$
$$a_m = -\frac{1}{12m}\left[\left(12m^2 - 12m + 1\right)a_{m-1} + \left(4m^3 - 12m^2 + 9m - 2\right)a_{m-2}\right], \qquad m \geq 2.$$

Similarly,

$$G(\mu) \sim -\frac{(4\mu)^{1/3}}{\sqrt{3\pi}}\mathrm{Im}\left\{e^{-z+i\pi/6}\left(1 + \sum_{m=1}^{\infty} c_m z^{-m}\right)\right\},$$

where

$$c_0 = 1, \qquad c_1 = \frac{5}{12}, \qquad c_2 = -\frac{35}{288},$$
$$c_m = \frac{1}{24m}\Big[\left(-48m^2 + 60m - 2\right)c_{m-1} + \left(-32m^3 + 108m^2 - 80m + 9\right)c_{m-2}$$
$$+ \left(-8m^4 + 52m^3 - 102m^2 + 67m - 14\right)c_{m-3}\Big], \qquad m \geq 3.$$

11. Preliminary Numerical Results

For all computations in this section we take $n = 2$. The numerical results and computation scheme are preliminary. The algorithm has not been optimized for efficiency, and not all estimates have been rigorously analyzed yet, and these will be published elsewhere. Nonetheless, the partial results show some important features of the integral equation approach.

11.1. **Test Case.** We first tested our code with the following test function:

$$(\text{Kida flow}) : v = (v^{(1)}, v^{(2)}, v^{(3)}),$$
$$v^{(1)}(x_1, x_2, x_3, t) = \frac{\sin x_1}{1+t}(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3),$$
$$v^{(1)}(x_1, x_2, x_3, t) = v^{(2)}(x_3, x_1, x_2, t) = v^{(3)}(x_2, x_3, x_1, t).$$

The forcing f corresponding to v was generated with $\nu = 1$ and equation (1.6) was solved without the knowledge of v. The computed solution was then compared to v.

For this test case, the startup routine computed the solution on $[0, q_m] = [0, 0.2]$ using $m_0 = 8$ terms and the Runge-Kutta solver advanced the solution to $q_0 = 1$. $2N = 16$ points (i.e. 8 Fourier modes) were used in each dimension (excluding the extra points for anti-aliasing).

We computed the solution for different step size δ and the errors at q_0

$$e_\delta = \max_{x \in \mathbb{T}^3} |U_\delta^{(N)}(x, q_0) - U(x, q_0)|$$

are listed in Table 1. To ensure the error decays at the right order $O(\delta^2)$, we also included in the table the numerical order of convergence:

$$\beta_\delta = \log_2 \frac{e_{2\delta}}{e_\delta}.$$

TABLE 1. Test case: errors at q_0.

δ	e_δ	β_δ
1/20	1.3399e-04	–
1/40	3.1987e-05	2.07
1/80	7.1462e-06	2.16
1/160	1.3620e-06	2.39

11.2. **Kida Flow.** Now we consider the Kida flow with the initial condition

$$v_0^{(1)}(x_1, x_2, x_3, 0) = \sin x_1(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3).$$

We computed the solution for $\nu = 0.1$ with zero forcing to $q_0 = 10$ using $2N = 128$ points in each dimension, and step size $\delta = 0.05$. The parameters for the startup procedure are the same as before: $q_m = 0.2$ and $m_0 = 8$. To investigate the growth of the solution $\hat{U}_\delta^{(N)}$ with q, we computed the l^1-norm

$$\|\hat{U}_\delta^{(N)}(\cdot, q)\|_{l^1} = \sum_{k \in [-N,N]^3} |\hat{U}_\delta^{(N)}(k, q)|$$

and plotted $\|\hat{U}_\delta^{(64)}(\cdot, q)\|_{l^1}$ vs. q in Fig.2. For comparison we also included in Fig.2 a plot of the solution to the original (unaccelerated) equation.

Fig.3 shows the plot of $\log \|\hat{U}_\delta^{(64)}(\cdot, q)\|_{l^1}$ vs. $q^{1/3}$. Note that $\|\hat{U}(\cdot, q)\|_{l^1} \sim c_1 e^{-c_2 q^{1/3}}$ for large q, where $c_2 = (0.3)^{2/3} 2^{-5/3} 3 \approx 0.42$.

11.3. **Longer Time Existence.** We next computed the constants in estimate (8.69). By taking $q_0 = 10$ and $\alpha_0 = 30$, we obtained

$$b \approx 0, \qquad \epsilon \approx 1.1403, \qquad \epsilon_1 \approx 13.6921.$$

This implies the existence of the solution for $\alpha \geq 32.7564$, which corresponds to an interval of existence $(0, \alpha^{-1/2}) = (0, 0.1747)$.

We compare with a classical estimate of the existence time. The formula

$$T_{cl} = \frac{1}{c_m \|D^m v_0\|_{L^2}}$$

(where c_m is known) was optimized in the range $m > 5/2$, giving a maximal value $T_{cl} \approx 0.01$ at $m \approx 3.2$, about 17 time shorter than the time obtained from the integral equation.

Furthermore, considerable optimization of code is expected to allow numerical calculation over much larger q-interval.

12. APPENDIX

12.1. **Derivation of the integral equation and of its properties.**

The integral equation. We start with the Fourier transformed equation (12.86):

$$\hat{u}_t + \nu|k|^2\hat{u} = -ik_jP_k[\hat{v}_{0,j}\hat{*}\hat{u} + \hat{u}_j\hat{*}\hat{v}_0 + \hat{u}_j\hat{*}\hat{u}] + \hat{v}_1(k)$$
$$=: -ik_j\hat{h}_j + \hat{v}_1(k) =: \hat{r} + \hat{v}_1(k),$$
$$\hat{u}(k,0) = 0, \tag{12.86}$$

where

$$\hat{v}_1(k) = \hat{f}(k) - \nu|k|^2\hat{v}_0 - ik_jP_k[\hat{v}_{0,j}\hat{*}\hat{v}_0].$$

For $n > 1$, look for a solution in the form

$$\hat{u}(k,t) = \int_0^\infty \hat{U}(k,q)e^{-q/t^n}\,dq \tag{12.87}$$

where

$$\hat{r}(k,t) = -ik_j\hat{h}_j(k,t) = -ik_j\int_0^\infty \hat{H}_j(k,q)e^{-q/t^n}\,dq$$
$$=: \int_0^\infty \hat{R}(k,q)e^{-q/t^n}\,dq. \tag{12.88}$$

Inversion of the left side of (12.86) and the change of variable $\tau = t^{-n}$ yield

$$\hat{u}(k,t) = \int_0^t e^{-\nu|k|^2(t-s)}\hat{r}(k,s)\,ds + \int_0^t e^{-\nu|k|^2(t-s)}\hat{v}_1(k)\,ds \tag{12.89}$$
$$= \int_0^1 te^{-\nu|k|^2t(1-s)}\hat{r}(k,ts)\,ds + \frac{\hat{v}_1(k)}{\nu|k|^2}\left(1 - e^{-\nu|k|^2t}\right)$$
$$= \int_0^1 \tau^{-1/n}e^{-\nu|k|^2\tau^{-1/n}(1-s)}\int_0^\infty \hat{R}(k,q')e^{-q's^{-n}\tau}\,dq'\,ds + \frac{\hat{v}_1(k)}{\nu|k|^2}\left(1 - e^{-\nu|k|^2\tau^{-1/n}}\right)$$
$$=: I(k,\tau) + J(k,\tau).$$

Inverse Laplace transform (formal for now) of I and J yield:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} I(k,\tau)e^{q\tau}\,d\tau \tag{12.90}$$
$$= \int_0^\infty \hat{R}(k,q')\int_0^1\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tau^{-1/n}e^{-\nu|k|^2\tau^{-1/n}(1-s)+(q-q's^{-n})\tau}\,d\tau\right\}ds\,dq'$$
$$= \int_0^\infty \hat{R}(k,q')\int_0^1 (q-q's^{-n})^{1/n-1}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^{-1/n}e^{\zeta-\mu\zeta^{-1/n}}\,d\zeta\right\}ds\,dq'$$
$$=: \int_0^\infty \hat{R}(k,q')\mathcal{G}(q,q';k)dq',$$

where

$$\zeta = (q-q's^{-n})\tau, \qquad \mu = \nu|k|^2(1-s)(q-q's^{-n})^{1/n},$$

while

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} J(k,\tau)e^{q\tau}\,d\tau = \frac{\hat{v}_1(k)}{\nu|k|^2}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{q\tau}\left(1 - e^{-\nu|k|^2\tau^{-1/n}}\right)d\tau\right\} \tag{12.91}$$
$$= \frac{\hat{v}_1(k)}{\nu|k|^2q}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(e^{\tilde{\zeta}} - e^{\tilde{\zeta}-\tilde{\mu}\tilde{\zeta}^{-1/n}}\right)d\tilde{\zeta}\right\} =: \hat{U}^{(0)}(k,q),$$

where

$$\tilde{\zeta} = q\tau, \qquad \tilde{\mu} = \nu|k|^2 q^{1/n}.$$

The Bromwich contour is homotopic to a contour C from $\infty e^{-i\pi}$ to the left of the origin, ending at $\infty e^{i\pi}$ encircling the origin, and we finally obtain the integral equation:

$$\hat{U}(k,q) = \int_0^q \mathcal{G}(q,q';k)\Big[-ik_j \hat{H}_j(k,q')\Big]\,dq' + \hat{U}^{(0)}(k,q),$$

where

$$\hat{H}_j(k,q) = P_k\Big[\hat{v}_{0,j}\hat{*}\hat{U} + \hat{U}_j\hat{*}\hat{v}_0 + \hat{U}_j\hat{*}\hat{U}\Big](k,q).$$

Rescaling the integration variable, $s \to s\gamma^{1/n}$, the kernel in (12.90) becomes

$$\text{(12.92)} \quad \mathcal{G}(q,q';k) = q^{1/n-1}\gamma^{1/n}\int_1^{\gamma^{-1/n}} (1-s^{-n})^{1/n-1}F(\mu)\,ds$$

$$= \frac{\gamma^{1/n}}{\nu^{1/2}|k|q^{1-1/(2n)}}\int_1^{\gamma^{-1/n}} (1-s^{-n})^{1/(2n)-1}(1-s\gamma^{1/n})^{-1/2}\mu^{1/2}F(\mu)\,ds,$$

where

$$\gamma = \frac{q'}{q}, \qquad \mu = \nu|k|^2 q^{1/n}(1-s\gamma^{1/n})(1-s^{-n})^{1/n},$$

and

$$F(\mu) = \frac{1}{2\pi i}\int_C \zeta^{-1/n}e^{\zeta-\mu\zeta^{-1/n}}\,d\zeta.$$

Furthermore, from (12.91) we have

$$\text{(12.93)} \qquad \hat{U}^{(0)}(k,q) = \frac{\hat{v}_1(k)}{\nu|k|^2 q}G(\nu|k|^2 q^{1/n}),$$

where

$$G(\mu) = -\frac{1}{2\pi i}\int_C e^{\zeta-\mu\zeta^{-1/n}}\,d\zeta, \qquad G(0)=0.$$

Power series representations of F and G. To show that F is entire, we start with the definition

$$\text{(12.94)} \qquad F(\mu) = \frac{1}{2\pi i}\int_C \zeta^{-1/n}e^{\zeta}e^{-\mu\zeta^{-1/n}}\,d\zeta$$

and expand $e^{-\mu\zeta^{-1/n}}$ into power series of $\zeta^{-1/n}$ to obtain

$$F(\mu) = \frac{1}{2\pi i}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\mu^j\int_C \zeta^{-(j+1)/n}e^{\zeta}\,d\zeta,$$

where the interchange of order of summation and integration is justified by the absolute convergence of the series. From the integral representation of the Gamma function (see [1]) we get

$$\int_C \zeta^{-(j+1)/n}e^{\zeta}\,d\zeta = 2i\sin\left(\frac{j+1}{n}\pi\right)\Gamma\left(1-\frac{j+1}{n}\right) = \frac{2\pi i}{\Gamma((j+1)/n)},$$

(where in the last step we have used the identity $\sin(\pi z)\Gamma(1-z)\Gamma(z)=\pi$) and thus F has the power series representation

$$F(\mu) = \sum_{j=0}^{\infty}F_j\mu^j, \qquad \text{where } F_j = \frac{(-1)^j}{j!\,\Gamma((j+1)/n)}.$$

Similarly, G is an entire function and has the power series representation

$$G(\mu) = \sum_{j=1}^{\infty} G_j \mu^j, \qquad \text{where } G_j = -\frac{(-1)^j}{j!\,\Gamma(j/n)}.$$

12.1.1. *The Asymptotics of F and G for $n \geq 2$ and large $\mu > 0$.* Elementary contour deformation and estimates at 0 show that

$$F(\mu) = \frac{1}{2\pi i}(I_1 - \bar{I}_1) = \frac{1}{\pi}\text{Im} I_1,$$

where

$$\begin{aligned}(12.95)\quad I_1(\mu) &= \int_0^{\infty} r^{-1/n} e^{i\pi/n} \exp\Big[-r - \mu r^{-1/n} e^{i\pi/n}\Big]\, dr \\ &= n\mu^{1-2/(n+1)} e^{i\pi/n} \int_0^{\infty} s^{n-2} \exp\Big[-\mu^{n/(n+1)}(s^n + e^{i\pi/n} s^{-1})\Big]\, ds \\ &= n\mu^{1-2/(n+1)} e^{2i\pi/(n+1)} \int_0^{\infty} x^{n-2} \exp\Big[-w\varphi(x)\Big]\, dx,\end{aligned}$$

where

$$w = \mu^{n/(n+1)} e^{i\pi/(n+1)}, \qquad \varphi(x) = x^n + \frac{1}{x}.$$

Similarly,

$$\bar{I}_1 = n\mu^{1-2/(n+1)} e^{-2i\pi/(n+1)} \int_0^{\infty} x^{n-2} \exp\Big[-\bar{w}\varphi(x)\Big]\, dx.$$

We now use the Laplace method to obtain the complete asymptotic expansion of I_1 for large w with $\arg w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\arg\mu \in \left(-\frac{(n+3)\pi}{2n}, \frac{(n-1)\pi}{2n}\right)$. We then show that I_1 solves a linear differential equation. It will follow, from standard results on asymptotics in ODEs, that the expansion is valid in a wider complex sector. First, it is easily seen that the only solution to the equation

$$\varphi'(x) = nx^{n-1} - \frac{1}{x^2} = 0$$

on the positive real axis is $x = x_0 = n^{-1/(n+1)}$. If we introduce a new variable

$$\xi = \varphi(x), \tag{12.96}$$

then clearly ξ decreases monotonically from $x = 0^+$ to $x = x_0$, where it attains the minimum value

$$\xi_0 = \varphi(x_0) = n^{-n/(n+1)}(n+1).$$

We denote this branch of φ^{-1} by $x_1(\xi)$. Further, as x increases beyond $x = x_0$ up to ∞, ξ increases from ξ_0 to ∞. We denote this branch of φ^{-1} by $x_2(\xi)$. It follows that

$$I_1 = n\mu^{1-2/(n+1)} e^{2i\pi/(n+1)} \left[-\int_{\xi_0}^{\infty} \frac{x_1^{n-2}(\xi)\, e^{-w\xi}}{n x_1^{n-1}(\xi) - x_1^{-2}(\xi)}\, d\xi + \int_{\xi_0}^{\infty} \frac{x_2^{n-2}(\xi)\, e^{-w\xi}}{n x_2^{n-1}(\xi) - x_2^{-2}(\xi)}\, d\xi\right].$$

To find an expansion of $x_i(\xi),\ i = 1, 2$, we note that

$$\xi - \xi_0 = \varphi(x) - \varphi(x_0) = \sum_{j=2}^{\infty} \varphi^{(j)}(x_0) \frac{(x - x_0)^j}{j!},$$

and thus

$$(\xi-\xi_0)-\sum_{j=3}^{\infty}\varphi^{(j)}(x_0)\frac{(x-x_0)^j}{j!}=\frac{1}{2}\varphi''(x_0)(x-x_0)^2,$$

or

$$x_\pm = x_0 \pm \sqrt{\frac{2}{\varphi''(x_0)}\left[(\xi-\xi_0)-\sum_{j=3}^{\infty}\varphi^{(j)}(x_0)\frac{(x-x_0)^j}{j!}\right]}, \tag{12.97}$$

where $x_-=x_1$ and $x_+=x_2$. By (12.97) we have

$$\frac{x_i^{n-2}(\xi)}{nx_i^{n-1}(\xi)-x_i^{-2}(\xi)}=\sum_{j=-1}^{\infty}b_j^{[i]}(\xi-\xi_0)^{j/2}.$$

Watson's lemma then implies that

(12.98)

$$\begin{aligned}I_1 &\sim n\mu^{1-2/(n+1)}e^{2i\pi/(n+1)}e^{-\xi_0 w}\sum_{j=-1}^{\infty}\int_0^{\infty}\left(b_j^{[2]}-b_j^{[1]}\right)\eta^{j/2}e^{-w\eta}\,d\eta \qquad (\eta=\xi-\xi_0)\\ &\sim n\mu^{1-2/(n+1)}e^{2i\pi/(n+1)}e^{-\xi_0 w}\sum_{j=-1}^{\infty}\left(b_j^{[2]}-b_j^{[1]}\right)\Gamma\left(1+\frac{j}{2}\right)w^{-1-j/2}.\end{aligned}$$

We see that

$$b_j^{[2]}-b_j^{[1]}=\begin{cases}0 & j \text{ even}\\ 2b_j^{[2]} & j\text{ odd}\end{cases}.$$

Similar analysis for $\bar{I}_1$ gives

$$\begin{aligned}\bar{I}_1 &\sim n\mu^{1-2/(n+1)}e^{-2i\pi/(n+1)}e^{-\xi_0\bar{w}}\sum_{j=-1}^{\infty}\int_0^{\infty}\left(b_j^{[2]}-b_j^{[1]}\right)\eta^{j/2}e^{-\bar{w}\eta}\,d\eta\\ &\sim n\mu^{1-2/(n+1)}e^{-2i\pi/(n+1)}e^{-\xi_0\bar{w}}\sum_{j=-1}^{\infty}\left(b_j^{[2]}-b_j^{[1]}\right)\Gamma\left(1+\frac{j}{2}\right)\bar{w}^{-1-j/2}.\end{aligned} \tag{12.99}$$

With $\xi_0 w$ replaced by z, we finally obtain for μ large and positive

$$\begin{aligned}F(\mu) &= \frac{1}{\pi}\mathrm{Im}I_1\\ &\sim \frac{n}{\pi}\mathrm{Im}\left\{\mu^{(n-2)/[2(n+1)]}e^{3i\pi/[2(n+1)]}e^{-z}\sum_{m=0}^{\infty}2b_{2m-1}^{[2]}\Gamma\left(m+\frac{1}{2}\right)\xi_0^m z^{-m}\right\},\end{aligned} \tag{12.100}$$

where

$$\xi_0=n^{-n/(n+1)}(n+1),\qquad z=\xi_0\mu^{n/(n+1)}e^{i\pi/(n+1)}. \tag{12.101}$$

A similar analysis shows that

$$G(\mu)\sim -\frac{n}{\pi}\mathrm{Im}\left\{\mu^{n/[2(n+1)]}e^{i\pi/[2(n+1)]}e^{-z}\sum_{m=0}^{\infty}2d_{2m-1}^{[2]}\Gamma\left(m+\frac{1}{2}\right)\xi_0^m z^{-m}\right\},$$

where $z,\ \xi_0$ are given by (12.101) and $d_j^{[i]}$ are coefficients of the expansion

$$\frac{x_i^{n-1}(\xi)}{nx_i^{n-1}(\xi)-x_i^{-2}(\xi)}=\sum_{j=-1}^{\infty}d_j^{[i]}(\xi-\xi_0)^{j/2}.$$

To obtain the leading asymptotics of F and G, we note that

$$\frac{x_2^{n-2}}{nx_2^{n-1}-x_2^{-2}}=\frac{x_2^{n-2}}{\varphi'(x_2)}=\frac{x_0^{n-2}}{\varphi''(x_0)(x_2-x_0)}+O(1)=\frac{x_0^{n-2}}{\sqrt{2\varphi''(x_0)}}(\xi-\xi_0)^{-1/2}+O(1).$$

It follows that

$$b_{-1}^{[2]}=\frac{x_0^{n-2}}{\sqrt{2\varphi''(x_0)}}=\frac{1}{\sqrt{2}}n^{3/[2(n+1)]-1}(n+1)^{-1/2}\qquad(\text{where }\varphi''(x_0)=n^{3/(n+1)}(n+1)).$$

Similarly

$$d_{-1}^{[2]}=\frac{x_0^{n-1}}{\sqrt{2\varphi''(x_0)}}=\frac{1}{\sqrt{2}}n^{1/[2(n+1)]-1}(n+1)^{-1/2}.$$

As a result, we have to the leading order,

$$F(\mu)\sim\sqrt{\frac{2}{\pi}}\,n^{3/[2(n+1)]}(n+1)^{-1/2}\mathrm{Im}\Big\{\mu^{(n-2)/[2(n+1)]}e^{3i\pi/[2(n+1)]}e^{-z}\Big\},$$
$$G(\mu)\sim-\sqrt{\frac{2}{\pi}}\,n^{1/[2(n+1)]}(n+1)^{-1/2}\mathrm{Im}\Big\{\mu^{n/[2(n+1)]}e^{i\pi/[2(n+1)]}e^{-z}\Big\}.$$

Differential equations for F and G for $n\in\mathbb{N}$ and extended asymptotics. To derive a differential equation satisfied by F, we differentiate (12.94) n times in μ (justified by dominated convergence)

$$F^{(n)}(\mu)=\frac{(-1)^n}{2\pi i}\int_C\zeta^{-1/n-1}e^{\zeta-\mu\zeta^{-1/n}}\,d\zeta.$$

Integrating by parts once, we get

$$F^{(n)}(\mu)=\frac{n}{2\pi i}(-1)^n\int_C\zeta^{-1/n}e^{\zeta-\mu\zeta^{-1/n}}\left(1+\frac{\mu}{n}\zeta^{-1/n-1}\right)d\zeta$$
$$=(-1)^n nF(\mu)-\mu F^{(n+1)}(\mu),$$

so the differential equation satisfied by F is

$$\mu F^{(n+1)}+F^{(n)}-(-1)^n nF=0.$$

Since $G'=F$, the differential equation satisfied by G is

$$\mu G^{(n+2)}+G^{(n+1)}-(-1)^n nG'=0.$$

Integrating once and using $G(0)=0$, we obtain

$$\mu G^{(n+1)}-(-1)^n nG=0. \tag{12.102}$$

We can make the same argument for

$$I_2(\mu)\equiv\int_{\infty e^{-i\pi}}^{0^+}e^{\zeta-\mu\zeta^{-1/n}}d\zeta-1 \tag{12.103}$$

or for

$$\bar{I}_2(\mu)\equiv\int_{\infty e^{i\pi}}^{0^+}e^{\zeta-\mu\zeta^{-1/n}}d\zeta-1. \tag{12.104}$$

It is to be noted that $G(\mu)=\frac{1}{2\pi i}\left[I_2(\mu)-\bar{I}_2(\mu)\right]$, while $I_2'(\mu)=I_1(\mu)$ and $\bar{I}_2'(\mu)=\bar{I}_1(\mu)$.

Equation (12.102) has $(n+1)$ independent solutions with the following asymptotic behavior for large μ (see [35]):
(12.105)

$$\mu^{n/[2(n+1)]} \exp\left[-ze^{-i2\pi j/(n+1)}\right]; \quad z := \xi_0 e^{i\pi/(n+1)} \mu^{n/(n+1)}, \ j = 0, 1, \ldots, n.$$

Thus, there is only one solution with the asymptotic behavior

$$-\sqrt{\frac{2}{\pi}}\, n^{1/[2(n+1)]}(n+1)^{-1/2}\mu^{n/[2(n+1)]} \exp[-z] \text{ for } \arg z = 0$$

(all solutions independent from it are larger). Since $I_2(\mu)$ has this asymptotics in particular for $\arg\mu = -\frac{\pi}{n}$, corresponding to $\arg z = 0$ as discussed already, I_2 is the only solution of (12.102) satisfying

$$(12.106) \quad I_2(\mu) \sim -\sqrt{\frac{2}{\pi}}\, n^{1/[2(n+1)]}(n+1)^{-1/2}\mu^{n/[2(n+1)]} \exp[-z] \text{ for } \arg\mu = -\frac{\pi}{n}.$$

As we rotate around in the counter-clockwise direction starting from $\arg z = 0$ in the complex z (or complex μ) plane, the classical asymptotics of I_2 can only change at antistokes lines. The first antistokes line is $\arg z = \frac{\pi}{2} + \frac{2\pi}{n+1}$, corresponding to $\arg\mu = \frac{(n+3)\pi}{2n}$.

Similarly, in a clockwise direction, the first antistokes line is $\arg z = -\frac{\pi}{2} - \frac{2\pi}{n+1}$, *i.e.* $\arg\mu = -\frac{(n+7)\pi}{2n}$.

Therefore, for $\arg\mu \in \left(-\frac{(n+7)\pi}{2n}, \frac{(n+3)\pi}{2n}\right)$ the asymptotic expansion I_2 is the same.

From the symmetry between $\bar{I}_2$ and I_2, it follows that

$$(12.107) \qquad \bar{I}_2(\mu) \sim -\sqrt{\frac{2}{\pi}}\, n^{1/[2(n+1)]}(n+1)^{-1/2}\mu^{n/[2(n+1)]} \exp[-\bar{z}]$$

for $\arg\mu \in \left(-\frac{(n+3)\pi}{2n}, \frac{(n+7)\pi}{2n}\right)$. Since $G(\mu) = \frac{1}{2\pi i}\left[I_2(\mu) - \bar{I}_2(\mu)\right]$, noting that $I_2(\mu)$ is dominant for $\arg\mu \in \left(0, \frac{(n+3)\pi}{2n}\right)$, it follows that in this range of $\arg\mu$, $G(\mu) \sim -\frac{i}{2\pi} I_2(\mu)$. while for $\arg\mu \in \left(-\frac{(n+3)\pi}{2n}, 0\right)$, since $\bar{I}_2$ is dominant, $G(\mu) \sim \frac{i}{2\pi}\bar{I}_2(\mu)$. Lemma 4.1 follows.

12.2. **Instantaneous smoothing.** The following result shows that the solution $\hat{v}(k,t)$ obtained from $\hat{U}(k,q)$ corresponds to a classical solution of (1.1) for $t \in (0,T]$, *i.e.* there is instantaneous smoothing due to viscous effects. This is a known result (See for instance [5]), but we include it for completeness.

Lemma 12.1. *Assume $\hat{v}_0, \hat{f} \in l^1(\mathbb{Z}^3)$, where $\hat{v}_0(0) = 0 = \hat{f}(0)$. Assume further that (1.1) has a solution $\hat{v}(k,t)$ with $\|\hat{v}(\cdot,t)\|_{l^1} < \infty$ for $t \in [0,T]$. Then $v(x,t) = \mathcal{F}^{-1}[\hat{v}(\cdot,t)](x)$ is a classical solution of (1.1) for $t \in (0,T]$.*

Proof. It suffices to show $|k|^2\hat{v}(\cdot,t) \in l^1$ for $t \in (0,T]$ since this implies $v \in C^2$ and usual arguments imply that v satisfies (1.1).

Consider the time interval $[\epsilon, T]$ for $\epsilon \geq 0$, $T < \alpha^{-1/n}$. Define

$$\hat{w}_\epsilon(k) = \sup_{\epsilon \leq t \leq T} |\hat{v}|(k,t).$$

Since $|\hat{v}(k,t)| \le \int_0^\infty |\hat{U}(k,q)| e^{-\alpha q} dq$, $\hat{w}_0$ (or $\hat{w}_\epsilon$) satisfies

$$\|\hat{w}_0\|_{l^1} \le \int_0^\infty \|\hat{U}(\cdot,q)\|_{l^1} e^{-\alpha q} dq.$$

On $[\epsilon, T]$ for $\epsilon > 0$, (1.2) implies
(12.108)

$$\hat{v}(k,t) = -ik_j \int_0^t e^{-\nu|k|^2(t-\tau)} P_k \left(\hat{v}_j \hat{*} \hat{v}\right)(k,\tau) d\tau + \hat{v}_0 e^{-\nu|k|^2 t} + \frac{\hat{f}}{\nu|k|^2}\left(1 - e^{-\nu|k|^2 t}\right).$$

Therefore,

$$|k||\hat{v}|(k,t) \le 2\left\{\hat{w}_0 \hat{*} \hat{w}_0\right\} \int_0^t |k|^2 e^{-\nu|k|^2(t-\tau)} d\tau + |k|\hat{v}_0 e^{-\nu|k|^2 t} + \frac{|\hat{f}|}{\nu|k|}\left(1 - e^{-\nu|k|^2 t}\right).$$

Since $\int_0^t \nu|k|^2 e^{-\nu|k|^2(t-\tau)} d\tau \le 1$, it follows that

$$|k|\hat{w}_{\epsilon/2} \le \frac{2}{\nu}\left\{\hat{w}_0 \hat{*} \hat{w}_0\right\} + \sqrt{\frac{2}{\nu\epsilon}}\left(\sup_{\gamma>0} \gamma e^{-\gamma^2}\right)|\hat{v}_0| + \left|\frac{\hat{f}}{\nu|k|}\right|. \tag{12.109}$$

Using now the bounds on $\hat{w}_0$ we get

$$\||k|\hat{w}_{\epsilon/2}\|_{l^1} \le \frac{2}{\nu}\left\{\|\hat{U}(\cdot,q)\|_1^\alpha\right\}^2 + \frac{C}{\epsilon^{1/2}\nu^{1/2}}\|\hat{v}_0\|_{l^1} + \nu^{-1}\left\|\frac{\hat{f}}{|k|}\right\|_{l^1}.$$

The evolution of $\hat{v}$ is autonomous in time, and thus, for $t \in \left[\frac{\epsilon}{2}, T\right]$ we have

$$\hat{v}(k,t) = -i\int_{\epsilon/2}^t e^{-\nu|k|^2(t-\tau)} P_k\left(\hat{v}_j \hat{*} [k_j \hat{v}]\right)(k,\tau) d\tau + \hat{v}(k,\epsilon/2) e^{-\nu|k|^2(t-\epsilon/2)} + \hat{f}(k)\frac{1 - e^{-\nu|k|^2(t-\epsilon/2)}}{\nu|k|^2}, \tag{12.110}$$

where we used the divergence condition $k \cdot \hat{v}(k,t) = 0$. Multiplying (12.110) by $|k|^2$ and using (12.109), it follows that for $t \in [\epsilon, T]$ we have

$$|k|^2|\hat{v}(k,t)| \le 2\hat{w}_{\epsilon/2}\hat{*}\left[|k|\hat{w}_{\epsilon/2}\right] \int_{\epsilon/2}^t |k|^2 e^{-\nu|k|^2(t-\tau)} d\tau + \frac{1}{\nu(t-\epsilon/2)}\left(\sup_{\gamma>0} \gamma e^{-\gamma}\right)|\hat{v}(k,\epsilon/2)| + \frac{|\hat{f}|}{\nu},$$

implying that

$$\||k|^2\hat{w}_\epsilon\|_{l^1} \le \frac{2}{\nu}\|\hat{w}_{\epsilon/2}\|_{l^1}\||k|\hat{w}_{\epsilon/2}\|_{l^1} + \frac{C}{\epsilon\nu}\|\hat{w}_{\epsilon/2}\|_{l^1} + \frac{\|\hat{f}\|_{l^1}}{\nu}.$$

Since $\epsilon > 0$ is arbitrary, it follows that $|k|^2\hat{v}(\cdot,t) \in l^1$ for $t \in (0,T]$. ∎

12.3. **Estimate of T_c beyond which Leray's weak solution becomes classical.** It is known that (1.2) is equivalent to the integral equation

$$\hat{v}(k,t) = \int_0^t e^{-\nu|k|^2(t-\tau)} P_k\left[-ik_j \hat{v}_j \hat{*} \hat{v}\right](k,\tau)\, d\tau + e^{-\nu|k|^2 t}\hat{v}_0 \equiv \mathcal{F}\left\{\mathcal{N}\left[v\right](\cdot,t)\right\}(k). \tag{12.111}$$

Applying $\mathcal{F}^{-1}$ in k to (12.111), it follows that

$$v(x,t) = e^{\nu t\Delta} v_0 - \int_0^t e^{\nu(t-\tau)\Delta} \mathcal{P}\left[(v\cdot\nabla)v\right] \equiv \mathcal{N}\left[v\right](x,t). \tag{12.112}$$

We first determine the value of ϵ such that, if $\|v_0\|_{H^1} \le \epsilon$, then classical solutions $v(\cdot,t)$ to Navier-Stokes exist for all time. The argument holds for real t as well as in

$$\tilde{S}_{\tilde{\delta}} := \left\{ t : \arg t \in (-\tilde{\delta}, \tilde{\delta}) \right\},$$

where $0 < \tilde{\delta} < \frac{\pi}{2}$. Sectorial existence of analytic solution in t with exponential decay for large $|t|$ was useful in proving Theorem 2.2. We denote by $\mathcal{A}_t$ the class of functions analytic in t for $t \in \tilde{S}_{\tilde{\delta}}$ for $0 < |t| < T$.

We consider the space of functions

$$X \equiv \left\{\mathcal{A}_t H^1_x\right\} \cap \left\{ L^2_{|t|} H^2_x \right\} := \left(\mathcal{A}_t \otimes H^1(\mathbb{T}^3[0,2\pi]) \right) \cap \left(L^2\left[e^{i\phi}(0,T)\right] \otimes H^2(\mathbb{T}^3[0,2\pi]) \right),$$

where $t = |t|e^{i\phi}$, $|\phi| < \tilde{\delta}$, and the weighted norm

$$\|v\|_X = \sup_{t\in\tilde{S}_{\tilde{\delta}}, 0<|t|<T} \|e^{\frac{3}{4}\nu t} v(\cdot,t)\|_{H^1_x} + \sup_{|\phi|<\tilde{\delta}} \left\{ \int_0^T \|e^{\frac{3}{4}\nu t} v(\cdot, |t|e^{i\phi})\|^2_{H^2_x} d|t| \right\}^{1/2}.$$

Note that

$$\|f\|_{H^1_x} = \left(\sum_k (1+|k|^2) |\hat{f}(k)|^2 \right)^{1/2}, \qquad \|f\|_{H^2_x} = \left(\sum_k (1+|k|^4) |\hat{f}(k)|^2 \right)^{1/2},$$

and $\hat{f}$ is the Fourier-Transform of f.

The arguments below are an adaptation of classical arguments, see [33]. We introduce exponential weights in time, allowing for estimates independent of T, and extend the analysis to a complex sector.

Lemma 12.2. *For $v_0 \in H^1_x$, with zero average over $\mathbb{T}^3[0,2\pi]$ we have*

$$\|e^{\nu t\Delta} v_0\|_X \le c_1 \|v_0\|_{H^1_x},$$

where $c_1 = \left(1 + \sqrt{\frac{2}{\nu\cos\tilde{\delta}}}\right)$.

Proof. First, take $f = v_0$ and $t \in [0,T]$. Note that zero average implies $\hat{f}(0) = 0$; so we only need to consider $|k| \ge 1$.

(12.113)

$$|e^{\frac{3}{2}\nu t}| \|e^{\nu t\Delta} f\|^2_{H^1_x} \le \sum_{k\neq 0} (1+|k|^2) e^{-2\nu(|k|^2 - 3/4)t} |\hat{f}_k|^2 \le \sum_{k\neq 0}(1+|k|^2)|\hat{f}_k|^2 = \|f\|^2_{H^1_x}.$$

Also, note that

$$\int_0^T \|e^{\frac{3}{4}\nu t} e^{\nu t\Delta} f\|^2_{H^2_x} dt \le \sum_{k\neq 0} (1+|k|^4)|\hat{f}_k|^2 \left(\int_0^T e^{-\nu(2|k|^2 - \frac{3}{2})t} dt \right) \le \sum_{k\neq 0} \frac{1+|k|^4}{\nu(2|k|^2 - \frac{3}{2})} |\hat{f}_k|^2 \le \frac{2}{\nu} \|f\|^2_{H^1_x}. \tag{12.114}$$

If $t \in \tilde{S}_{\tilde{\delta}}$, we integrate along the ray $|t|e^{i\phi}$. It is clear all the steps go through when ν is replaced by $\nu\cos\phi$. A bound, uniform in $\tilde{S}_{\tilde{\delta}}$, is obtained by replacing $\frac{2}{\nu}$ in (12.114) by $\frac{2}{\nu\cos\tilde{\delta}}$. The result follows. ∎

Lemma 12.3. *If $e^{\frac{3}{4}\nu t}F \in L^2_{|t|}L^2_x$ uniformly in $\phi \in (-\tilde{\delta}, \tilde{\delta})$, then*

$$\left\| \int_0^t e^{\nu(t-\tau)\Delta} F(x,\tau)d\tau \right\|_X \leq c_2 \sup_{|\phi|<\tilde{\delta}} \|e^{\frac{3}{4}\nu t}F\|_{L^2_{|t|}L^2_x}, \tag{12.115}$$

with

$$c_2 = \left(\frac{2\sqrt{2}}{\sqrt{\nu\cos\tilde{\delta}}} + \frac{4\sqrt{2}}{\nu\cos\tilde{\delta}} \right).$$

Proof. We first show this for $t \in [0,T]$. The function

$$v(x,t) = \int_0^t e^{\nu(t-\tau)\Delta} F(x,\tau)d\tau$$

satisfies

$$v_t - \nu\Delta v = F, \qquad v(x,0) = 0. \tag{12.116}$$

Multiplying (12.116) by v^*, the conjugate of v, integrating over $x \in \mathbb{T}^3[0,2\pi]$ and combining with the equation for v^* we obtain

$$\frac{d}{dt}\|v(\cdot,t)\|^2_{L^2_x} + 2\nu\|Dv(\cdot,t)\|^2_{L^2_x} \leq \frac{4}{\nu}\|F(\cdot,t)\|^2_{L^2_x} + \frac{\nu}{4}\|v(\cdot,t)\|^2_{L^2_x}. \tag{12.117}$$

Similarly, taking the gradient in x of (12.116), taking the dot product with ∇v^* and combining with the equation satisfied by ∇v^*, we obtain

$$\frac{d}{dt}\|Dv(\cdot,t)\|^2_{L^2_x} + 2\nu\|D^2v(\cdot,t)\|^2_{L^2_x} = \int_{\mathbb{T}^3}(DF)\cdot(Dv^*)dx + \int_{\mathbb{T}^3}(DF^*)\cdot(Dv)dx.$$

Integration by parts and Cauchy's inequality give

$$\frac{d}{dt}\|Dv(\cdot,t)\|^2_{L^2_x} + 2\nu\|D^2v(\cdot,t)\|^2_{L^2_x} \leq \frac{4}{\nu}\|F(\cdot,t)\|^2_{L^2_x} + \frac{\nu}{4}\|\Delta v(\cdot,t)\|^2_{L^2_x}. \tag{12.118}$$

Combining (12.117) and (12.118) and using Poincaré's inequality, we have

$$\frac{d}{dt}\|v(\cdot,t)\|^2_{H^1_x} + \frac{3}{2}\nu\|v(\cdot,t)\|^2_{H^1_x} + \frac{\nu}{4}\|Dv(\cdot,t)\|^2_{H^1_x} \leq \frac{8}{\nu}\|F(\cdot,t)\|^2_{L^2_x}. \tag{12.119}$$

Therefore, using (12.119) and the fact that $v(x,0) = 0$,

$$\|e^{\frac{3}{4}\nu t}v(\cdot,t)\|^2_{H^1_x} \leq \frac{8}{\nu}\int_0^t \|e^{\frac{3}{4}\nu\tau}F(\cdot,\tau)\|^2_{L^2_x}d\tau.$$

Hence,

$$\sup_{t\in[0,T]} \|e^{\frac{3}{4}\nu t}v(\cdot,t)\|_{H^1_x} \leq \frac{2\sqrt{2}}{\sqrt{\nu}}\|e^{\frac{3}{4}\nu t}F\|_{L^2_{|t|}L^2_x}. \tag{12.120}$$

Integration of (12.119), using $v(x,0) = 0$, gives

$$\int_0^t \|e^{\frac{3}{4}\nu\tau}v(\cdot,\tau)\|^2_{H^2_x}d\tau \leq \frac{32}{\nu^2}\int_0^t \|e^{\frac{3}{4}\nu\tau}F(\cdot,\tau)\|^2_{L^2_x}d\tau.$$

Therefore, for $t \in [0,T]$, we obtain

$$\left[\int_0^t \|e^{\frac{3}{4}\nu\tau} v(\cdot,\tau)\|^2_{H^2_x} d\tau\right]^{1/2} \le \frac{4\sqrt{2}}{\nu}\left[\int_0^t \|e^{\frac{3}{4}\nu\tau} F(\cdot,\tau)\|^2_{L^2_x} d\tau\right]^{1/2}. \tag{12.121}$$

Now (12.120) and (12.121) together imply

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\{\sum_{k\neq 0}(1+|k|^2)\left|e^{\frac{3}{4}\nu t} t\int_0^1 e^{-\nu|k|^2 t(1-s)}\hat{F}(k,ts)ds\right|^2\right\}^{1/2} \\
&+\left\{\int_0^T d|t|\sum_{k\neq 0}(1+|k|^4)\left|e^{\frac{3}{4}\nu t} t\int_0^1 \hat{F}(k,ts)e^{-\nu|k|^2 t(1-s)}ds\right|^2\right\}^{1/2} \\
&\qquad \le \left(\frac{2\sqrt{2}}{\sqrt{\nu}}+\frac{4\sqrt{2}}{\nu}\right)\left\{\int_0^T \sum_{k\neq 0}|e^{\frac{3}{4}\nu t}\hat{F}|^2(k,t)|dt|\right\}^{1/2},
\end{aligned} \tag{12.122}$$

and replacing $t \in [0,T]$ by $t \in e^{i\phi}[0,T] \in \tilde{S}_{\tilde{\delta}}$ is equivalent to replacing ν by $\nu\cos\tilde{\delta}$. ∎

Lemma 12.4. *If $F = -\mathcal{P}[v\cdot\nabla v]$, then for $v \in X$, and $t \in e^{i\phi}[0,T] \subset \tilde{S}_{\tilde{\delta}}$,*

$$\sup_{|\phi|<\tilde{\delta}} \|e^{\frac{3}{4}\nu t} F\|_{L^2_{|t|}L^2_x} \le c_3\|v\|^2_X,$$

where $c_3 = \frac{c_4^{3/2}}{(3\nu\cos\tilde{\delta})^{1/4}}$ for $t \in \tilde{S}_{\tilde{\delta}}$, and c_4 is the Sobolev constant bounding $\|\cdot\|_{L^6}$ by $\|\cdot\|_{H^1}$ (see for instance [2]*, page 75).*

Proof. First consider $t \in [0,T]$. Hölder's inequality implies

$$\|e^{\frac{3}{4}\nu t}F\|^2_{L^2_{|t|}L^2_x} \le \left[\int_0^T |e^{-3\nu\tau}|d|\tau|\right]^{1/2}\left[\int_0^T \|e^{\frac{3}{2}\nu\tau}|F(\cdot,\tau)|\|^4_{L^2_x} d|\tau|\right]^{1/2}.$$

Hence,

$$\|e^{\frac{3}{4}\nu t}F\|_{L^2_{|t|}L^2_x} \le \frac{1}{(3\nu)^{1/4}}\|e^{\frac{3}{2}\nu t}F\|_{L^4_{|t|}L^2_x}.$$

If we replace $t \in [0,T]$ by $t \in \tilde{S}_{\tilde{\delta}}$ in this argument, the effect is simply that $\frac{1}{(3\nu)^{1/4}}$ gets replaced by $\frac{1}{(3\nu\cos\tilde{\delta})^{1/4}}$.

For nonnegative u, w, repeated use of Hölder's inequality gives

$$\begin{aligned}
\int_{\mathbb{T}^3} w^2u^2dx &\le \left(\int_{\mathbb{T}^3} w^6dx\right)^{1/3}\left(\int_{\mathbb{T}^3} u^3dx\right)^{2/3} \\
&\le \left\{\int_{\mathbb{T}^3} w^6dx\right\}^{1/3}\left\{\int_{\mathbb{T}^3} u^2dx\right\}^{1/2}\left\{\int_{\mathbb{T}^3} u^6dx\right\}^{1/6} \le \|w\|^2_{L^6_x}\|u\|_{L^2_x}\|u\|_{L^6_x}.
\end{aligned}$$

Therefore, it follows that

$$\|e^{\frac{3}{2}\nu t}F(\cdot,t)\|_{L^2_x} \le \|e^{\frac{3}{2}\nu t}|v(\cdot,t)||\nabla v(\cdot,t)|\|_{L^2_x} \le \|e^{\frac{3}{4}\nu t}v\|_{L^6_x}\|e^{\frac{3}{4}\nu t}\nabla v\|^{1/2}_{L^2_x}\|e^{\frac{3}{4}\nu t}\nabla v\|^{1/2}_{L^6_x},$$

and

$$\|e^{\frac{3}{2}\nu t}F\|_{L^4_{|t|}L^2_x} \le \|e^{\frac{3}{4}\nu t}v\|_{L^\infty_{|t|}L^6_x}\|e^{\frac{3}{4}\nu t}\nabla v\|^{1/2}_{L^\infty_{|t|}L^2_x}\|e^{\frac{3}{4}\nu t}\nabla v\|^{1/2}_{L^2_{|t|}L^6_x}.$$

Using Sobolev inequalities, we have

$$\|v(\cdot,t)\|_{L^6_x} \le c_4 \|v(\cdot,t)\|_{H^1_x},$$
$$\|Dv(\cdot,t)\|_{L^6_x} \le c_4 \|Dv(\cdot,t)\|_{H^1_x}.$$

Thus

$$\|e^{\frac{3}{2}\nu t} F\|_{L^4_{|t|} L^2_x} \le c_4^{3/2} \|e^{\frac{3}{4}\nu t} v\|^{3/2}_{L^\infty_{|t|} H^1_x} \|e^{\frac{3}{4}\nu t} Dv\|^{1/2}_{L^2_{|t|} H^1_x} \le c_4^{3/2} \|v\|_X^2.$$

Therefore,

$$\|e^{\frac{3}{4}\nu t} F\|_{L^2_{|t|} L^2_x} \le \frac{c_4^{3/2}}{(3\nu \cos\tilde{\delta})^{1/4}} \|v\|_X^2.$$

Since the right hand side is independent of ϕ, taking the supremum of the left side over ϕ for $|\phi| < \tilde{\delta}$, the Lemma follows. ∎

Lemma 12.5. *The operator $\mathcal{N}$ defined in (12.112) satisfies the following estimate:*

$$\|\mathcal{N}[v]\|_X \le c_1 \|v_0\|_{H^1_x} + c_2 c_3 \|v\|_X^2,$$
$$\|\mathcal{N}[v^{(1)}] - \mathcal{N}[v^{(2)}]\|_X \le c_2 c_3 \left(\|v^{(1)}\|_X + \|v^{(2)}\|_X \right) \|v^{(1)} - v^{(2)}\|_X.$$

Proof. Note that

$$\mathcal{N}[v] = e^{\nu t \Delta} v_0 + \int_0^t e^{\nu (t-\tau)\Delta} F(\cdot,\tau) d\tau,$$

where $F = -\mathcal{P}[v \cdot \nabla v]$. By Lemmas 12.2, 12.3 and 12.4 it follows that

$$\|\mathcal{N}[v]\|_X \le c_1 \|v_0\|_{H^1_x} + c_2 c_3 \|v\|_X^2.$$

For the second part, we note that

$$v^{(1)} \cdot \nabla v^{(1)} - v^{(2)} \cdot \nabla v^{(2)} = \left(v^{(1)} - v^{(2)} \right) \cdot \nabla v^{(1)} + v^{(2)} \cdot \left(\nabla v^{(1)} - \nabla v^{(2)} \right).$$

Using Lemmas 12.2, 12.3 and 12.4 again, we obtain the desired estimate. ∎

Lemma 12.6. *If*

$$\|v_0\|_{H^1_x} < \hat{\epsilon} \equiv \frac{1}{4c_1 c_2 c_3} = \frac{3^{1/4} \nu^{7/4} [\cos\tilde{\delta}]^{7/4}}{8\sqrt{2}\, c_4^{3/2} (\sqrt{\nu \cos\tilde{\delta}} + \sqrt{2})(2 + \sqrt{\nu \cos\tilde{\delta}})},$$

$v(x,t)$ exists in X for any T. $v(\cdot,t)$ is analytic in $t \in \tilde{S}_{\tilde{\delta}}$ and decays exponentially in that sector as $|t| \to \infty$, with

$$\|v(\cdot,t)\|_{H^1_x} < 2c_1 \hat{\epsilon} e^{-\frac{3}{4}\nu \mathrm{Re} t}.$$

Further, this solution is smooth in x. If

$$\|v_0\|_{H^1_x} < \epsilon_0 \equiv \frac{3^{1/4} \nu^{7/4}}{8\sqrt{2}\, c_4^{3/2} (\sqrt{\nu} + \sqrt{2})(2 + \sqrt{\nu})},$$

then $v(x,t)$ is a classical solution for all $t \in \mathbb{R}^+$.

Proof. If $\|v_0\|_{H^1_x} < \hat{\epsilon}$, Lemma 12.5 implies that the operator $\mathcal{N}$ (defined in Lemma 12.5) is contractive and hence a solution to Navier-Stokes equation exists in X. Since the estimates are uniform in t, it follows that this solution exists for all $t \in \tilde{S}_{\tilde{\delta}}$. Known results (or Theorem 2.1 above) imply that if the initial data is in H^1_x, then the solution becomes smooth (in fact, analytic for periodic data, [17]) instantly, and thus it is a classical solution when $t > 0$. Analyticity and decay in t follows from the definition of X, the arbitrariness in the choice of T and the observation that $\mathcal{N}$ in Lemma 12.5 is contractive in a ball of radius $2c_1\|v_0\|_{H^1_x}$. Further, by taking the $\lim_{\tilde{\delta}\to 0^+}\hat{\epsilon} = \epsilon_0$, we obtain the less restrictive condition on $\|v_0\|_{H^1_x}$ that ensures existence of classical solution only for $t \in \mathbb{R}^+$. ∎

Lemma 12.7. *If $\|v_0\|_{H^2_x} \leq \epsilon_2$ for sufficiently small ϵ_2,*

$$\|v(\cdot,t)\|_{H^2_x} \leq 2c_1\|v_0\|_{H^2_x} e^{-\frac{3}{4}\nu \mathrm{Re} t}$$

for any $t \in \tilde{S}_{\tilde{\delta}}$.

Proof. This is similar to the proof of Lemma 12.6 with X replaced by

$$X \equiv \left\{\mathcal{A}_t H^2_x\right\} \cap \left\{L^2_{|t|} H^3_x\right\} := \left(\mathcal{A}_t \otimes H^2(\mathbb{T}^3[0,2\pi])\right) \cap \left(L^2\left[e^{i\phi}(0,T)\right] \otimes H^3(\mathbb{T}^3[0,2\pi])\right),$$

for $|\phi| < \tilde{\delta}$. ∎

Theorem 12.1. *A weak solution to (1.1) becomes classical when $t > T_c$, where*

$$T_c = \frac{256 E c_4^3 (\sqrt{\nu} + \sqrt{2})^2 (2 + \sqrt{\nu})^2}{3^{1/2}\nu^{9/2}}.$$

This solution is analytic in t for $(t - T_{c,a}) \in \tilde{S}_{\tilde{\delta}}$, where

$$T_{c,a} = \frac{256 E c_4^3 (\sqrt{\nu\cos\tilde{\delta}} + \sqrt{2})^2 (2 + \sqrt{\nu\cos\tilde{\delta}})^2}{3^{1/2}\nu[\nu\cos\tilde{\delta}]^{7/2}}.$$

Further, for any constant C, there exists T_2 so that for $(t - T_2) \in \tilde{S}_{\tilde{\delta}}$,

$$\|\hat{v}(\cdot,t)\|_{l^1} < C \exp\left[-\frac{3}{4}\nu \mathrm{Re}\{t - T_2\}\right].$$

Proof. Leray's energy estimate implies

$$\|\nabla v\|_{L^2_{|t|}L^2_x} \leq \sqrt{\frac{E}{\nu}},$$

where $E = \frac{1}{2}\|v_0\|^2_{L^2_x}$. From a standard pigeon-hole argument, it follows that there exists $T_1 \in (0, T]$ so that

$$\|\nabla v(\cdot, T_1)\|_{L^2_x} \leq \sqrt{\frac{E}{\nu T}}.$$

Therefore, Poincaré's inequality implies

$$\|v(\cdot, T_1)\|_{H^1_x} \leq \sqrt{\frac{2E}{\nu T}}.$$

This means there exists some $T_1 \in [0, T_c]$, where

$$T_c = \frac{256 E c_4^3 (\sqrt{\nu} + \sqrt{2})^2 (2 + \sqrt{\nu})^2}{3^{1/2}\nu^{9/2}}$$

for which

$$\|v(\cdot, T_1)\|_{H^1_x} < \frac{3^{1/4}\nu^{7/4}}{8\sqrt{2}c_4^{3/2}(\sqrt{\nu} + \sqrt{2})(2 + \sqrt{\nu})}.$$

Replacing t by $t - T_1$ in Lemma 12.6, we see that the solution is classical and smooth for $t - T_1 \in \mathbb{R}^+$, therefore necessarily for $t > T_c$.

Further, from these arguments, it is clear that there exists a $T_{1,a} \in [0, T_{c,a}]$ so that

$$\|v(\cdot, T_{1,a})\|_{H^1_x} \leq \frac{3^{1/4}[\nu \cos\tilde{\delta}]^{7/4}}{8\sqrt{2}c_4^{3/2}(\sqrt{\nu\cos\tilde{\delta}} + \sqrt{2})(2 + \sqrt{\nu\cos\tilde{\delta}})}.$$

Replacing t by $t - T_{1,a}$ in Lemma (12.6), we see that the classical solution is analytic in $t - T_{1,a} \in \tilde{S}_{\tilde{\delta}}$ (which includes the region $t - T_{c,a} \in \tilde{S}_{\tilde{\delta}}$).

Further, since for $t > T_1$ we have

$$\int_{T_1}^{\infty} \|v(\cdot, t)\|^2_{H^2_x} dt \leq \sup_{T > T_1} \|v\|^2_X \leq (2c_1\epsilon_0)^2,$$

it follows from a pigeon-hole argument that given ϵ_2, there exists a $T_2 > T_1$ such that

$$\|v(\cdot, T_2)\|_{H^2_x} < \epsilon_2.$$

From Lemma 12.7, it follows that v exists for $t - T_2 \in \tilde{S}_{\tilde{\delta}}$ and

$$\|v(\cdot, t)\|_{H^2_x} < 2c_1\epsilon_2 e^{-\frac{3}{4}\nu \mathrm{Re}(t - T_2)}.$$

The last part of the theorem follows from (recall $\hat{v}(0) = 0$)

$$\|\hat{v}(\cdot, t)\|_{l^1} \leq c_5 \||k|^2 \hat{v}(\cdot, t)\|_{l^2} \leq c_5 \|v(\cdot, t)\|_{H^2_x}.$$

∎

Remark 12.8. The decay rate $e^{-\frac{3}{4}\nu t}$ for $\|\hat{v}(\cdot, t)\|_{l^1}$ is not sharp. A more refined argument can be given, to estimate away the nonlinear terms and obtain a $e^{-\nu t}$ decay.

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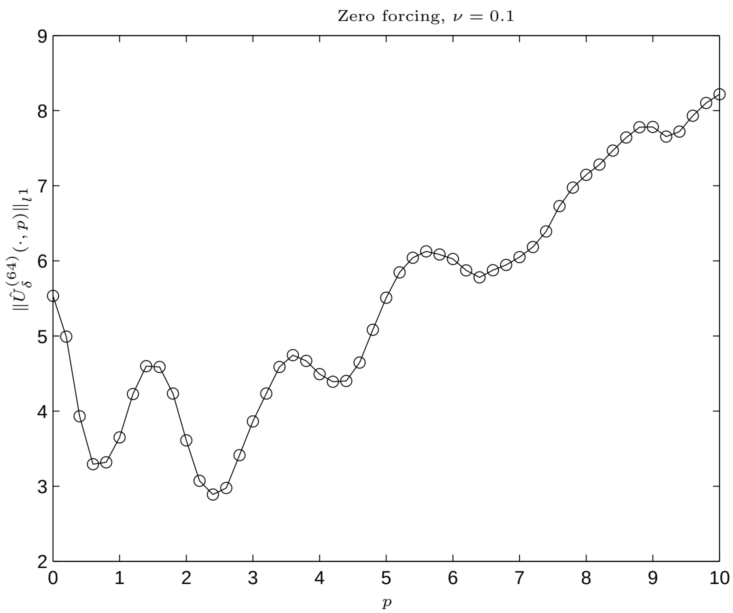

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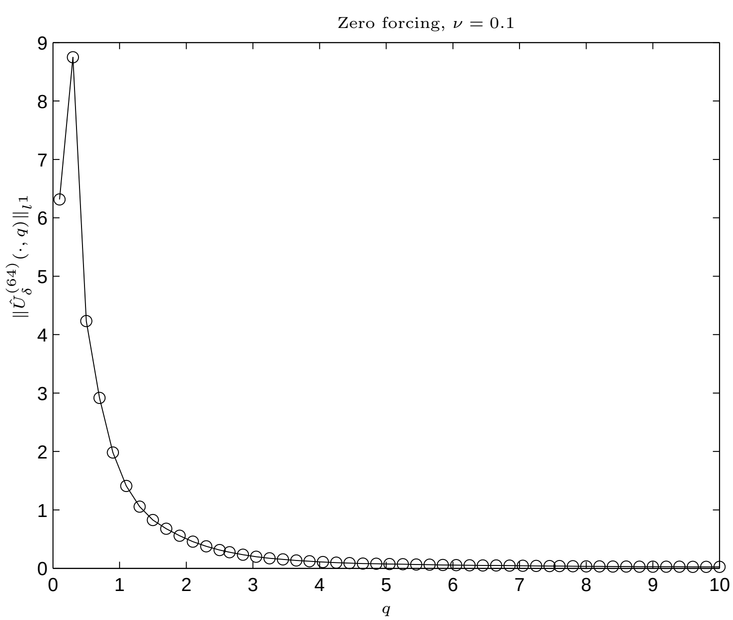
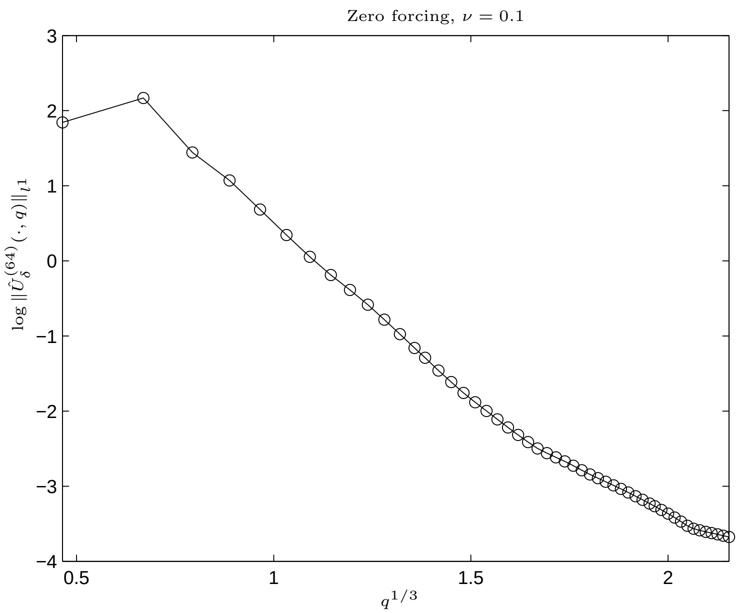
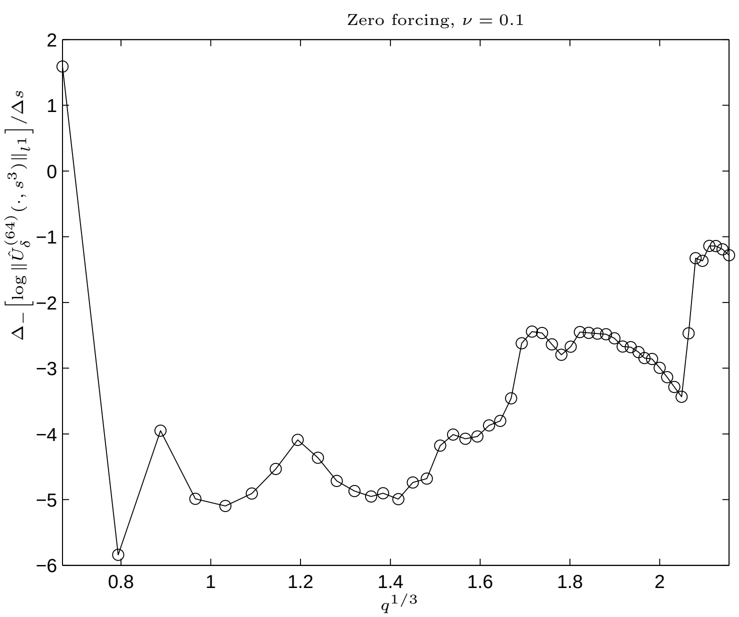


(b)

FIGURE 2. For zero forcing and $\nu = 0.1$: (a). The original (unaccelerated) equation, $\|\hat{U}_\delta^{(64)}(\cdot,p)\|_{l^1}$ vs. p. (b). Accelerated equation with $n = 2$, $\|\hat{U}_\delta^{(64)}(\cdot,q)\|_{l^1}$ vs. q.



(a)



(b)

FIGURE 3. Asymptotic behavior of $\|\hat{U}_\delta^{(64)}(\cdot, q)\|_{l^1}$. (a). $\log \|\hat{U}_\delta^{(64)}(\cdot, q)\|_{l^1}$ vs. $q^{1/3}$. (b) $\Delta_-\left[\log \|\hat{U}_\delta^{(64)}(\cdot, s^3)\|_{l^1}\right]/\Delta s$ vs. s, where $s = q^{1/3}$ and Δ_- is the backward difference operator in s.