Finals, Math 716, Tanveer, Due: March 16th, 5 p.m.

Instructions: No collaboration or any discussion of any kind relating to finals. Do not consult material outside of class notes, text and homework solutions. For clarifications, please send e-mail to me.

1. Calculate and plot the characteristic curves of

$$yu_x + xu_y = 0$$

where $(x, y) \in \mathbb{R}^2$. Afterwards, derive the general solution $u(x, y)$ under the additional assumption that u is even in x and y. On which of the coordinate axes do we have to impose initial conditions to make the solution unique.

2. Consider the problem of a vibrating drum confined in a sector geometry, described in polar coordinates (r, θ) by $r < 1$, $\theta \in (0, \alpha)$, with $\alpha < 2\pi$. Determine displacement u satisfying

$$u_{tt} = \Delta u \ \ , \ u(r, 0, t) = u(r, \alpha, t) = u(1, \theta, t) = 0 \ \ , \ \text{ and } u(r, \theta, 0) = f(r, \theta), \ u_t(r, \theta, 0) = g(r, \theta)$$

Hint: You may use the fact that $y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$ has independent solutions in the form of Bessel functions $J_\nu(x)$ and $Y_\nu(x)$, where ν need not be an integer. J_ν is the only one that is finite at the origin. You may use the general theory for eigenfunctions of $-\Delta$ for Dirichlet BC to deduce appropriate properties of the zeros $\{k_{n,\nu}\}_{n=1}^{\infty}$ of J_ν:

3. Solve for $u(x, t)$ in the domain $t > 0$ and $0 < x < 1$, where u satisfies:

$$u_t = u_{xx} \ ; \quad u_x(0, t) = 0 \ \ , \ \ u(1, t) = 1 \ ; \ \ u(x, 0) = x^2$$

Determine the $t \to \infty$ asymptotic behavior of $u(x, t)$.

4. Consider the following PDE for $u(x, y)$, for $(x, y) \in \Omega$, where Ω a bounded domain in $\mathbb{R}^2$ with $\partial\Omega$ appropriately regular.

$$u_{xx} + au_{xy} + u_{yy} + bu_x + cu_y = 0 \ ,$$

where a, b, c are constants. Determine restrictions on a, b and c that will make the problem elliptic. State and prove for appropriate set of values of a, b, c a weak maximum principle.

5. Suppose, we seek to find the eigenvalues of the operator $-\Delta + V(\mathbf{x})$, where $V(\mathbf{x})$ is a given function, with *Neumann* boundary conditions:

$$-\Delta u + Vu = \lambda u \ \text{ for } \mathbf{x} \in \Omega \subset \mathbb{R}^n \ , \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega$$

Formulate the appropriate minimization principle that will give rise to eigenvectors and eigenvalues for this problem. Assuming completeness, find a representation of the solution to the the initial value problem in Ω:

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi + V\Psi \ \ ; \ \ \Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x}) \ \ ; \ \ \frac{\partial \Psi}{\partial n} = 0 \ \text{ on } \partial\Omega$$