

Homework Set 1: Math 716, Due: Friday, January 23rd

1. Calculate and plot the characteristic curves for

$$u_{x_1} + 2u_{x_2} + (2x_1 - x_2)u = x_1x_2, \quad (x_1, x_2) \in \mathbb{R}^2$$

Afterwards, derive the general solution $u(x_1, x_2)$ and in particular find solution satisfying $u(0, x_2) = e^{x_2}$.

2. Solve

$$u_x - u_y + u = e^{x-2y}$$

with $u(x, 0) = 0$.

3. Use the method of characteristics to find representation for solution to

$$u_t + uu_x = u \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad u(x, 0) = e^{-x}$$

What is the restriction on t , if any, for solution to be smooth in x ?

4. By direct verification show that for any integer n , $u(x, t) = e^{-n}e^{\kappa n^2 t} \sin n(x - t)$ is a solution to the backwards heat equation with advection:

$$u_t + u_x = -\kappa u_{xx} \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \quad \text{with } u(x, 0) = e^{-n} \sin nx$$

Use this to prove that the solution is unstable to variations in initial conditions and therefore the problem is *not* well-posed in the $\|\cdot\|_\infty$ (sup norm in x).

5. A function $u(x, t)$, defined for $(x, t) \in \mathbb{R}^2$, is said to be a weak solution of the linear second order wave equation: $u_{tt} - c^2 u_{xx} = 0$ if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) [\phi_{tt}(x, t) - c^2 \phi_{xx}(x, t)] dx dt = 0$$

for all test functions $\phi(x, t)$ with compact support, *i.e.* for any smooth function $\phi(x, t)$ that is defined for $(x, t) \in \mathbb{R}^2$ and that vanishes outside a bounded region of \mathbb{R}^2 . Verify that

$$u(x, t) = f(x - ct) + g(x + ct)$$

is a weak solution for any continuous functions f and g .

Hint: You may think of transforming independent variables from $(x, t) \rightarrow (\xi, \eta)$, where $\xi = x - ct$, $\eta = x + ct$.