Homework Set 4: Math 716, Due Wednesday, February 18th

1. Using the two approaches listed below as (a) and (b), find two different representation of solution to the following Initial-Boundary value problem for the heat equation:

$$u_t - \kappa u_{xx} = 0 \;\text{ for }\; 0 < x < l \;\; ; \;\; u(x,0) = \phi(x) \;,\;\; u(0,t) = 0 \; = \; u(l,t)$$

   Prove that for $t > 0$, u is infinitely differentiable both in x and t. Which representation is suitable for evaluation of solution for small t? Which one is suitable for large t? Explain.

   (a) Use method of successive reflection of source solutions.

   (b) Separation of variables.

   For the separation of variable method, determine a suitable bound for the size of the maximum error uniformly for $t > 0$, if the series representation is truncated to N terms, when $\phi \in \mathbf{C}^1(0,1)$, with $\phi(0) = 0 = \phi(1)$.

2. Find solution to the problem of a circular vibrating membrane for 2-D :

$$u_{tt} - c^2 \Delta u = 0 \;\text{ for }\; |\mathbf{x}| < 1; \; u(\mathbf{x},0) = \phi(\mathbf{x}) \;,\; u_t(\mathbf{x},0) = 0, \;,\; u(\mathbf{x},t) = 0 \text{ for } |\mathbf{x}| = 1$$

   What is the restriction on ϕ needed? **Hint:** You may want to use the information that the solution to $v'' + \frac{1}{r} v' + \left(\lambda^2 - \frac{m^2}{r^2} \right) v(r) = 0$ is given by a linear combination of Bessel functions $J_m(\lambda r)$ and $Y_m(\lambda r)$, where Y_m blows up at the origin. J_m is regular at the origin and has countably infinite zeros on the real line similar to the sin function.

3. Find solution in series form to Laplace's equation in 2-D in an annular domain $a < |\mathbf{x}| < 1$:

$$\Delta u = 0 \;\; ; \;\; u(\mathbf{x}) = \phi(\mathbf{x}) \text{ for } |\mathbf{x}| = a \;\text{ and }\; u(\mathbf{x}) = \psi(\mathbf{x}) \;\text{ for } |\mathbf{x}| = 1$$

   Show that $u(\mathbf{x})$ is infinitely differentiable for $a < |\mathbf{x}| < 1$, even when boundary data ϕ and ψ are simply continuous.

4. Determine a representaton for solution to Laplace's equation in a semi-circular domain

$$\mathcal{H} := \left\{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1, x_2 > 0 \right\}$$

$$\Delta u \;=\; 0 \quad \text{with } \; u = 0 \;\text{ for } x_2 = 0, \; \frac{\partial u}{\partial n} + u = f(\mathbf{x}) \text{ on } |\mathbf{x}| = 1$$

   Make appropriate assumptions on f and apply whatever theory you need to justify your solution represention and to show that the solution is $\mathbf{C}^\infty$ inside $\mathcal{H}$.