Homework Set 5: Math 716, Due Monday, February 25th

1. Show that the partial differential operator $\mathcal{A}$ defined by:
$$\mathcal{A}u \equiv = -\nabla \cdot (p \nabla u) + qu$$
in a bounded $\Omega \subset \mathcal{R}^n$ for $p(\mathbf{x}) > 0$ is symmetric, with respect to the usual $\mathcal{L}_2$ inner-product. What condition on $q(\mathbf{x})$ makes $\mathcal{A}$ positive. Suppose we consider the eigenfunctions u satisfying
$$\mathcal{A}u = \lambda m u$$
for $m(\mathbf{x}) > 0$ in Ω. Prove the orthogonality of eigenfunctions corresponding to unequal eigenvalues with respect to inner-product $\langle .,. \rangle$ defined by
$$\langle u, v \rangle = \int_\Omega muv d\mathbf{x}$$

2. Show that minimization of
$$\frac{\|\nabla w\|^2 + \int_{\partial\Omega} a w^2(\mathbf{x}) d\mathbf{x}}{\|w\|^2}$$
for $w \neq 0$ for $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ leads to smallest eigenvalue and corresponding eigenfunction for $-\Delta$ operator with Robin boundary condition $\frac{\partial w}{\partial n} + aw = 0$ on $\partial\Omega$. What is the analogous minimization principle for n-the eigenvalue. Notice no boundary conditions on w on $\partial\Omega$, unlike the Dirichlet problem (where $w = 0$).

3. a. Use orthogonality of the eigenfunctions in the $\mathcal{L}_2$ sense of the operator $-\Delta$ with Robin-Boundary condition $\frac{\partial u}{\partial n} + au = 0$ ($a > 0$) in a disk of radius 1 in 2-D to prove the following properties of the Bessel-function
$$\int_0^1 r J_m(k_i r) J_m(k_j r) dr = 0 \quad \text{for } i \neq j$$
for any integer $m \geq 0$, where k_j is the j-th positive root of the transcendental equation:
$$k_j J_m'(k_j) + a J_m(k_j) = 0$$

   b. What does completeness of the eigen functions of $-\Delta$ with Robin boundary conditions imply about class of functions f expressible as:
$$\sum_{j=1}^{\infty} c_j J_m(k_j r)$$
How is c_j determined from f? How does k_j behave like as $j \to \infty$?

   Hint: You might want to use polar coordinates separation of variables to express eigenfunctions of $-\Delta$ in terms of Bessel-function J_m.

4. For a rectangular box in 3 dimensions show using explicit computation of eigenvalues through separation of variable of the operator $-\Delta$ with Neumann boundary conditions that
$$\lim_{n \to \infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{\text{Volume of rectangular box}}$$