Solution to Homework Set 1, Math 716

1. Calculate and plot the caracteristic curves for

$$u_{x_1} + 2u_{x_2} + (2x_1 - x_2)u = x_1 x_2 \quad , \quad (x_1, x_2) \in \mathbb{R}^2$$

Afterwards, derive the general solution $u(x_1, x_2)$ and in particular find solution satisfying $u(0, x_2) = e^{x_2}$.

Solution: Characteristic curves are determined from $\frac{dx_1}{dt} = 1$, $\frac{dx_2}{dt} = 2$ and are set of all straight lines with slope 2 in the (x_1, x_2) plane. (I am avoiding drawing the characteristics in this case since the set of straight lines with slope 2 is obvious.) Parametrically, they can be represented without loss of generality as $x_1 = t + g(s)$, $x_2 = 2t + s$, where $g(s)$ is the value of x_1 when $t = 0$. The initial curve Γ here is parametrized by $(g(s), s)$. By taking $g(s)$ arbitrary we can describe essentially all initial curves needed to characterize a general solution (In the nongeneric case when the initial curve is horizontal in the $x_1 - x_2$ plane, we can switch around the role of x_1 and x_2 and then the initial curve can be parametrized by $(s, g(s))$).

On the characteric curves, we have from the equation

$$\frac{du}{dt} = -(2x_1 - x_2)u - x_1 x_2 = -(2g(s) - s)u - (t + g(s))(2t + s)$$

or

$$\frac{du}{dt} + (2g(s) - s)u = 2t^2 + [s + 2g(s)]t + sg(s)$$

We use initial condition $u(0) = f(s)$, where $f(s)$ is the initial value of u on the initial curve Γ, characterized by s. This ordinary differential equation can be solved for fixed s by first seeking a *particular* solution in the form of a quadratic in t and then using the fact that the associated homogeneous solution has a solution in the form $\exp[-(2g(s) - s)t]$. One obtains through in a straight forward (but laborious) manner:

$$u = U(t, s) = A(s) \exp[-(2g - s)t] + \frac{1}{(-2g + s)^3} \times \big(-2(s - 2g)^2 t^2 \\ + (2g - s)(s^2 - 4g^2 + 4)t + 4s^2 g^2 - s^3 g + 4g^2 - 4sg^3 - 4 - s^2 \big)$$

where

$$f(s) = A(s) + \frac{4s^2[g(s)]^2 - s^3 g(s) + 4[g(s)]^2 - 4s[g(s)]^3 - 4 - s^2}{(-2g(s) + s)^3}$$

The general solution is given by $u(x_1, x_2) = U(T(x_1, x_2), S(x_1, x_2))$ where $(T(x_1, x_2), S(x_1, x_2))$ is obtained by inverting the relation $(x_1, x_2) = (t + g(s), 2t + s)$.

When the initial curve Γ is given by $(0, x_2)$, in parametric form, this corresponds to $x_{2,0} = s$, $x_{1,0} = 0 = g(s)$. Further, the relation determining $A(s)$ becomes,

$$f(s) = e^s = A(s) - \frac{4 + s^2}{s^3} \quad ; \quad \text{implying } A(s) = e^s + \frac{4 + s^2}{s^3}$$

From the relation, $(x_1, x_2) = (t, 2t+s)$, we obtain the inversion $(T(x_1,x_2), S(x_1,x_2)) = (x_1, x_2 - 2x_1)$. So, from the general expression for $U(t,s)$, for $g(s) = 0$ and $A(s)$ as above, we obtain

$$U(t,s) = \left(e^s + \frac{4+s^2}{s^3}\right) e^{st} + \frac{1}{s^3}\left[-2s^2t^2 - s(s^2+4)t - 4 - s^2\right]$$

Substituting $(t,s) = (T(x_1,x_2), S(x_1,x_2)) = (x_1, x_2 - 2x_1)$, we obtain the solution to the particular problem posed.

2. Solve

$$u_x - u_y + u = e^{x-2y}$$

with $u(x,0) = 0$.

Solution: Using the method of characteristics,

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -1 \quad , \quad \frac{du}{dt} = -u + e^{x-2y}$$

Therefore, $x = s + t$, $y = -t$, where $(x,y) = (s,0)$ is the parameterization of the initial curve Γ where data is specified. For fixed s, we obtain

$$\frac{du}{dt} + u = \mathrm{e}^{s+3t} \text{ with } u(0) = 0 \text{ whose solution is } \quad u = U(t;s) = \frac{1}{4}\left(\mathrm{e}^{s+3t} - \mathrm{e}^{s-t}\right)$$

But, since $x = s + t$, $y = -t$, it follows $t = -y = T(x,y)$ and $s = x + y = S(x,y)$. Therefore, solution to the initial value problem is

$$u(x,y) = \frac{1}{4}\left(\mathrm{e}^{x-2y} - \mathrm{e}^{x+2y}\right)$$

3. Use the method of characteristics to find representation for solution to

$$u_t + uu_x = u \text{ for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \quad u(x,0) = e^{-x}$$

What is the restriction on t, if any, for solution to be smooth in x?

Solution We take initial curve parameter to be s, so that $(x,y) = (s,0)$ on the initial curve. Using the method of characteristics, we have for fixed s,

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u \quad , \quad \frac{du}{d\tau} = u$$

Solving for u first, with initial condition $u(0) = e^{-s}$ we obtain $u = U(\tau;s) = e^{-s}e^{\tau}$. Substituting into the equation for x, we obtain

$$\frac{dx}{d\tau} = e^{-s}e^{\tau}, \quad \text{with} \quad x(0) = s$$

implying

$$x = X(\tau;s) = e^{-s}\left(e^{\tau} - 1\right) + s$$

Also, the equation for t gives $t = \tau$, since $t(0) = 0$. Therefore, in the above, we obtain

$$x = X(t; s) = e^{-s}\left(e^t - 1\right) + s \tag{1}$$

We need to invert this, using implicit function theorem to solve for $s = S(x,t)$. This is possible when t is small enough so that $X_s = -e^{-s}(e^t - 1) + 1 > 0$. Once $s = S(x,t)$ is found by inverting (1), we obtain solution

$$u(x,t) = e^{-S(x,t)} e^t$$

The smallest value of t, call it t_0, for which the inversion fails is given by

$$t_0 = \log\left[1 + e^s\right]$$

Notice this has as $s \to -\infty$, $t_0 \to 0$; in otherwords, there is no time interval over which the solution is smooth uniformly for all x. This is not the case, if x were restricted to some finite interval.

4. By direct verification show that for any integer n, $u(x,t) = e^{-n} e^{\kappa n^2 t} \sin n(x - t)$ is a solution to the backwards heat equation with advection:

$$u_t + u_x = -\kappa u_{xx} \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \;\text{ with }\; u(x,0) = e^{-n} \sin nx$$

Use this to prove that the solution is unstable to variations in initial conditions and therefore the problem is *not* well-posed in the $\|.\|_\infty$ (sup norm in x).

Solution: On plugging in, it is readily verified that

$$u_t + u_x = \kappa n^2 e^{-n} e^{\kappa n^2 t} \sin n(x - ct) = -\kappa u_{xx}$$

Therefore, for any n, $u(x,t)$ is a solution to the *backwards* heat equation with advection. It is easy to verify $u(x,0) = e^{-n} \sin\ nx$ and $\|u(.,0)\|_\infty$ tends to 0 as $n \to +\infty$. We claim that there exists some ϵ_0 ($\epsilon_0 = 1$ suffices here), so that for any $\delta > 0$ and $t > 0$, there exists soluton $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ so that

$$\|\tilde{u}(.,0) - \hat{u}(.,0)\|_\infty < \delta; \quad \text{but} \quad \|\tilde{u}(.,t) - \hat{u}(.,t)\|_\infty \geq \epsilon_0 \tag{2}$$

This is true because we choose $\tilde{u}(x,t) = 0$ and $\hat{u}(x,t) = u(x,t)$ for n large enough so that $\|e^{-n} \sin nx\|_\infty = e^{-n} < \delta$, but $\|e^{-n} e^{\kappa n^2 t} \sin n(x - ct)\|_\infty = e^{-n} e^{\kappa n^2 t} > 1 = \epsilon_0$. Clearly this is possible for n sufficiently large. Therefore, the backwards heat equation with advection is not stable to small perturbation of initial condition, and hence is ill-posed.

5. A function $u(x,t)$, defined for $(x,t) \in \mathbb{R}^2$, is said to be a weak solution of the linear second order wave equation: $u_{tt} - c^2 u_{xx} = 0$ if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) \left[\phi_{tt}(x,t) - c^2 \phi_{xx}(x,t)\right] dx dt = 0$$

for all test functions $\phi(x,t)$ with compact support, *i.e.* for any smooth function $\phi(x,t)$ that is defined for $(x,t) \in \mathbb{R}^2$ and that vanishes outside a bounded region of $\mathbb{R}^2$. Verify that

$$u(x,t) = f(x - ct) + g(x + ct)$$

is a weak solution for any continuous functions f and g.

Solution We introduce change of variable $(x, y) \rightarrow (\xi, \eta) \equiv (x - ct, x + ct)$. Note the Jacobian of the transformation $= 2c$. Then, if we define $\phi(x(\xi, \eta), t(\xi, \eta)) = \Phi(\xi, \eta)$, it follows from chain rule,

$$\phi_x = \Phi_\xi + \Phi_\eta \quad , \quad \phi_{xx} = \Phi_{\xi\xi} + 2\Phi_{\xi\eta} + \Phi_{\eta\eta}$$

$$\phi_t = -c\Phi_\xi + c\Phi_\eta \quad , \quad \phi_{tt} = c^2\Phi_{\xi\xi} - 2c^2\Phi_{\xi\eta} + c^2\Phi_{\eta\eta}$$

Therefore,

$$\phi_{tt} - c^2\phi_{xx} = -4c^2\Phi_{\xi\eta}$$

Therefore, for any test function Φ that is smooth and has compact support in the (x, t) and therefore (ξ, η) plane, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\phi_{tt} - c^2\phi_{xx}\right] f(x - ct)dxdt = -2c\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Phi_{\xi\eta}d\eta\right] f(\xi)d\xi = 0$$

since the first integration is zero, as Φ and its derivatives are zero outside a compact interval in η. Again

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\phi_{tt} - c^2\phi_{xx}\right] g(x + ct)dxdt = -2c\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Phi_{\xi\eta}d\xi\right] g(\eta)d\eta = 0$$

since the first integration is zero, as Φ and its derivatives are zero outside a compact interval in ξ. Therefore, for $u(x, t) = f(x - ct) + g(x + ct)$, with f, $g \in \mathbf{C}^0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\phi_{tt} - c^2\phi_{xx}\right] u(x, t)dxdt = 0$$

and we have demonstrated it is a weak solution to the linear wave equation.