Homework Set 2: Math 716, Due: Friday, January 26th

1. Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

for $x \in \mathbb{R}$ with $c \neq 0$. Assume $\phi \in \mathbf{C}^2$ and $\psi \in \mathbf{C}^1$ have compact support (*i.e.* there is a number $R > 0$ such that $\phi(x) = \psi(x) = 0$ for $|x| > R$). Denote d'Alembert's solution of (1) by $u(x,t)$. Using d'Alembert's solution, show that the solutions to (1) depend in a stable fashion on the initial data ϕ and ψ. More precisely: if u_1 and u_2 satisfy (1) with initial data (ϕ_1, ψ_1) and (ϕ_2, ψ_2), respectively, show that

$$\sup_{x\in\mathbb{R}} |u_1(x,t) - u_2(x,t)| \leq C(t) \left[\sup_{x\in\mathbb{R}} |\phi_1(x) - \phi_2(x)| + \sup_{x\in\mathbb{R}} |\psi_1(x) - \psi_2(x)| \right]$$

for some constant $C(t)$ that may depend on t, but not on the initial data. What is a good estimate for $C(t)$? Can you find similar estimates for $\partial_x(u_1 - u_2)$ and $\partial_{xx}(u_1 - u_2)$?

Solution: Consider $v = u_1 - u_2$. It satisfies

$$v_{tt} - c^2 v_{xx} = 0, \quad v(x,0) = \phi_1(x) - \phi_2(x) \equiv \phi(x), \quad v_t(x,0) = \psi_1(x) - \psi_2(x) \equiv \psi(x)$$

We know from D'Alembert solution

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

So,

$$|v(x,t)| \leq \|\phi\|_\infty + \|\psi\|_\infty \frac{1}{2c}\int_{x-ct}^{x+ct} dy \leq (1+t)\left[\|\phi\|_\infty + \|\psi\|_\infty\right] \qquad (1)$$

from which the result follows, with $C(t) = 1+t$. A stricter choice of $C(t)$ will be $\max\{1, t\}$. These results even for ψ without compact support. However, since in our problem $\psi(y) = 0$ for $|y| > R$, we have

$$\frac{1}{2c}\int_{x-ct}^{x+ct} |\psi(y)|dy = \min\left\{t, \frac{R}{c}\right\} \|\psi\|_\infty$$

Therefore, a strict choice of $C(t)$ would be

$$C(t) = \max\left[1, \min\left\{t, \frac{R}{c}\right\}\right]$$

For estimates on $\partial_x(u_1 - u_2)$ and $\partial_{xx}(u_1 - u_2)$, we just take $v = \partial_x(u_1 - u_2)$ (or $v = \partial_{xx}(u_1 - u_2)$), and it is clear that it solves the wave equation, with $v(x,0) = \phi(x) = \partial_x\phi_1 - \partial_x\phi_2$ and $v_t(x,0) = \psi(x) = \partial_x\psi_1 - \partial_x\partial_x\psi_2$ (or similar expression with second derivatives). Then, using (1), we get very similar bounds for $\partial_x(u_1 - u_2)$ (or $\partial_{xx}(u_1 - u_2)$, except the estimate on the right hand side involves first (or second) x-derivatives of $\phi_1 - \phi_2$ and $\psi_1 - \psi_2$.

2. Find an integral representation of the solution to advection-diffusion equation

$$u_t + a u_x = u + \kappa u_{xx} \quad , -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \phi(x) \quad , -\infty < x < \infty$$

where a and κ are constants.

Solution: With change of variable $u(x,t) = e^t v(\xi, t)$, where $\xi = x - at$, we get $u_t = e^t v + e^t [v_\xi \xi_t + v_t] = e^t (v - a v_\xi + v_t)$. $u_x = e^t v_\xi \xi_x = e^t v_\xi$. $u_{xx} = e^t v_{\xi\xi} \xi_x = e^t v_{\xi\xi}$. Therefore,

$$0 = u_t + a u_x - u - \kappa u_{xx} = e^t \left\{ (v - a v_\xi + v_t) + a v_\xi - v - \kappa v_{\xi\xi} \right\}$$

Therefore, we have

$$v_t - \kappa v_{\xi\xi} = 0 \ \text{ for } \xi \in \mathbb{R} \ \ , v(\xi, 0) = u(\xi, 0) = \phi(\xi)$$

Therefore,

$$v(\xi, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{\mathbb{R}} \exp\left[-\frac{(\xi - y)^2}{4\kappa t} \right] \phi(y) dy$$

Hence, solution of the advection-diffusion equation is:

$$u(x,t) = \frac{e^t}{\sqrt{4\pi\kappa t}} \int_{\mathbb{R}} \exp\left[-\frac{(x - ct - y)^2}{4\kappa t} \right] \phi(y) dy$$

3. Consider the equation

$$u_t = u_{xx} \qquad 0 < x < 1, \quad t \ > \ 0$$

$$u(0,t) = 0 = u(1,t) \qquad t > 0$$

$$u(x,0) = 4x(1-x) \qquad 0 < x < 1$$

Show that:

1) $0 < u(x,t) < 1$ for all $0 < x < 1$ and $t > 0$.

2) $u(x,t) = u(1-x,t)$ for all $0 < x < 1$ and $t > 0$.

3) The energy $\int_0^1 u^2(x,t) dx$ decreases strictly with t for $t > 0$.

Solution: Note from simple calculus, $\sup_{x \in (0,1)} 4x(1-x) = 1$, and $\inf_{x \in (0,1)} 4x(1-x) = 0$. Therefore in the set

$$\Sigma \equiv \{(x,t) : t = 0 \ \text{ or } \ x = 0 \ \text{ or } \ x = 1\}$$

$\sup_{(x,t)\in\Sigma} u(x,t) = 1$ and $\inf_{(x,t)\in\Sigma} u(x,t) = 0$. From strong maximum principle, for $(x,t) \in (0,1) \times (0,T)$ for any T, $0 < u(x,t) < 1$.

For second part note that $\tilde{u}(x,t) = u(1-x,t)$ solves the same diffusion equation, since replacing x by $-x$, followed by shift of 1 still leads to solution of diffusion equation. However, $\tilde{u}(0,t) = 0 = u(1,t)$, and $\tilde{u}(1,t) = 0 = u(0,t)$ and initially, $\tilde{u}(x,0) = u(1-x,0) = 4(1-x)x = 4x(1-x) = u(x,0)$. Therefore, $\tilde{u}$ is the solution to the same initial boundary value problem as does $u(x,t)$. From uniqueness results proved in class for classical solutions, we obtain $u(1-x,t) = \tilde{u}(x,t) = u(x,t)$.

For third part, we note that

$$\frac{d}{dt}\int_0^1 u^2(x,t)dx = 2\int_0^1 uu_t dx = 2\int_0^1 uu_{xx} = -2\int_0^1 u_x^2(x,t)dx < 0$$

unless u is identically constant in x for some $t > 0$. If it is a constant, then it has to be 0, since as $x \to 0^+$, $u(x,t) \to 0$. But, $u(x,t) = 0$ for $t > 0$ is incompatible with $0 < u(x,t) < 1$ proved in the first part.

4. Prove the comparison principle: If $u(x,t)$ and $v(x,t)$ are two solutions of the heat equation for $0 < x < l$ and $t \geq 0$ such that $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for all $0 \leq x \leq l$ and $t > 0$.

   Solution: Define $w(x,t) = v(x,t) - u(x,t)$. As before, define

   $$\Sigma \equiv \{(x,t) : t = 0 \quad \text{or} \quad x = 0 \quad \text{or} \quad x = l\}$$

   and

   $$\mathcal{D} \equiv \{(x,t) : t > 0 \quad , \quad x \in (0,l)\}$$

   Then with given information, $\sup_{(x,t)\in\Sigma} w(x,t) \geq 0$. From maximum principle, $\sup_{(x,t)\in\mathcal{D}} w(x,t) \geq 0$. Therefore, $u(x,t) \leq v(x,t)$ for $0 \leq x \leq l$, $t > 0$.