Homework Set 3: Math 716, Due: Wednesday, February 7th

1. Use energy method to prove uniqueness of classical solution to the initial value problem for the damped wave equation ($\epsilon > 0$):

   $$u_{tt}+\epsilon u_t = \Delta u \quad \text{for } \mathbf{x} \in \Omega \subset \mathbb{R}^n, \ t > 0 \,, \text{with } \ u(\mathbf{x},0) = \phi(\mathbf{x}), \ \ u_t(\mathbf{x},0) = \psi(\mathbf{x}) \ \ , \ \ u(\mathbf{x},t) = 0 \text{ on } \partial\Omega$$

   Solution: Define the Energy the same way as for the undamped ($\epsilon = 0$ case) wave equation:

   $$E(t) = \frac{1}{2} \int_{\Omega} \left[u_t^2 + (\nabla u)^2 \right] dx$$

   Then, using equation,

   $$\frac{dE}{dt} = \int_{\Omega} \left[u_t u_{tt} + [\nabla u] \cdot [\nabla u_t] \right] dx = \int_{\Omega} \left[u_t \Delta u - \epsilon u_t^2 + [\nabla u] \cdot [\nabla u_t] \right] dx$$

   However,

   $$u_t \Delta u + [\nabla u] \cdot [\nabla u_t] = \nabla \cdot [u_t \nabla u]$$

   Assuming $\partial\Omega$ is regular enough to apply divergence theorem, we obtain from the boundary condition $u(x,t) = 0$ on $\partial\Omega$ (and hence $u_t(x,t) = 0$ on $\partial\Omega$), we obtain

   $$\int_{\Omega} \left[u_t \Delta u + [\nabla u] \cdot [\nabla u_t] \right] dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} u_t dx = 0$$

   Therefore,

   $$\frac{dE}{dt} = -\epsilon \int_{\Omega} u_t^2 dx \leq 0$$

   Therefore,

   $$E(t) \leq E(0)$$

   To prove uniqueness, consider two solutions u_1, u_2 of the wave equation satisfying homogeneous BC on $\partial\Omega$ and same initial condition. Then $u = u_1 - u_1$ will satisfy zero initial and boundary conditions. Hence from the above calculation

   $$E(t) \leq E(0) = \frac{1}{2} \int_{\Omega} \left[u_t^2(x,0) + (\nabla u)^2(x,0) \right] dx = 0$$

   Therefore, $E(t) = 0$ implying from the expression for E that for any classical solution, $u_t(x,t) = 0 = \nabla u(x,t)$. This means $u(x,t) =$ constant, independent of x and t. But since $u(x,0) = 0$, $u(x,t) = 0$, proving uniqueness of initial value problem.

2. Find representation of solution to heat equation with Neumann boundary conditions on a half-line

   $$u_t = u_{xx} \quad 0 < x < \infty, \ t \ > \ 0, \quad \text{with} \ \ u(x,0) = 0, \ \ u_x(0,t) = \sin \ t$$

   Solution: We note that $v(x,t) = -e^{-x} \sin t$ satisfies given Boundary condition, though not the PDE. Nonethless, define $w(x,t) = u(x,t) - v(x,t)$. Then, $w_x(0,t) = 0$, $w(x,0) = u(x,0) - v(x,0) = 0$, and

   $$w_t - w_{xx} = -v_t + v_{xx} = e^{-x} (\cos t - \sin t) \equiv f(x,t)$$

Now, use Duhammel's principle shown in class for the Neumann boundary condition. We have with $f(x,t)$ as defined above, and $S(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right]$,

$$w(x,t) = \int_0^t \left\{ \int_0^\infty \left[S(x-y,t-\tau) + S(x+y,t-\tau)\right] f(y,\tau) dy \right\} d\tau$$

3. Find a representation for the solution to wave equation on a half-line

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad x \in \mathbb{R}^+, t \in \mathbb{R}$$

with initial and boundary conditions:

$$u(x,0) = \sin x \quad ; \quad u_t(x,0) = 0 \quad ; \quad u_x(0,t) = 1$$

Solution:

We note that if we decompose $u(x,t) = \sin x + v(x,t)$, then $v_x(0,t) = 0$, while $v(x,0) = 0$, $v_t(x,0) = 0$ and

$$v_{tt} - c^2 v_{xx} = -c^2 \sin x$$

We do an even extension $f(x)$ of the function $-c^2 \sin x$ to $\mathbb{R}$, i.e

$$f(x) = -c^2 \sin |x|$$

Note that the even extension leaves $f \in \mathbf{C}^0(\mathbb{R})$, but $f \notin \mathbf{C}^1(\mathbb{R})$. So, the solution given below is to be understood in a weak sense (see first homework set). Then applying Duhammel's principle on $\mathbb{R}$ as shown in the next exercise, we obtain

$$v(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y) dy ds$$

We can directly verify

$$v(x,0) = 0 \quad ; \quad v_t(x,0) = 0 \quad ; \quad v_x(0,t) = 0$$

Therefore,

$$u(x,t) = \sin x + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y) dy ds$$

4. Find a solution to the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

for $x \in \mathbb{R}$ with $c \neq 0$. Assume $\phi \in \mathbf{C}^2$, $\psi \in \mathbf{C}^1$, and $f \in \mathbf{C}^0$ are given bounded functions.

Solution: In the spirit of applying Duhammel's principle let's first solve the homogeneous problem:

$$v_{tt} - c^2 v_{xx} \quad = 0 \quad t > \tau \quad ;\text{with} \quad v(x,\tau) = 0 \quad ; \quad v_t(x,\tau) = f(x,\tau)$$

$$v(x,t;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)dy$$

We now claim that $u_p(x,t) = \int_0^t v(x,t;\tau)$ is a particular solution to the inhomogeneous problem satisfying $u_p(x,0) = 0$ and $\partial_t u_p(x,0) = 0$. To prove this, we note that

$$\partial_t u_p(x,t) = v(x,t;t) + \int_0^t v_t(x,t;\tau)d\tau = \int_0^t v_t(x,t;\tau)d\tau$$

$$\partial_{tt} u_p(x,t) = v_t(x,t;t) + \int_0^t v_{tt}(x,t;\tau)d\tau = f(x,t) + \int_0^t v_{tt}(x,t;\tau)d\tau$$

$$\partial_{xx} u_p(x,t) = \int_0^t v_{xx}(x,t;\tau)$$

So,

$$\partial_{tt} u_p - c^2 \partial_{xx} u_p = f(x,t)$$

Further, $u_p(x,0) = \int_0^0 v(x,0;\tau)d\tau = 0$, and

$$\partial_t u_p(x,0) = \int_0^0 v_t(x,0;\tau)d\tau + v(x,t;t) = 0$$

Therefore, to find solution of given initial value problem, we decompose

$$u(x,t) = u_p(x,t) + w(x,t)$$

Then $w(x,t)$ solves

$$w_{tt} - c^2 w_{xx} \quad = 0 \quad t > 0 \quad ;\text{with} \;\; v(x,0) = \phi(x) \;\; ; \; v_t(x,\tau) = \psi(x)$$

D'Alembert solution is

$$w(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

Therefore, we have solution to the initial value problem:

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,s)dyd\tau + \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

5. Prove the weak maximum principle for Laplace's equation. Thus, assume that $\mathcal{D}$ is an open and bounded subset of $\mathbb{R}^n$ with boundary ∂D. Assume also that $u(\mathbf{x})$ is a solution to Laplace's equation $\Delta u = 0$ in $\mathcal{D}$ and that u is continuous on $\bar{\mathcal{D}}$, twice differentiable in D. Show that

$$\sup_{\bar{D}} u = \sup_{\partial D} u$$

Solution:

Let $u(\mathbf{x}$ be any solution to $\Delta u = 0$. Consider for $\epsilon > 0$

$$v_\epsilon(\mathbf{x}) = \epsilon|\mathbf{x}|^2 + u(\mathbf{x})$$

We claim first that v_ϵ cannot attain an interior maximum. First note that

$$\Delta v_\epsilon = 2n\epsilon > 0$$

If $\mathbf{x}_m \in \mathcal{D}$ is an interior maximum, then (see class note discussion in the context of heat equation)

$$\Delta v_\epsilon(\mathbf{x}_m) \leq 0$$

The contradiction above implies that no interior maximum is possible for v_ϵ. So, in particular

$$\sup_{\mathbf{x}\in\mathcal{D}} v_\epsilon \leq \sup_{\mathbf{x}\in\partial D} v_\epsilon \leq \sup_{\mathbf{x}\in\partial D} u + \sup_{\mathbf{x}\in\partial D} \epsilon|\mathbf{x}|^2$$

Taking the limit of $\epsilon \to 0^+$, we get

$$\limsup_{\epsilon\to 0^+} \sup_{\mathbf{x}\in\mathcal{D}} v_\epsilon \leq \sup_{\mathbf{x}\in\partial D} u$$

However, since $u(\mathbf{x}) < v_\epsilon$ for any $\epsilon > 0$,

$$\sup_{\mathbf{x}\in\partial\mathcal{D}} u(\mathbf{x}) \leq \limsup_{\epsilon\to 0^+} \sup_{\mathbf{x}\in\mathcal{D}} v_\epsilon \leq \sup_{\mathbf{x}\in\partial D} u$$

and hence the weak maximum principle follows.

6. a. Prove that if there exists a solution of the Neumann problem

$$\Delta u = f \ \text{ for } \ x \in \mathcal{D} \subset \mathbb{R}^n, \ \ \frac{\partial u}{\partial n} = h(x) \ \text{ for } \ x \in \partial D,$$

then it is unique up to adding an arbitrary constant.

b. Consider the Robin problem

$$\Delta u = f \ \text{ for } \ x \in \mathcal{D}, \ \ \frac{\partial u}{\partial n} + a(x)u = h(x) \ \text{ for } \ x \in \partial D$$

Show that its solutions are unique.

Solution, Part a: Assume u_1, u_2 be wo solutions to the problem. Define $v = u_1 - u_2$. Then from linearity of the problem v satisfies

$$\Delta v = 0 \ \text{ for } \ x \in \mathcal{D} \subset \mathbb{R}^n, \ \ \frac{\partial v}{\partial n} = 0 \ \text{ for } \ x \in \partial D,$$

Now, multiplying $\Delta v = 0$ by v and integrating in $\mathbf{x}$ over $\mathcal{D}$, we get

$$0 = \int_{\mathcal{D}} v \Delta v d\mathbf{x} = \int_{\mathcal{D}} \nabla \cdot (v \nabla v)\, d\mathbf{x} = \int_{\partial D} v \frac{\partial v}{\partial n} d\mathbf{x} - \int_{\mathcal{D}} (\nabla v)^2 d\mathbf{x} = -\int_{\mathcal{D}} (\nabla v)^2 d\mathbf{x}$$

This implies that $\nabla v = 0$ for all $\mathbf{x} \in \mathcal{D}$ since for classical solutions $\nabla v \in \mathbf{C}^0(\mathcal{D})$. So v is at most a constant.

Solution, Part b: Assume u_1, u_2 be wo solutions to the problem. Define $v = u_1 - u_2$. Then from linearity of the problem v satisfies

$$\Delta v = 0 \quad \text{for} \quad x \in \mathcal{D} \subset \mathbb{R}^n, \quad \frac{\partial v}{\partial n} + a(\mathbf{x}) v = 0 \quad \text{for} \quad x \in \partial D,$$

Now, multiplying $\Delta v = 0$ by v and integrating in $\mathbf{x}$ over $\mathcal{D}$, we get

$$\begin{aligned} 0 = \int_{\mathcal{D}} v \Delta v d\mathbf{x} &= \int_{\mathcal{D}} \nabla \cdot (v \nabla v)\, d\mathbf{x} = \int_{\partial D} v \frac{\partial v}{\partial n} d\mathbf{x} - \int_{\mathcal{D}} (\nabla v)^2 d\mathbf{x} \\ &= - \int_{\partial D} a(\mathbf{x}) v^2 d\mathbf{x} - \int_{\mathcal{D}} (\nabla v)^2 d\mathbf{x} \end{aligned}$$

This is impossible for $a(\mathbf{x}) > 0$ for any $\nabla v \in \mathbf{C}^0(\mathcal{D})$, unless $v = 0$. Note that unlike part a, the constant solution is ruled out by the term $-\int_{\partial D} a(\mathbf{x}) v^2 d\mathbf{x}$, which would otherwise be negative.