Homework Set 4: Math 716, Due Friday, February 16th

1. Using the two approaches listed below as (a) and (b), find two different representation of solution to the following Initial-Boundary value problem for the heat equation:

$$u_t - \kappa u_{xx} = 0 \text{ for } 0 < x < l \ ; \ u(x,0) = \phi(x) \ , \ u(0,t) = 0 \ = \ u(l,t)$$

Prove that for $t > 0$, u is infinitely differentiable both in x and t. Which representation is suitable for evaluation of solution for small t? Which one is suitable for large t? Explain.

(a) Use method of successive reflection of source solutions.

(b) Separation of variables.

For the separation of variable method, determine a suitable bound for the size of the maximum error uniformly for $t > 0$, if the series representation is truncated to N terms, when $\phi \in \mathbf{C}^1[0,l]$, with $\phi(0) = 0 = \phi(l)$.

Solution: Method a: Define as in class, $S(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{x^2}{4\kappa t}\right]$. Then, if was source at a location y, the correponding solution would be $S(x-y,t)$. Now we go through the reflection process in the following sequence:

(a) Reflection about origin of source at $x = y$ causes a sink (source with negative sign) at $x = -y$.

(b) Reflection about $x = l$ of source at y causes a sink at $x = 2l - y$.

(c) Reflection about $x = l$ of sink at $x = -y$ causes a source at $x = 2l + y$.

(d) Reflection about origin of sink at $x = 2l - y$ causes a source $x = -2l + y$.

(e) Keep going.

Collecting the effect of all sources and sinks, we get a source solution that satisfies the Dirichlet boundary conditions at $x = 0$ and $x = l$:

$$\mathcal{K}(x,y;t) = S(x-y,t) - S(x+y,t) - S(x-(2l+y),t) + S(x-(2l+y),t) + S(x-(y-2l),t) + ..$$

$$= \sum_{j=-\infty}^{\infty} S(x-y+2jl,t) - \sum_{j=-\infty}^{\infty} S(x+y+2jl,t)$$

We will prove that this series converges for any $t > 0$ and further satisfies zero boundary condition at $x = 0$ and $x = l$.

We note that for $x, y \in (0,l)$ that for $|j| \geq 2$, $\frac{(x \pm y + 2jl)^2}{4\kappa t} > \frac{(|j|-1)l}{\kappa t}$, and so

$$\exp\left[-\frac{(x \pm y + 2jl)^2}{4\kappa t}\right] \ < \ \exp\left[-\frac{(|j|-1)l}{\kappa t}\right]$$

Thus, the series for $\mathcal{K}$ is bounded above by a convergent geometric series, which is independent of x and y. Hence it follows we have the series $\mathcal{K}(x,y;t)$ converging both absolutely and uniformly in $x, y \in [0,l]$ for any fixed $t > 0$. Clearly

$$\mathcal{K}(0,y;t) = \sum_{j=-\infty}^{\infty} S(-y+2jl,t) - \sum_{j=-\infty}^{\infty} S(y+2jl,t) = \sum_{j'=-\infty}^{\infty} S(-y-2j'l,t) - \sum_{j=-\infty}^{\infty} S(y+2jl,t) = 0$$

since $S(x,t)$ is even. Further,

$$\mathcal{K}(l,y;t) = \sum_{j=-\infty}^{\infty} S(l-y+2jl,t) - \sum_{j=-\infty}^{\infty} S(l+y+2jl,t) = \sum_{j'=-\infty}^{\infty} S(-l-y+2j'l,t) - \sum_{j=-\infty}^{\infty} S(l+y+2jl,t)$$

$$= \sum_{\hat{j}=-\infty}^{\infty} S(-l-y-2\hat{j}l,t) - \sum_{j=-\infty}^{\infty} S(l+y+2jl,t) = 0$$

Therefore, having produced a source solution that satisfies boundary conditions, it follows from discussions in class that solution to the initial value problem is

$$u(x,t) = \int_0^l \mathcal{K}(x,y;t)\phi(y)dy \tag{1}$$

To prove directly that this satisfies heat equation, we need to justify taking the x-derivatives inside the integral and inside the summation. We note that there exists constant A, so that

$$|\frac{e^{-\eta-h} - e^{-\eta}}{h}| \le Ae^{-\eta} \quad \text{for } h > 0$$

$$|\frac{e^{-\eta-h} - e^{-\eta}}{h}| \le Ae^{-\eta-h} \quad \text{for } h < 0$$

Using this, we get for $x, y \in (0,l)$,

$$|\frac{1}{\delta}\left\{\exp\left[-\frac{(x+\delta \pm y+2jl)^2}{4\kappa t}\right] - \exp\left[-\frac{(x+\pm y+2jl)^2}{4\kappa t}\right]\right\}| \le (A+B|j|)\exp\left[-\frac{(|j|-1)^2 l^2}{\kappa t}\right]$$

Since for any fixed $t > 0$

$$\sum_j (A+B|j|)\exp\left[-\frac{(|j|-1)^2 l^2}{\kappa t}\right] < \infty$$

It follows that $\lim_{\delta\to 0} \frac{u(x+\delta,t)-u(x,t)}{\delta}$ exists and from using (1) equals

$$u_x(x,t) = \int_0^l \phi(y)\left\{\sum_{j=-\infty}^{\infty}\left[S_x(x-y+2jl,t) - S_x(x+y+2jl,t)\right]\right\}dy$$

The same argument applies again in taking a divided difference with respect to t or of u_x. Thus, in the expression (1), we can justify commuting differentiation with respect to x with integration with respect to y and summation with respect to j any number of times. Since each S solves the heat equation, it follows that u will also solve the heat equation, and we can take derivative of u any number of times with respect to x and t.

This representation (1) is very useful representation for t small because of the rapid covergence rate. Only a few term in the j summation are needed for very accurate representation.

Method b: If we try simple solution in the form $X(x)T(t)$ and plug into heat equation, we get

$$X''(x) + \lambda X(x) = 0 \quad ; \quad X(0) = 0 = X(l) \tag{2}$$

$$T'(t) + \kappa \lambda T(t) = 0 \tag{3}$$

As discussed in class, nonzero solution to (2) is only possible for $\lambda = \frac{n^2 \pi^2}{l^2} = \lambda_n$, for $n = 1, 2, ...$, with corresponding

$$X(x) = \sin \frac{n \pi x}{l} \equiv X_n(x)$$

The solution to (6) with λ as respected above is

$$T_n = b_n \exp \left[- \frac{n^2 \pi^2}{l^2} \kappa t \right]$$

Therefore, summing over all $n \geq 1$, we get a general representation of solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp \left[- \frac{n^2 \pi^2}{l^2} \kappa t \right] \sin \frac{n \pi x}{l} \tag{4}$$

with initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l}$$

We know from theory that if $\phi \in \mathcal{L}_2(0,l)$, we obtain

$$b_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n \pi x}{l} dx$$

Wince $\phi \in \mathcal{L}_2(0,l)$, it follows from Bessel's inequality that b_n has to go to zero as $n \to \infty$ sufficiently fast and in particular is bounded with n. Further since for any j,

$$\sum_{n=1}^{\infty} n^j |b_n| \exp \left[- \frac{n^2 \pi^2}{l^2} \kappa t \right] < \infty$$

it follows that we can take any number of derivatives of the Fourier representation (4), with respect to x and t and term by term differentiation of Fourier representation is valid. It follows in particular that (4) solves the heat equation since each part $X_n(x) T_n(x)$ is a solution by construction. Further, it is in $\mathbf{C}^\infty$ with respect to both x and t for any $t > 0$.

To get a uniform estimate of the For the separation of variable method, determine a suitable bound for the size of the maximum error uniformly for $t > 0$, if the series representation is truncated to N terms, when $\phi \in \mathbf{C}^1(0,l)$, with $\phi(0) = 0 = \phi(l)$.

To get a uniform estimate of the error in truncation in separation of variable method, we note that

$$|u(x,t) - S_N(x,t)| \leq \sum_{n=N+1}^{\infty} \sum_{n=N+1}^{\infty} |b_n| \leq \left\{ \sum_{n=(N+1)} (n |b_n|)^2 \right\}^{1/2} \left(\sum_{N+1}^{\infty} \frac{1}{n^2} \right)^{1/2}$$

$$\leq \frac{1}{l\sqrt{N}} \int_0^l \left[\phi'(x)\right]^2 dx,$$

on using Bessel's inequality and condition that $\phi' \in \mathcal{L}^2(0,l)$. Thus, the error is $O(N^{-1/2})$. (Note, this is just an upper-bound and in practice, the decay rate will be faster).

2. Find solution to the problem of a circular vibrating membrane for 2-D :

$$u_{tt} - c^2 \Delta u = 0 \text{ for } |\mathbf{x}| < 1;\ u(\mathbf{x},0) = \phi(\mathbf{x})\ ,\ u_t(\mathbf{x},0) = 0,\ ,\ u(\mathbf{x},t) = 0 \text{ for } |\mathbf{x}| = 1$$

What is the restriction on ϕ needed? **Hint:** You may want to use the information that the solution to $v'' + \frac{1}{r}v' + \left(\lambda^2 - \frac{m^2}{r^2}\right)v(r) = 0$ is given by a linear combination of Bessel functions $J_m(\lambda r)$ and $Y_m(\lambda r)$, where Y_m blows up at the origin. J_m is regular at the origin and has countably infinite zeros on the real line similar to the sin function.

Solution: Use polar coordinates (r,θ), then $u = u(r,\theta,t)$ and

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2 u_{rr} - \frac{c^2}{r} u_r - \frac{c^2}{r^2} u_{\theta\theta} = 0$$

Then we seek simple solution in the separable form $T(t)R(r)\Theta(\theta)$, to get (see analogous heat equation analysis in class):

$$-\frac{\Theta''}{\Theta} = r^2 \left\{ \frac{T''(t)}{c^2 T(t)} + \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} \right\} = m^2 \tag{5}$$

where m^2 is some constant independent of θ, r and t. Therefore,

$$\Theta'' + m^2 \Theta = 0 \ \ ; \ \ \text{with } \Theta(\theta + 2\pi) = \Theta(theta)$$

Then,

$$\Theta(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

where m must be an integer to satisfy periodicity condition. For cos terms $m \geq 0$, while for sin terms $m > 0$ without loss of generality. Then going back to (5), we obtain

$$\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} - \frac{m^2}{r^2} = -\frac{T''}{c^2 T} = \lambda^2 \tag{6}$$

Therefore, we have

$$R''(r) + \frac{1}{r}R'(r) + \left(\lambda^2 - \frac{m^2}{r^2}\right) R(r) = 0$$

With given information, we have $R(r) = J_m(\lambda r)$, since $Y_m(\lambda r)$ is singular at $r = 0$. In order for Dirichlet condition to be satisfied on the unit disk, $R(1) = 0$ and therefore $J_m(\lambda) = 0$. So $\lambda = \lambda_{n,m}$, which is defined to be the n-th postive zero of J_m. This means

$$R(r) = J_m(\lambda_{n,m} r) \equiv R_{m,n}(r)$$

Going back to (6), we get

$$T''(t) + \lambda_{n,m}^2 T(t) = 0\ , \quad \text{and hence wlog } \ T(t) = \cos \lambda_{n,m} t + B \sin \lambda_{n,m} t = T_{m,n}(t)$$

Therefore superposing simple separable solutions in the form $T_{m,n}(t)\Theta_m(\theta)R_{m,n}(r)$ over all possible m and n, we obtain

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \cos m\theta \ \left(a_{m,n}\cos[c\lambda_{n,m}t] + b_{m,n}\sin[c\lambda_{m,n}t]\right) \ J_n(\lambda_{m,n}r)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin m\theta \ \left(c_{m,n}\cos[c\lambda_{n,m}t] + d_{m,n}\sin[c\lambda_{m,n}t]\right) \ J_n(\lambda_{m,n}r)$$

It satisfies initial conditions

$$\phi(r,\theta) = u(r,\theta,0) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{m,n} \ \cos m\theta J_m(\lambda_{m,n}r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} \ \sin m\theta J_m(\lambda_{m,n}r) \tag{7}$$

$$0 = u_t(r,\theta,0) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} +c\lambda_{m,n}b_{m,n}\cos m\theta \ J_n(\lambda_{m,n}r)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\lambda_{m,n}d_{m,n}\sin m\theta \ \ J_n(\lambda_{m,n}r) \tag{8}$$

implying $b_{m,n} = 0 = d_{m,n}$.

Thus,

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \cos m\theta \ a_{m,n}\cos[c\lambda_{n,m}t] \ J_n(\lambda_{m,n}r)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin m\theta \ c_{m,n}\cos[c\lambda_{n,m}t] \ J_n(\lambda_{m,n}r)$$

where $a_{m,n}$ and $c_{m.n}$ are chosen in accordance to the relation (7). From Theory done recently in class we know that the series representation on the right of (7) is complete and hence there is no restriction on ϕ other than that it is in $\mathcal{L}_2$ inside the unit disk.

3. Find solution in series form to Laplace's equation in 2-D in an annular domain $a < |\mathbf{x}| < 1$:

$$\Delta u = 0 \ \ ; \ \ u(\mathbf{x}) = \phi(\mathbf{x}) \text{ for } |\mathbf{x}| = a \ \text{ and } \ u(\mathbf{x}) = \psi(\mathbf{x}) \ \text{ for } |\mathbf{x}| = 1$$

Show that $u(\mathbf{x})$ is infinitely differentiable for $a < |\mathbf{x}| < 1$, even when boundary data ϕ and ψ are simply continuous.

Solution: Using separation of variable, in polar coordinates, we get as in class $u(r,\theta) = R(r)\Theta(\theta)$, where

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0 \tag{9}$$

$$\Theta'' + m^2\Theta = 0 \ \ ; \ \ \text{with} \ \ \Theta(\theta + 2\pi) = \Theta(\theta) \tag{10}$$

As discussed in class periodicity implies $m \geq 0$ an integer, and

$$\Theta(\theta) = a_m \cos m\theta + b_m \sin m\theta \equiv \Theta_m \tag{11}$$

Going back to (9), we get for $m > 0$

$$R(r) = c_m r^{-m} + d_m r^m \equiv R_m \tag{12}$$

and for $m = 0$,

$$R(r) = c_0 \log r + d_0 \equiv R_0 \tag{13}$$

Therefore, superposition over $R_m \Theta_m$ for all possible m gives rise to the expression, without any loss of generality, we may write (after redefining a_m, b_m, c_m and d_m):

$$u(r,\theta) = c_0 \log r + d_0 + \sum_{m=1}^{\infty} \cos m\theta \left\{ a_m \left(\frac{r}{a}\right)^{-m} + c_m r^m \right\} + \sum_{m=1}^{\infty} \sin m\theta \left\{ b_m \left(\frac{r}{a}\right)^{-m} + d_m r^m \right\} \tag{14}$$

In order to satisfy boundary conditions we get

$$\psi(\theta) = (c_0 \log a + d_0) = \sum_{m=1}^{\infty} \cos m\theta \left\{ a_m + c_m a^m \right\} + \sum_{m=1}^{\infty} \sin m\theta \left\{ b_m + d_m a^m \right\}$$

$$\phi(\theta) = d_0 + \sum_{m=1}^{\infty} \cos m\theta \left\{ a_m a^m + c_m \right\} + \sum_{m=1}^{\infty} \sin m\theta \left\{ b_m a^m + d_m \right\}$$

So, Fourier series coefficients:

$$d_0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta$$

$$a_m + c_m a^m = \frac{1}{\pi} \int_0^{2\pi} \phi(\theta) \cos m\theta d\theta$$

$$b_m + d_m a^m = \frac{1}{\pi} \int_0^{2\pi} \phi(\theta) \sin m\theta d\theta$$

$$d_0 + c_0 \log a = \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta) d\theta$$

$$a_m a^m + c_m = \frac{1}{\pi} \int_0^{2\pi} \phi(\theta) \cos m\theta d\theta$$

$$b_m a^m + d_m = \frac{1}{\pi} \int_0^{2\pi} \phi(\theta) \sin m\theta d\theta$$

Since $\phi, \psi \in \mathcal{L}_2$, it follows that $a_m + c_m a^m$ is in l_2, as are $b_m + d_m a^m$, $a_m a^m + c_m$ and $b_m a^m + d_m$ and in particular bounded in m. This implies that each of a_m, d_m are bounded. Since for any $r \in [a(1+\epsilon), 1-\epsilon]$

$$\sum_{m=1}^{\infty} \left\{ m|a_m| \left(\frac{r}{a}\right)^{-m} + m|c_m| r^m \right\} + \sum_{m=1}^{\infty} \left\{ m|b_m| \left(\frac{r}{a}\right)^{-m} + d_m r^m \right\}$$

$$< \sum_{m=1}^{\infty} \left\{ m|a_m|(1+\epsilon)^{-m} + m|c_m|(1-\epsilon)^m \right\} + \sum_{m=1}^{\infty} \left\{ m|b_m|(1+\epsilon)^{-m} + m|d_m|(1-\epsilon)^m \right\}$$

But the bounds above imply that derivatives of the series representation (14) with respect to r and θ are possible since the formal series obtained by differentiation is indeed bounded by multiples of the above bounds. Also, we note that the geometric factors $(1-\epsilon)^m$ and $(1+\epsilon)^{-m}$ decay faster than any polynomial in m. So we can take the derivatives of the series for $u(r,\theta)$ any number of times and it will converge from the fact that

$$\sum_{m=1}^{\infty} \left\{ m^j|a_m|(1+\epsilon)^{-m} + m^j|c_m|(1-\epsilon)^m \right\} + \sum_{m=1}^{\infty} \left\{ m^j|b_m|(1+\epsilon)^{-m} + m^j|d_m|(1-\epsilon)^m \right\} < \infty$$

Since $\epsilon > 0$ is arbitrarily, this proves the infinite differentiability of $u(r,\theta)$ with respect to both r and θ for $a < r < 1$. Hence $u(\mathbf{x})$ is infinitely differentiable in the annular region.

4. Determine a representaton for solution to Laplace's equation in a semi-circular domain

$$\mathcal{H} := \left\{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1, x_2 > 0 \right\}$$

$$\Delta u \; = \; 0 \quad \text{with} \;\; u = 0 \;\text{ for } x_2 = 0, \; \frac{\partial u}{\partial n} + u = f(\mathbf{x}) \text{ on } |\mathbf{x}| = 1$$

Make appropriate assumptions on f and apply whatever theory you need to justify your solution represention and to show that the solution is $\mathbf{C}^\infty$ inside $\mathcal{H}$.

Solution: We have from separation of variable $u(r,\theta) = R(r)\Theta(\theta)$, the same equation as in the previous problem:

$$\Theta'' + \lambda\Theta = 0$$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0$$

From the boundary condition, $\Theta(0) = 0 = \Theta(\pi)$. So, separation constant $\lambda = -m^2$ for integer $m > 0$, and $\Theta(\theta) = \Theta_m(\theta) \equiv \sin m\theta$. From the R equation, solution:

$$R = R_m(r) = c_m r^m + d_m r^{-m}$$

If solution is well-behaved at the boundary at $r = 0$, as we assume, then $d_m = 0$. Therefore, a general solution to the PDE satisfying boundary conditions in θ will be

$$u(r,\theta) = \sum_{m=1}^{\infty} \Theta_m(\theta)R_m(r) = \sum_{m=1}^{\infty} c_m r^m \sin m\theta$$

Now, consider the boundary conditions at $r = 1$. Formally, differentiating under the summation, we have

$$\partial_r u(1,\theta) + u(1,\theta) = \sum_{m=1}^{\infty} (mc_m + c_m) \sin m\theta = f(\theta)$$

We do an odd extension of $f(\theta)$ to the interval $[-\pi, 0)$ and therefore it follows that $(m+1)c_m$ is the Fourier sine cofficients of $f(\theta)$, the cosine coefficient and the constant being identically 0. If $f \in \mathbf{L}_2(0,\pi)$, then it follows that in the mean-square sense ($\mathbf{L}^2$ sense)

$$(m+1)c_m = \frac{2}{\pi}\int_0^\pi f(\theta)\sin m\theta d\theta$$

Therefore, solution for $r < 1$ is given by

$$u(r,\theta) = \sum_{m=1}^{\infty} c_m r^m \sin m\theta$$

where c_m is as determined above. Note that Bessel's inequality implies that $(m+1)c_m \in l^2$ hence

$$\sum_{m=1}^{\infty} |c_m| < \infty$$

implying pointwise convergence of Fourier series $\sum_{m=1}^{\infty} c_m \sin m\theta$ to

$$\lim_{r\to 1^-} u(r,\theta) = u(1,\theta)$$

On the otherhand, following the methods of previous exercise, it is easily seen that $u(r,\theta)$ as above is infinitely differentiable for $r < 1$.