Solution to Set 5: Math 716

1. Show that the partial differential operator $\mathcal{A}$ defined by:

$$\mathcal{A}u \equiv = -\nabla \cdot (p \nabla u) + qu$$

in a bounded $\Omega \subset \mathcal{R}^n$ for $p(\mathbf{x}) > 0$ is symmetric, with respect to the usual $\mathcal{L}_2$ inner-product. What condition on $q(\mathbf{x})$ makes $\mathcal{A}$ positive. Suppose we consider the eigenfunctions u satisfying

$$\mathcal{A}u = \lambda m u$$

for $m(\mathbf{x}) > 0$ in Ω. Prove the orthogonality of eigenfunctions corresponding to unequal eigenvalues with respect to inner-product $\langle ., . \rangle$ defined by

$$\langle u, v \rangle = \int_\Omega m u v d\mathbf{x}$$

Solution: We note that

$$(\mathcal{A}u, v) = \int_\Omega v^* \left\{ -\nabla \cdot (p \nabla u) + qu \right\} d\mathbf{x} = -\int_{\partial\Omega} p \frac{\partial u}{\partial n} d\mathbf{x} + \int_\Omega \left\{ p \nabla u \cdot \nabla v^* + qu \right\} d\mathbf{x}$$
$$= -\int_{\partial\Omega} p \left\{ \frac{\partial u}{\partial n} v^* - u \frac{\partial v^*}{\partial n} \right\} + \int_\Omega u \left\{ -\nabla \cdot (p \nabla v^*) + q v^* \right\} d\mathbf{x}$$

For Dirichlet BC, the integral term over $\partial\Omega$ clearly drops out, and we clearly obtain from the above expression, $(\mathcal{A}u, v) = (u, \mathcal{A}v)$. For Robin boundary condition (which includes Neumann as a special case (when $a = 0$), we obtain

$$-\int_{\partial\Omega} p \left\{ \frac{\partial u}{\partial n} v^* - u \frac{\partial v^*}{\partial n} \right\} = \int_{\partial\Omega} p \left\{ a u v^* - u a v^* \right\} = 0$$

Therefore, in all cases, the boundary term drops out, we get $(\mathcal{A}u, v) = (u, \mathcal{A}v)$, implying that $\mathcal{A}$ is a symmetric operator in every case. From the above calculation, we obtain

$$(u, \mathcal{A}u) = -\int_{\partial\Omega} p u^* \frac{\partial u}{\partial n} d\mathbf{x} + \int_\Omega \left\{ p \nabla u \cdot \nabla u^* + qu \right\} d\mathbf{x}$$

For Dirichlet condition, the boundary term drops out, and we have

$$(u, \mathcal{A}u) = \int_\Omega \left\{ p |\nabla u|^2 + q |u|^2 \right\} d\mathbf{x} > 0$$

for arbitrary $u \in \mathbf{C}^2$ iff $q(\mathbf{x}) > 0$ almost everywhere (*i.e.* it can be zero at most on a set of measure 0).

For Robin BC (including Neumann as a special case), we get

$$(u, \mathcal{A}u) = \int_{\partial\Omega} a |u|^2 d\mathbf{x} + \int_\Omega \left\{ p |\nabla u|^2 + q |u|^2 \right\} d\mathbf{x} > 0$$

for arbitary nonzero $u \in \mathbf{C}^2$, iff $a \geq 0$ and $q > 0$. Thus, under these conditions $\mathcal{A}$ is positive.

If we have only $a \geq 0$ and $q \geq 0$, then we will obtain $\mathcal{A}$ semi-positive in all three cases.

If u is an eigenvector of $\mathcal{A}$ corresponding to eigenvalue λ, it follows that

$$\lambda < u, u >= (\mathcal{A}u, u) = (u, \mathcal{A}u) = \lambda^*(u, mu) = \lambda^* < u, u >$$

This implies that λ is real. Further, if u, v are eigenvectors corresponding to two unequal eigenvalues λ and μ, it follows that

$$\lambda < u, v >= (\mathcal{A}u, v) = (u, \mathcal{A}v) = \mu < u, v >,$$

since both eigenvalues λ and μ must be real. Therefore,

$$(\lambda - \mu) < u, v >= 0 \quad \text{implying} \quad < u, v > -0$$

For equal eigenvalues, we can use Gram-Schmidt orgonalization procedure to obtain orthogonality. Thus all eigenvectors are orthogonal.

2. Show that minimization of
$$\frac{\|\nabla w\|^2 + \int_{\partial\Omega} a w^2(\mathbf{x}) d\mathbf{x}}{\|w\|^2}$$
for $w \neq 0$ for $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ leads to smallest eigenvalue and corresponding eigenfunction for $-\Delta$ operator with Robin boundary condition $\frac{\partial w}{\partial n} + aw = 0$ on $\partial\Omega$. What is the analogous minimization principle for n-the eigenvalue. Notice no boundary conditions on w on $\partial\Omega$, unlike the Dirichlet problem (where $w = 0$).

Solution:

Assume that minimum m is assumed by $u \in \mathcal{C}^2(\Omega) \cap \mathbf{C}^1(\bar{\Omega})$. Then we take $w = u + \epsilon v$ in the same class of functions and define

$$f(\epsilon) = \frac{\|\nabla w\|^2 + \int_{\partial\Omega} a w^2(\mathbf{x}) d\mathbf{x}}{\|w\|^2}$$

Then

$$0 = f'(0) = \frac{2(\nabla u, \nabla v) + 2\int_{\partial\Omega} a(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})}{\|u\|^2} - 2(u, v) \frac{\|\nabla u\|^2 + \int_{\partial\Omega} a u^2(\mathbf{x}) d\mathbf{x}}{\|u\|^4}$$

Therefore,

$$(\nabla u, \nabla v) + \int_{\partial\Omega} a(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = (u, v) m$$

On integration by parts,

$$(-\Delta u, v) + \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\mathbf{x} + \int_{\partial\Omega} a(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = m(u, v)$$

$$(-\Delta u - mu, v) + \int_{\partial\Omega} v \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = 0 \tag{1}$$

for arbitrary v. First we take class of v for which $v = 0$ on the boundary. For that class, the above reduces to

$$(-\Delta u - mu, v) = 0$$

implying

$$-\Delta u = mu \quad \text{for} \quad \mathbf{x} \in \Omega \tag{2}$$

Now, we enlarge the class of v to include functions that are not zero on $\partial\Omega$. Using (2) in (1), we obtain

$$\int_{\partial\Omega} v \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = 0$$

Since this is true for any $v \in \mathbf{C}^1(\partial\Omega)$, it follows that

$$\frac{\partial u}{\partial n} + au = 0 \quad \text{for} \quad \mathbf{x} \in \partial\Omega \tag{3}$$

Thus, minimizaton over the class of functions leads to the eigenfunction u_1 corresponding to the smallest eigenvalue λ_1 of the $-\Delta$, with Robin boundary conditions (3) on $\partial\Omega$. Note that we did not have to necessarily restrict trial functions w to satisfy any particular boundary conditions on $\partial\Omega$. The boundary conditions (3) simply followed on minimization of an appropriate quotient.

For the n-th eigenfunction, corresponding to eigenvalue λ_n, we will have to consider

$$\tilde{m} = \inf_{w \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^1(\partial\Omega)} \left\{ \frac{\|\nabla w\|^2 + \int_{\partial\Omega} a w^2(\mathbf{x}) d\mathbf{x}}{\|w\|^2} \right\} \quad \text{with} \quad w \quad \text{orthogonal to} \quad u_1, u_2, ..u_{n-1}$$

We assume that the infimum is actually attained by u in the class of trial functions given above. We now take in the above $w = u + \epsilon \hat{v}$. Calculating $f'(0)$ as before, and setting it to zero, we obtain

$$(-\Delta u - \tilde{m} u, \hat{v}) + \int_{\partial\Omega} \hat{v} \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = 0 \tag{4}$$

where u, $\hat{v}$ are additionally orthogonal to first $(n-1)$ eigenfunction u_1, u_2, $..u_{n-1}$. If we take a general v, we can decompose

$$v = \hat{v} + \sum_{j=1}^{n-1} \frac{(v, u_j) u_j}{\|u_j\|^2} \tag{5}$$

It is clear that $\hat{v}$ is orthogonal to u_1, u_2, $..u_{n-1}$. Further we note that

$$(-\Delta u - \tilde{m} u, u_j) + \int_{\partial\Omega} u_j \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = (u, -\Delta u_j) - \tilde{m}(u, u_j) - \int_{\partial\Omega} \left\{ u_j \frac{\partial u}{\partial n} - u \frac{\partial u_j}{\partial n} \right\} d\mathbf{x}$$

$$+ \int_{\partial\Omega} u_j \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = -\tilde{m}(u, u_j) + \lambda_j (u, u_j) = 0$$

since the boundary terms completely cancel out since it is known $\frac{\partial u_j}{\partial n} = -a u_j$ on $\partial\Omega$ and $(u, u_j) = 0$. Therefore, using above, (5) and (4), we obtain for any trial function v

$$(-\Delta u - \tilde{m} u, v) + \int_{\partial\Omega} \hat{v} \left\{ \frac{\partial u}{\partial n} + au \right\} d\mathbf{x} = 0$$

As for the minimal eigenvalue problem above, the arbitrariness of v implies

$$-\Delta u = \tilde{m} u \quad \text{in} \ \ \Omega \ \ \text{and} \ \ \frac{\partial u}{\partial n} + au = 0 \ \ \text{on} \ \partial\Omega$$

Since $\tilde{m}$ was obtained in a constrained minimization problem, where we required the trial functions to be orthogonal to u_1, u_2, $..u_{n-1}$, it follows from induction that $\tilde{m} \geq \lambda_{n-1}$. Since from construction, $\tilde{m}$ is the smallest such eigenvalue of $-\Delta$ with Robin boundary conditions, $\lambda_n = \tilde{m}$, with with corresponding eigenfunction being u, which is the minimizer of the quotient given above.

3. a. Use orthogonality of the eigenfunctions in the $\mathcal{L}_2$ sense of the operator $-\Delta$ with Robin-Boundary condition $\frac{\partial u}{\partial n} + au = 0$ $(a > 0)$ in a disk of radius 1 in 2-D to prove the following properties of the Bessel-function

$$\int_0^1 r J_m(k_i r) J_m(k_j r) dr = 0 \ \ \text{for } i \neq j$$

for any integer $m \geq 0$, where k_j is the j-th positive root of the transcendental equation:

$$k_j J_m'(k_j) + a J_m(k_j) = 0$$

Solution: We notice, as we did in last week's homework, in using separation of variable for the heat problem in a unit circle, that the separation of variable gives rise to eigenfunctions in the form

$$u(r,\theta) = \cos m\theta \ \ J_m(kr) \ \ ; \ \ \text{or} \ \ u(r,\theta) = \sin m\theta \ \ J_m(kr)$$

which will satisfy

$$-\Delta u = -u_{rr} - \frac{1}{r} u_r - \frac{1}{r^2} u_{\theta\theta} = -k^2 \cos m\theta \left\{ J_m''(kr) + \frac{1}{kr} J_m'(kr) - \frac{m^2}{k^2 r^2} J_m(kr) \right\} = k^2 u$$

since Bessel's function J_m satisfies

$$J_m''(z) + \frac{1}{z} J_m'(z) + \left(1 - \frac{m^2}{z^2}\right) J_m(z) = 0$$

So, $u(r,\theta) = \cos m\theta J_m(kr)$ is an eigenvector of $-\Delta$ corresponding to eigenvalue $\lambda = k^2$. Similarly, we can check that $u(r,\theta) = \sin m\theta J_m(kr)$ is also an eigenvector corresponding to $\lambda = k^2$. To find restriction on λ, we note that $u_r + au = 0$ for any $\theta \in (0, 2\pi)$ for $r = 1$. This means for any integer $m \geq 0$

$$\{k J_m'(k) + a J_m(k)\} \cos m\theta = 0 \ \ ; \ \ \{k J_m'(k) + a J_m(k)\} \sin m\theta = 0 \ \ \text{for } \theta \in (0, 2\pi)$$

implying for each integer $m \geq 0$

$$k J_m'(k) + a J_m(k) = 0$$

We denote the i-th zero of above equation by $k_{i,m}$. So λ_n is to be associated with the set of eigenvalues $\{k_{i,m}^2\}_{i=1,m=0}^{\infty}$ in order of increasing size. Orthogonality of eigenfunctions imply for any $(i', m') \neq (i, m)$,

$$\int_0^{2\pi} \int_0^1 r dr d\theta \left\{ \sin m\theta J_m(k_{i,m} r) \sin m'\theta J'_m(k_{i',m'}) \sin m'\theta \right\} = 0$$

$$\int_0^{2\pi} \int_0^1 r dr d\theta \left\{ \sin m\theta J_m(k_{i,m} r) \sin m'\theta J'_m(k_{i',m'}) \sin m'\theta \right\} = 0$$

If $m = m'$ in the above relation, using $\int_0^{2\pi} \sin^2 m\theta d\theta = \pi$ or $\int_0^{2\pi} \cos^2 m\theta d\theta = \pi$, it follows that

$$\int_0^1 r dr J_m(k_{i,m} r) J'_m(k_{i',m} r) = 0$$

b. What does completeness of the eigen functions of $-\Delta$ with Robin boundary conditions imply about class of functions f expressible as:

$$\sum_{j=1}^{\infty} c_j J_m(k_j r)$$

How is c_j determined from f? How does k_j behave like as $j \to \infty$?

Solution: Completeness implies that any function g of (r, θ) can be expressed as a series involving the the eigenfunctions u determined above. In particular if we have $g(r, \theta) = \cos m\theta f(r)$, then completeness implies

$$\cos n\theta f(r) = \sum_{m,j} \left\{ c_{m,j} \cos m\theta + d_{m,j} \sin m\theta \right\} J_m(k_{j,m} r)$$

$$(\cos n\theta f(r), \cos m\theta J_m(k_{j,m} r)) = c_{m,j} \int_0^{2\pi} \cos^2 m\theta d\theta \int_0^1 J_m^2(k_{j,m} r) r dr$$

$$(\cos n\theta f(r), \sin m\theta J_m(k_{j,m} r)) = d_{m,j} \int_0^{2\pi} \sin m\theta \cos n\theta d\theta \int_0^1 J_m^2(k_{j,m} r) r dr$$

Using the fact that $\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = 0$ for $m \neq n$ and equals π for $m = n$ and the fact that in all cases $\int_0^{2\pi} \sin m\theta \cos n\theta d\theta = 0$, we obtain $d_{m,j} = 0$ and $c_{m,j} = 0$ for $m \neq n$. For $m = n$, we obtain

$$c_{n,j} = \frac{\int_0^1 r f(r) J_n(k_{j,n} r) dr}{\int_0^1 r J_n^2(k_{j,n} r)}$$

and we have

$$f(r) = \sum_{j=1}^{\infty} c_{n,j} J_n(k_{j,n} r)$$

for arbitary function $f(r)$ for which $\int_0^1 r |f(r)|^2 dr < \infty$. We know that $\lambda_n \to \infty$ as $n \to \infty$; but this λ_n is formed from the sequence of eigenvalues $\{k_{i,m}^2\}_{i=1,m=0}^{\infty}$ by arranging them in increasing order. For fixed m, as $i \to \infty$, $k_{i,m}^2 \to \infty$, since we note that $J_m(k_{i,m}r)$ is the eigen function corresponding to eigenvalue $k_{i,m}^2$ for the the 1-D symmetric operator (Sturm-Liouville) $\mathcal{L}$ satisfying Robin boundary conditions at $r = 1$, where

$$\mathcal{L}u \equiv -(ru'(r))' + \frac{m^2}{r}u(r) = \lambda r u(r) \quad \text{for} \quad r \in (0,1)$$

with weight r. Therefore, from the properties of such symmetric operators (mentioned in class), the eigenvalues $k_{i,m}^2 \to \infty$ as $i \to \infty$.

4. For a rectangular box in 3 dimensions show using explicit computation of eigenvalues through separation of variable of the operator $-\Delta$ with Neumann boundary conditions that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{\text{Volume of rectangular box}}$$

Solution We know from separation of variables that the eigenfunctions of $-\Delta$ with homogeneous Neumann condition in the domain $x \in (0,a)$, $y \in (0,b)$, $z \in (0,c)$ is given by

$$\cos\frac{k\pi x}{a}\cos\frac{l\pi y}{b}\cos\frac{m\pi z}{c}$$

with corresponding eigenvalue

$$\lambda = \pi^2\left\{\frac{k^2}{a^2} + \frac{l^2}{b^2} + \frac{m^2}{c^2}\right\}$$

Therefore, introducing the enumeration functon $N(\lambda)$ we have to find the number of grid points in the first octant of an ellipsoid bounded by

$$\frac{\pi^2 k^2}{\lambda a^2} + \frac{\pi^2 l^2}{\lambda b^2} + \frac{\pi^2 m^2}{\lambda c^2} = 1$$

with semi-axes lengths $\frac{\sqrt{\lambda}a}{\pi}$, $\frac{\sqrt{\lambda}b}{\pi}$, $\frac{\sqrt{\lambda}a}{\pi}$ in the x,y and z-directions. Therefore, from geometric consideration for large λ, $N(\lambda)$ asymptotically approaches 1/8th the volume of this ellipsoid. Hence for large λ,

$$N(\lambda) \sim \frac{\pi}{6}\frac{\lambda^{3/2}abc}{\pi^3}$$

Therefore, taking $N(\lambda) = n$ and $\lambda = \lambda_n$, it follows that

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{abc} = \frac{6\pi^2}{\text{Volume of rectangular box}}$$