

Solution to homework Set 6: Math 716

1. Determine the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ for Dirichlet condition for Laplace's equation in 3-D dimensions in the hemispherical domain:

$$\Omega = \{\mathbf{x} : |\mathbf{x}| < 1, x_3 > 0\}$$

Use this to determine an integral expression for $u(r, \theta, \phi)$ in spherical polar-coordinates satisfying

$$\Delta u = 0 \text{ in } \Omega, \text{ with } u = 0 \text{ for } \theta = 0, \text{ and } u(1, \theta, \phi) = \psi(\theta, \phi) \text{ for } \theta \in [0, \frac{\pi}{2}], \phi \in [0, 2\pi]$$

Solution: Note that if $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3})$ be the location of a unit positive charge inside the hemisphere, to satisfy the boundary condition at $x_3 = 0$, we need a negative unit charge at $\bar{\mathbf{x}}_0 = (x_{0,1}, x_{0,2}, -x_{0,3})$. The image of positive charge \mathbf{x}_0 outside the unit sphere is a negative unit charge at $\frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}$, while the image of negative unit charge at $\bar{\mathbf{x}}_0$ outside the unit sphere is a positive unit charge at $\frac{\bar{\mathbf{x}}_0}{|\bar{\mathbf{x}}_0|^2}$. Combining, we have

$$G(\mathbf{x}; \mathbf{x}_0) = -\frac{1}{4\pi} \left\{ \frac{1}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{|\mathbf{x} - \bar{\mathbf{x}}_0|} - \frac{1}{|\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|} + \frac{1}{|\mathbf{x} - \frac{\bar{\mathbf{x}}_0}{|\bar{\mathbf{x}}_0|^2}|} \right\}$$

We note that on $|\mathbf{x}| = 1$,

$$\frac{\partial}{\partial n} \frac{-1}{|\mathbf{x} - \mathbf{x}_0|} = \mathbf{x} \cdot \nabla \left\{ \frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right\} = \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^3}$$

If $\mathbf{x}_0 = (r_0, \theta_0, \phi_0)$, and on the spherical surface $\mathbf{x} = (1, \theta, \phi)$, then $\mathbf{x} \cdot (\mathbf{x} - \mathbf{x}_0) = 1 - r_0 \cos \Psi$, where Ψ is the angle between \mathbf{x} and \mathbf{x}_0 and is given by (using dot products):

$$\begin{aligned} \cos \Psi &= \sin \theta_0 \sin \theta \cos \phi \cos \phi_0 + \sin \theta_0 \sin \theta \sin \phi \sin \phi_0 + \cos \theta_0 \cos \theta \\ &= \sin \theta_0 \sin \theta \cos(\phi - \phi_0) + \cos \theta_0 \cos \theta \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial n} \frac{-1}{|\mathbf{x} - \mathbf{x}_0|} = \frac{1 - r_0 \cos \Psi}{(1 + r_0^2 - 2r_0 \cos \Psi)^{3/2}}$$

We note that the image point $\bar{\mathbf{x}}_0$ has spherical coordinates $(r_0, \pi - \theta_0, \phi_0)$, and so using above formula makes an angle $\hat{\psi}$ with \mathbf{x} , given by

$$\cos \hat{\Psi} = \sin \theta_0 \sin \theta \cos(\phi - \phi_0) - \cos \theta_0 \cos \theta$$

Hence,

$$\frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \bar{\mathbf{x}}_0|} = \frac{1 - r_0 \cos \hat{\Psi}}{1 + r_0^2 - 2r_0 \cos \hat{\Psi}}^{3/2}$$

The image point $\frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}$ has spherical coordinates $(\frac{1}{r_0}, \theta_0, \phi_0)$ and makes the same angle Ψ with \mathbf{x} as does \mathbf{x}_0 . Therefore,

$$\frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|} = \frac{1 - r_0^{-1} \cos \Psi}{(1 + r_0^{-2} - 2r_0^{-1} \cos \Psi)^{3/2}}$$

and similarly

$$\frac{\partial}{\partial n} \frac{1}{|\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|} = \frac{1 - r_0^{-1} \cos \hat{\Psi}}{(1 + r_0^{-2} - 2r_0^{-1} \cos \hat{\Psi})^{3/2}}$$

So combining and using it in Poisson's formula on a sphere, we obtain

$$\phi(r_0, \theta_0, \phi_0) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\phi \psi(\theta, \phi) \left\{ \frac{1 - r_0 \cos \Psi}{(1 + r_0^2 - 2r_0 \cos \Psi)^{3/2}} - \frac{1 - r_0 \cos \hat{\Psi}}{(1 + r_0^2 - 2r_0 \cos \hat{\Psi})^{3/2}} \right. \\ \left. - \frac{1 - r_0^{-1} \cos \Psi}{(1 + r_0^{-2} - 2r_0^{-1} \cos \Psi)^{3/2}} + \frac{1 - r_0^{-1} \cos \hat{\Psi}}{(1 + r_0^{-2} - 2r_0^{-1} \cos \hat{\Psi})^{3/2}} \right\}$$

2. If $\mathcal{S}(\mathbf{x}, t)$ is the source solution to the heat equation given by

$$\mathcal{S}(\mathbf{x}, t) = \left(\frac{1}{4\pi\kappa t} \right)^{n/2} \exp \left[-\frac{|\mathbf{x}|^2}{4\kappa t} \right],$$

then show that

$$R(\mathbf{x}, t; \mathbf{x}_0, t_0) = \mathcal{S}(\mathbf{x} - \mathbf{x}_0, t - t_0) \text{ for } t > t_0 \text{ and } R = 0 \text{ for } t < t_0 \quad \mathbf{x} \in \mathbb{R}^n$$

satisfies

$$R_t - \kappa \Delta R = \delta(\mathbf{x} - \mathbf{x}_0) \delta(t - t_0)$$

Solution: Instead of taking R as related to \mathcal{S} , let us define it as the solution to

$$R_t - \kappa \Delta R = \delta(\mathbf{x} - \mathbf{x}_0) \delta(t - t_0) \quad ; \quad R(\mathbf{x}, t) = 0 \text{ for } t < t_0$$

The solution to this is unique since the difference of two such solution satisfies homogeneous heat equation with zero initial condition. From maximum principle, this difference is zero. We will show that $R(\mathbf{x}, t; \mathbf{x}_0, t_0)$ is indeed related to \mathcal{S} as stated above.

We know from Duhammel's principle that if we seek to solve

$$u_t - \kappa \Delta u = g(\mathbf{x}, t) \text{ for } t > 0 \text{ with } u(\mathbf{x}, 0) = 0,$$

Then solution is given by

$$u(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) g(\mathbf{y}, \tau) d\mathbf{y} d\tau \quad (1)$$

But, in terms of R , from principle of linear superposition discussed in class, we know that solution must be

$$u(\mathbf{x}, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} R(\mathbf{x}, t; \mathbf{y}, \tau) g(\mathbf{y}, \tau) d\mathbf{y} d\tau \quad (2)$$

Since $g(\mathbf{y}, \tau)$ can be chosen to be an arbitrary test function, it follows from uniqueness of solution to inhomogeneous heat equation and therefore from equality of expressions (1) and (2) that

$$R(\mathbf{x}, t; \mathbf{y}, \tau) = \mathcal{S}(\mathbf{x} - \mathbf{y}, t - \tau) \text{ for } t > \tau \quad ; \quad R(\mathbf{x}, t; \mathbf{y}, \tau) = 0 \text{ for } t < \tau$$

3. Consider the wave equation

$$u_{tt} - \Delta u = 0 \text{ for } \mathbf{x} \in \mathbb{R}^n, \quad t > 0, \quad \text{with } u(\mathbf{x}, 0) = \phi(\mathbf{x}), u_t(\mathbf{x}, 0) = \psi(\mathbf{x}),$$

with compactly supported ϕ and ψ .

a. Show that for $n = 3$, then there exists a constant C such that

$$|u(\mathbf{x}, t)| < \frac{C}{1+t} \text{ for } \mathbf{x} \in \mathbb{R}^3, \quad t > 0$$

Solution: The solution is expressible as

$$u(\mathbf{x}, t) = \frac{1}{4\pi t^2} \int_{\partial B(\mathbf{x}, t)} [t\psi(\mathbf{y}) + \phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla \phi(\mathbf{y})] dS_{\mathbf{y}}$$

We note that because of compact support \mathcal{K} of ϕ and ψ , the surface area of $\partial B(\mathbf{x}, t) \cap \mathcal{K}$ is bounded above independent of t . So,

$$\left| \int_{\partial B(\mathbf{x}, t)} [t\psi(\mathbf{y}) + \phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla \phi(\mathbf{y})] dS_{\mathbf{y}} \right| \leq \int_{\mathcal{K} \cap \partial B(\mathbf{x}, t)} [t|\psi(\mathbf{y})| + |\phi(\mathbf{y})| + |(\mathbf{y} - \mathbf{x})| |\nabla \phi(\mathbf{y})|] dS_{\mathbf{y}} \leq C$$

Therefore, for $t \geq 1$,

$$\|u(\cdot, t)\|_{\infty} \leq \frac{C(1+t)}{4\pi t^2} \quad (3)$$

On the otherhand for $t \leq 1$, it is clear that

$$\begin{aligned} |u(\mathbf{x}, t)| &\leq \left| \frac{1}{4\pi t^2} \int_{\partial B(\mathbf{x}, t)} [t\psi(\mathbf{y}) + \phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla \phi(\mathbf{y})] dS_{\mathbf{y}} \right| \leq \\ &\left| \frac{1}{4\pi} \int_{\partial B(0,1)} [t\psi(\mathbf{x} + t\mathbf{z}) + \phi(\mathbf{x} + t\mathbf{z}) + t\mathbf{z} \cdot \nabla \phi(\mathbf{x} + t\mathbf{z})] dS_{\mathbf{z}} \right| \leq C \end{aligned} \quad (4)$$

Combining (3), (4), it follows that there exists some other constant \tilde{C} so that

$$\|u(\cdot, t)\|_{\infty} \leq \frac{\tilde{C}}{1+t} \quad (5)$$

b. Show that for $n = 3$, at a fixed point \mathbf{x} outside the support of ϕ and ψ , there exists an earliest impact time $t_i > 0$ that depends on \mathbf{x} so that $u(\mathbf{x}, t) = 0$, for $0 \leq t < t_i$, and that there also exists a final impact time $t_f > t_i$ so that for $t > t_f$, $u(\mathbf{x}, t) = 0$.

For a fixed point \mathbf{x} outside the compact support \mathcal{K} of ϕ and ψ , since $\mathbb{R}^n \setminus \mathcal{K}$ is open, there exists a small ball around \mathbf{x} which together with its boundary is completely contained in $\mathbb{R}^n \setminus \mathcal{K}$. Therefore, there exists t_i (depending on \mathbf{x}) so that if $t < t_i$, $\partial B(\mathbf{x}, t) \cap \mathcal{K} = \emptyset$, and therefore,

$$u(\mathbf{x}, t) = \frac{1}{4\pi t^2} \int_{\partial B(\mathbf{x}, t)} [t\psi(\mathbf{y}) + \phi(\mathbf{y}) + (\mathbf{y} - \mathbf{x}) \cdot \nabla \phi(\mathbf{y})] dS_{\mathbf{y}} = 0$$

Again, for fixed \mathbf{x} , if t keeps on increasing, it is clear that the boundary $\partial B(\mathbf{x}, t)$ keeps expanding in t and therefore there exists a time t_f (depending on \mathbf{x}) so that if $t > t_f$, $\partial B(\mathbf{x}, t) \cap \mathcal{K} = \emptyset$, and therefore $u(\mathbf{x}, t) = 0$.

c. Show that for $n = 2$, for \mathbf{x} outside the support of ψ and ϕ , there exists an initial impact time $t_i > 0$ depending on \mathbf{x} but no final impact time t_f .

In 2-D, the expression for the solution is

$$u(\mathbf{x}, t) = \frac{1}{2\pi t^2} \int_{B(\mathbf{x}, t)} \frac{t\psi(\mathbf{y}) + t^2\psi(\mathbf{y}) + t(\mathbf{y} - \mathbf{x}) \cdot \nabla\phi(\mathbf{y})}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{1/2}} d\mathbf{y}$$

Now, the same argument as in 3-D shows that if \mathbf{x} outside \mathcal{K} , the support of ϕ and ψ , we can a small ball $B(\mathbf{x}, t)$ of radius t centered around \mathbf{x} will not intersect \mathcal{K} . Hence there exists a time $t_i > 0$ so that if $t < t_i$, $u(\mathbf{x}, t) = 0$. However, sufficiently large t , regardless of how large it is, $B(\mathbf{x}, t)$ will contain points in \mathcal{K} and hence for general ϕ and ψ will not be zero. So there is no final time t_f beyond which $u(\mathbf{x}, t) = 0$.