Week 1 notes: Math 716

1 PDE: Order, Linear & Nonlinear

A differential equation that involves more than one independent variable is called a *partial differential equation*, abbreviated as PDE. The order of the highest derivative occurring in the PDE is defined as the *order* of the PDE. For instance, the following

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

is a second order PDE for $u(x, y)$, which is usually called the *Laplace's equation* in two variables. Another example of a second order linear differential equation is the so-called *heat* equation for $u(x, t)$ in one space variable with a source:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + g(x, t) \quad \text{for some constant} \quad \kappa \tag{2}$$

As with ODEs, if the differential operator $\mathcal{L}$ is linear, *i.e.* for constant c_1 and c_2,

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L} u_1 + c_2 \mathcal{L} u_2 \tag{3}$$

then $\mathcal{L}u = g$ is called a linear differential differential equation. For instance, equations (1) and (2) above are linear differential equations since in those cases we can check easily the linearity of the corresponding differential operators $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or $\mathcal{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$, acting on appropriate class of functions[1] Note, that in (1), $g = 0$. In that case, the PDE is linear and homogeneous.

Partial differential equations that are not linear are called *nonlinear*; the following *semi-linear* heat equation is an example:

$$u_t - u_{xx} = u^2, \tag{4}$$

where for notational brevity, the partial derivatives are denoted by subscripts.

As for ODEs, linear PDEs are usually simpler to analyze/solve than nonlinear PDEs. Nonlinear PDEs arise in many applications; but general theory rarely exists.

2 Some physical applications where PDEs arise

PDEs abound in physical sciences and engineering. Here we give some examples.

2.1 Dispersion of pollutants in a river stream

Consider predicting concentration ρ (measured in some units, say Kg/m^3) of some pollutant as a function of position $\mathbf{x}$ and time t, in a river. We denote the $\mathbf{x}$ domain by $\Omega \subset \mathbb{R}^3$. Let the fluid velocity field in the river (domain Ω) be given by $\mathbf{u}(\mathbf{x}, t)$ (measured in some units, say m/sec). We will assume that the pollutant is passive, meaning it's inertia is neglected. Consider a small but fixed volume V centered around at a point $\mathbf{x}$ (See Figure 1), entirely contained in Ω. The

[1] Classically, this would be $\mathbf{C}^2(\mathbb{R}^2)$ in the first case and $\mathbf{C}^2$ in x and $\mathbf{C}^1$ in time for the second. However, this set can be extended further by introducing the concept of weak solutions.

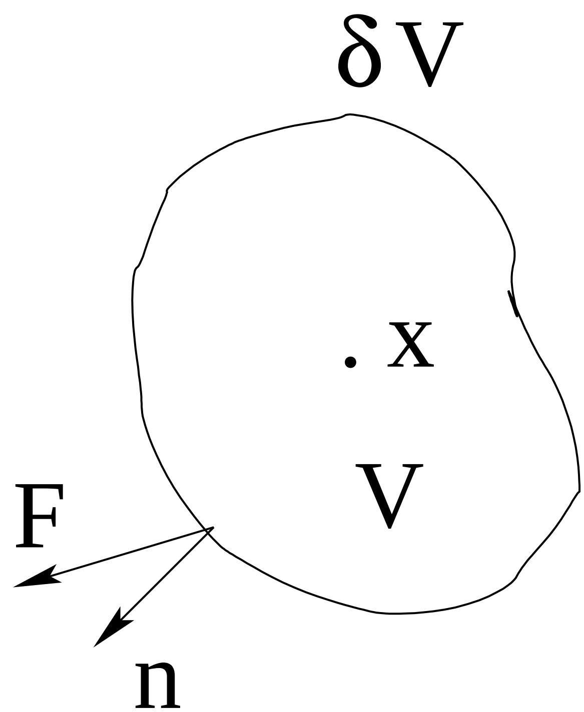

Figure 1: Control Volume V for pollutant

rate of change of mass of fluid inside that volume equals $\frac{d}{dt}\int_V \rho(\mathbf{x},t)dV$. Assuming no pollutant is created or destroyed within V, this must equal the net inward flow of pollutants into V:

$$-\int_{\partial V} \mathbf{F}\cdot\mathbf{n}dA$$

where $\mathbf{F}(\mathbf{x},t)$ is the flux (measured in units like $Kg/m^2/sec$) of pollutants at any point $\mathbf{x}$ and $\mathbf{n}$ is the outward normal to the boundary of volume V. A reasonable expression for flux is:

$$\mathbf{F} = \mathbf{u}\rho - \kappa\nabla\rho$$

where the first term is due to fluid motion carrying pollutants, usually called *advection* , while the second term is the flux due to molecular diffusion, called *Fix's law*, where κ is the diffusion rate. Therefore, conservation of mass of pollutants imply

$$\frac{d}{dt}\int_V \rho(\mathbf{x},t)dV = -\int_{\partial V} \mathbf{n}\cdot(\mathbf{u}\rho - \kappa\nabla\rho)\,dA \tag{5}$$

This is the case with no source. If there is a source emitting pollutants at a rate $q(\mathbf{x},t)$ per unit volume (measured in, say, $Kgm/m^3/sec$ units) within V, then (5) is replaced by

$$\frac{d}{dt}\int_V \rho(\mathbf{x},t)dV = -\int_{\partial V} \mathbf{n}\cdot(\mathbf{u}\rho - \kappa\nabla\rho)\,dA + \int_V q(\mathbf{x},t)dV \tag{6}$$

If we assume $\rho(\mathbf{x}, t)$ is $\mathbf{C}^1$ in time t and $\mathbf{C}^2$ in space $\mathbf{x}$, and q is $\mathbf{C}$ in $\mathbf{x}$, then it follows from taking t-derivative inside the volume integral and using Gauss's divergence theorem that

$$\int_V \left(\frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \nabla \cdot [(\mathbf{u}\rho - \kappa \nabla \rho] - q \right) dV = 0$$

Since this is true for any control volume V, it follows that

$$\frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \nabla \cdot [(\mathbf{u}\rho - \kappa \nabla \rho] = q \tag{7}$$

If the fluid is incompressible, *i.e.* $\nabla \cdot \mathbf{u} = 0$ and κ is a constant, (7) reduces to

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \kappa \Delta \rho + q \tag{8}$$

where $\Delta = \nabla \cdot (\nabla) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ for $\mathbf{x} = (x_1, x_2, x_3)$ in 3-D, is referred to as the *Laplacian* operator. Equation (8) is referred to as the diffusion-advection equation with source term. In the special case, when there is no source and diffusion is small and we have a one-dimensional constant flow $\mathbf{u}(\mathbf{x}, t) = c\, \mathbf{e}_1$, then (8) reduces to

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho}{\partial x_1} = 0 \tag{9}$$

whose solution is given by

$$\rho(x_1, x_2, x_3, t) = f(x_1 - ct, x_2, x_3) \tag{10}$$

for some arbitrary $\mathbf{C}^1$ function f. This is a wave travelling along the positive x_1 axis with constant speed c. This means that in the absence of diffusion, whatever the concentration is at point (x_1, x_2, x_3) at $t = 0$ is now transported downstream to the point $(x_1 + ct, x_2, x_3)$, as expected physically. Instead of guessing and checking, we will learn more later about systematic ways of solving first order PDEs through method of characteristics.

Another interesting limit of (8) is when advection is negligible compared to diffusion, *i.e.* when we set fluid velocity $\mathbf{u} = 0$. We obtain the *diffusion* or *heat* equation[2]

$$\rho_t = \Delta \rho + q \tag{11}$$

If $q = 0$, equilibrium solution in this case is found by seeking solution to Laplace equation:

$$\Delta \rho = 0 \tag{12}$$

For constant $\mathbf{u}$ and q, we can find explicit expression for the full equation (8) in simple geometries, as we will learn later. For more complicated geometries, there exists no explicit method for finding solutions; however, using mathematical analysis, we can prove existence, uniqueness and some valuable properties of such solutions.

Note that the problem specification in (8) (or (10) is not complete. One must append to it *initial* conditions at $t = 0$, *i.e.* specify $c(\mathbf{x}, 0)$ and *boundary* conditions on the boundary $\partial\Omega$; for

[2]So named because temperature also satisfies the same equaton when heat flux occurs through molecular diffusion only

instance, if pollutants are not allowed to move out of domain Ω, then we specify $\mathbf{F} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

It is to be noted that in the derivation of (7) from (6), we assumed *a priori* that solution was indeed smooth enough in $\mathbf{x}$ and t to allow derivatives inside the integral and use divergence theorem. When these conditions are not met, a more fundamental equation based on the physics is given by (6). Indeed, in a general mathematical study of equations such as (7), the assumption of differentiability in $\mathbf{x}$ and t is weakened by introducing the notion of *weak solution*. Instead of satisfying (7), we may require for instance that that $\rho(\mathbf{x}, t)$ satisfy

$$\int_0^\infty \int_\Omega \left[\phi_t \rho + \nabla\phi \cdot (\mathbf{u}\rho + \kappa \nabla \rho) + q\phi \right] dV \ dt = 0 \tag{13}$$

for any smooth function ϕ with *compact support*, *i.e.* vanishes outside a bounded set in $\Omega \times \mathbb{R}^+$. If we require that (13) is satisfied and it turns out $\rho \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^1(\mathbb{R}^+)$, then clearly on integration by parts (7) is satisfied. However, the weak solutions satisfiying (13) is more general since they do *not* require solutions to be as differentiable. We will discuss weak solutions in more details later in the quarter.

2.2 Vibrating String

Consider a flexible, elastic homogeneous string of length l with constant linear density ρ, measured in units of mass/length, of length. Suppose it undergoes transverse vibration like a guitar string or a plucked violin string. At a given instant of time t, the shape locally looks like that shown in thick outline in Figure 2. Assume that the string motion is restricted to a plane. Let $u(x,t)$ be the vertical displacement from equilibrium at time t at position x. For a perfectly flexible string, the tension (force) is directed tangentially along the string. Let $T(x,t)$ denote the magnitude of the tension tension vector $\mathbf{T}(x,t)$. Then, T is a constant along the string because the string is homogeneous. We shall apply Newton's law of motion for the part of the string between $x = x_0$ and $x = x_1$. The slope of the string at x_1 is clearly $u_x(x_1, t)$. The component of forces in the horizontal direction must be in balance since there is no motion in that direction. The upward component of the net forces must equal mass times acceleration of the portion of string between x_0 and x_1. Hence, using arclength s increasing with x,

$$\frac{T u_x(x_1,t)}{\sqrt{1 + u_x^2(x_1,t)}} - \frac{T u_x(x_0,t)}{\sqrt{1 + u_x^2(x_0,t)}} = \int_{x_0}^{x_1} u_{tt} \rho \frac{ds}{dx} dx \tag{14}$$

In the case, when $u(x,t)$ is $\mathbf{C}^2$ in x, we may write (14) as

$$\int_{x_0}^{x_1} \left\{ \left(\frac{T u_x}{\sqrt{1+u_x^2}} \right)_x - \rho \sqrt{1+u_x^2}\, u_{tt} \right\} dx$$

Since this is true for any interval (x_0, x_1), it follows that

$$\left(\frac{T u_x}{\sqrt{1+u_x^2}} \right)_x - \rho \sqrt{1+u_x^2}\, u_{tt} = 0 \tag{15}$$

In the case when u_x is small, *i.e.* small slope, (15) we obtain as an approximation the the linear wave equation: $(Tu_x)_x = \rho u_{tt}$, or

$$u_{tt} - c^2 u_{xx} \;=\; 0 \;\; \text{where} \;\; c = \sqrt{\frac{T}{\rho}} \tag{16}$$

General solution to (16) is of the form

$$u(x,t) = f(x - ct) + g(x + ct) \tag{17}$$

where f and g are arbitrary $\mathbf{C}^2$ functions of x and t. This corresponds to superposition of a wave moving to the right and a wave moving to the left. f and g that are determined completely by *initial* and *boundary* conditions. If the string corresponds to $x \in (0,l)$, then fixed end points would correspond to $u(0,t) = u(l,t) = 0$. From physical considerations, initial conditions would correspond to specifying both the initial displacement $u(x,0)$ and initial velocity $u_t(x,0)$. We will discuss more in details later on how to find f and g from initial and boundary conditions.

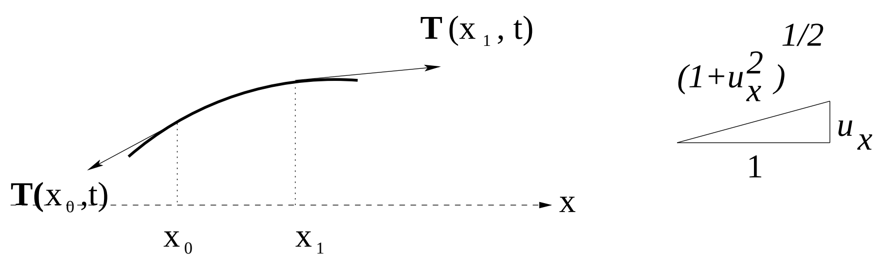

Figure 2: Section of Vibrating string between $x = x_0$ and $x = x_1$

Variations of (16) including friction, transverse elastic force or externally applied forces are given in the text (Page 12). Also, it is possible to generalize by introducing concept of appropriate *weak* solution so that (18) would indeed be the weak solution even when f and g are not in $\mathbf{C}^2$.

2.3 Electro-magnetic waves

Consider Maxwell's equations in free medium for electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$. In the Gaussian CGS system of units, they are given by:

$$\nabla \cdot \mathbf{E} = 0 \tag{18}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\mathbf{E}_t = 0 \tag{19}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\mathbf{B}_t = 0 \tag{20}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{21}$$

Taking curl of (20), using (18), (19) and the vector identity $\nabla \times (\nabla \times \mathbf{E}) = -\Delta \mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$, it follows that

$$\Delta \mathbf{E} = \frac{1}{c^2}\mathbf{E}_{tt} \tag{22}$$

In a similar manner, eliminating $\mathbf{E}$ between (19) and (20) and using (21), we obtain

$$\Delta \mathbf{B} = \frac{1}{c^2}\mathbf{B}_{tt} \tag{23}$$

Therefore, each scalar component of $\mathbf{E}$ and $\mathbf{B}$ satisfy the linear wave equation in 2-D or 3-D:

$$\Delta u = \frac{1}{c^2}u_{tt} \tag{24}$$

The solution to (22) and (23) have to be subject again to initial and boundary conditions. A physically reasonable initial condition will be to specify both $\mathbf{E}(\mathbf{x}, 0)$ and $\mathbf{B}(\mathbf{x}, 0)$. For a finite domain Ω in $\mathbf{x}$, we may specify for instance $\mathbf{E}$ or $\mathbf{B}$ on the boundary $\partial\Omega$ or a combination of components of $\mathbf{E}$ and $\mathbf{B}$ at the boundaries, the latter is common for instance with a conducting boundary.

As in 1-D, the higher dimensional wave equation allows for propagating waves, in this case propagating with speed c. If the electro-magnetic wave is emitted due oscillation of current in a loop, then there is a current term on the right of (19). This results in a forcing term on the right of (22) and (23) and we obtain an inhomogeneous wave equation. You can check out texts in electrodynamics (for instance Jackson) for other interesting examples.

2.4 Dirichlet, Neumann and Robin Boundary Conditions for 2nd order PDEs

In a more general context of solving wave equation (24), or for that matter any problem with second order spatial derivative like diffusion equation (11) or Laplace equation (12), if we specify u on $\partial\Omega$, this constitutes the *Dirichlet* Boundary conditions. If the normal derivative $\frac{\partial u}{\partial n}$ is specified instead, it is referred to as the *Neumann* BC. A boundary condition in the form of specified $\frac{\partial u}{\partial n} + au = c$ on $\partial\Omega$ will be called a *Robin* boundary condition.

More general discussions of initial and boundary conditions are given in §1.4 in the text. We will get to them in the context of specific problems as they arise.

3 Well-Posed Problems

A PDE in a domain Ω, together with initial and boundary conditions, is well-posed if the following properties are valid:

1. *Existence*: There is at least one solution to the problem that satisfies all conditions.
2. *Uniqueness*: There is at most one solution.
3. *Stability*: The unique solution depends continuously, with respect to some norm, on initial and boundary conditions. More precisely, let solutions u_1 and u_2 correspond to initial/boundary data $u_{1,0}$ and $u_{2,0}$. Stability implies that for any $\epsilon > 0$, there exists $\delta > 0$ so that if $\|u_{1,0} - u_{2,0}\| < \delta$, then $\|u_1 - u_2\| < \epsilon$.

For a physical problem modeled by a PDE, one has to formulate physically realistic auxiliary conditions (like initial and boundary conditions) which together makes a well-posed problem.

The mathematician has to prove if a problem is well-posed or not, since modeling usually involves approximation and it is *a priori* unclear whether the approximations are all consistent. Reliance on physical intuition alone is not always enough. If too few auxiliary conditions are given, the problem may not have unique solution and is therefore *underdetermined*. If too many are specified, the solution may not have any soluton and the problem is *overdetermined*.

The stability property (iii) is required in models of physical problems, since data is never measured exactly. You cannot distinguish between a set of data from a tiny perturbation of it (in sense of any physically relevant norm). The solution ought not to be signficantly affected by such tiny perturbations, as otherwise one loses all predictability. A model is not physically sensible without this stability.

For example, if we consider inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t) \quad \text{for } (x,t) \in (0,L) \times (0,\infty) \tag{25}$$

with auxiliary conditions

$$u(x,0) = \phi(x) \quad , \quad u_t(x,0) = \psi(x) \quad ; \quad u(0,t) = g(t) \quad ; \quad u(L,t) = h(t) \tag{26}$$

The data for this problem involves five functions f, g, ϕ, ψ and h. So, in order for problem to be well-posed, (25) with auxiliary conditions (26) must have a unique solution and the solution has to depend continuously with respect to small changes in each of these five functions. In §2.3, 3.4 and 5.5 of the text, it is shown that this is indeed the case for an appropriately chosen norm.

As another example, consider solving Laplace equation in 2-D: $u_{xx}+u_{yy} = 0$ in the region $\mathcal{D} = (-\infty,\infty)\times(0,\infty)$. It is not well-posed if both $u(x,0)$ and $u_y(x,0)$ are specified on part of ∂D. To see this, we verify that each member of the the following sequence $u_n(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin x \ \sinh ny$ and satisfies initial boundary data $u_n(x,0) = 0$ and $\frac{\partial u_n}{\partial y}(x,0) = e^{-\sqrt{n}} \sin x$, which tends to 0 as $n \to \infty$ uniformly in x. But, for $y > 0$, the solution does not tend to 0 as $n \to \infty$. Thus, the stability condition (iii) is violated.