Week 10 Lectures, Math 716, Tanveer

1 Examples of Fourier-Transform of Distributions

Example: Using the same argument, as for $\mathcal{F}[1]$, except with $\mathbf{k}$ replaced by $\mathbf{k}-\boldsymbol{\eta}$, we find:

$$\mathcal{F}\left[\exp[i\boldsymbol{\eta}\cdot\mathbf{x}]\right](\mathbf{k}) = (2\pi)^{m/2}\delta(\mathbf{k}-\boldsymbol{\eta})$$

If f is a 2π-periodic distribution represented by a Fourier-series:

$$f(x) = \sum_{n=1}^{\infty} c_n e^{inx},$$

we find that

$$\mathcal{F}[f](k) = \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n \delta(k-n)$$

Definition 1 *A distribution f is said to have a compact support, if there exists a compact $\mathcal{K}$ so that for all test function ϕ with support in $\mathbb{R}^n\backslash\mathcal{K}$, $(f,\phi)=0$. An example of this is $\delta(\mathbf{x})$, whose support is only $\{\mathbf{0}\}$.*

Example Let f be a distribution with compact support. Then for any $\phi \in \mathbf{C}^\infty(\mathbb{R}^n)$, we set $(f,\phi)=(f,\phi_0)$, where $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ and ϕ_0 agrees with ϕ is a neighborhood of the support of f. If ϕ_1 also has similar property as ϕ_0, it is clear from definition of f that $(f,\phi_0-\phi_1)=0$ since from construction, the support of $\phi_0-\phi_1$ is outside the support of f. Thus, (f,ϕ) can be defined unambigously (not depending on which ϕ_0 is used).

We claim that $\mathcal{F}[f]$ is the function

$$\mathcal{F}[f](\mathbf{k}) = (2\pi)^{-m/2}\left(\overline{f(\mathbf{x})}, e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

Here $(\bar{f},\phi)$ is defined as the complex conjugate of $(f,\bar{\phi})$. This follows since for any $\phi\in\mathcal{S}(\mathbb{R}^n)$, we have

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(f(\mathbf{x}), e^{i\mathbf{k}\cdot\mathbf{x}}\right)\phi(\mathbf{k})d\mathbf{k} = (f,\mathcal{F}^{-1}[\phi])) = (2\pi)^{-n/2}\left(f, \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k})d\mathbf{k}\right)$$

Example: Let $\delta(|\mathbf{x}|-a)$ represent a uniform mass distribution on a sphere of radius a, *i.e.*

$$(\delta(|\mathbf{x}|-a),\phi) = \int_{|\mathbf{x}|=a}\phi(\mathbf{x})dS$$

Then,

$$\mathcal{F}[\delta(|\mathbf{x}|-a)](\mathbf{k}) = (2\pi)^{-n/2}\int_{|\mathbf{x}|=a} e^{-i\mathbf{k}\cdot\mathbf{x}}dS$$

For $n=3$, using spherical polar coordinates, we get

$$\mathcal{F}[\delta(|\mathbf{x}|-a)](\mathbf{k}) = (2\pi)^{-n/2}\int_0^\pi\int_0^{2\pi} e^{-ia\rho\cos\theta}\sin\theta d\phi d\theta = \sqrt{\frac{2}{\pi}}a\frac{\sin a|\mathbf{k}|}{|\mathbf{k}|}$$

2 The Source Solution (fundamental solution) for the wave equation:

Consider solution to

$$S_{tt} = \Delta S \quad , \quad \text{for} \;\; \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}, \quad \text{with} \;\; S(\mathbf{x}, 0) = 0, \quad S_t(\mathbf{x}, 0) = \delta(\mathbf{x}) \tag{1}$$

Fourier-transforming, we obtain

$$\hat{\mathcal{S}}_{tt} = -\mathbf{k}^2 \hat{S} \quad \text{with} \;\; \hat{S}(\mathbf{k}, 0) = 0 \;\; \hat{S}_t(\mathbf{k}, 0) = (2\pi)^{-n/2} \tag{2}$$

Therefore,

$$\mathcal{S}(\mathbf{k}, t) = (2\pi)^{-n/2} \frac{\sin |\mathbf{k}| t}{|\mathbf{k}|} \tag{3}$$

For $n = 3$, from one of the previous examples, it follows that

$$S(\mathbf{x}, t) = \frac{\delta(|\mathbf{x}| - t)}{4\pi t} \tag{4}$$

This method of finding Green's function is generally valid for any constant coefficient system in free-space. For example, if we have have a PDE of the form

$$\mathcal{P}(\partial_{\mathbf{x}}) G = \delta(\mathbf{x})$$

where $\mathcal{P}$ is a polynomial, then application of Fourier-Transform leads to an algebraic relation:

$$\hat{G}(\mathbf{k}) = \frac{1}{(2\pi)^{n/2} \mathcal{P}(i\mathbf{k})}$$

We can then recover G by Fourier-transform. Note that since the above is true for any dimension, it can accomodate PDEs involving both t and $\mathbf{x}$, but just considering a higher dimensional variable $\overline{\mathbf{x}} = (\mathbf{x}, t)$.

3 Laplace Transform:

If $f \in \mathcal{S}'(\mathbb{R})$ have support contained in $\{x \geq 0\}$. Then obviously $e^{-\mu x} f(x)$ is also in $\mathcal{S}'(\mathbb{R})$ for every $\mu > 0$. Formally, we have

$$\mathcal{F}\left[e^{-\mu x} f\right](\mathbf{k}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) e^{-ikx} e^{-\mu x} dx = \mathcal{F}[f](k - i\mu)$$

Hence, it is sensible to define $\mathcal{F}[f](k - i\mu) = \mathcal{F}[f e^{-\mu x}]$. This defines $\mathcal{F}[f]$ in the lower half of the complex k-plane–as a generalized function of $\Re k$, depending of $\Im k$ as a paraemeter. Actually, however, this function is analytic k in the lower-half plane ($\Im k < 0$). The Laplace-transform is defined as

$$\mathcal{L}[f](s) = \sqrt{2\pi} \mathcal{F}[f](-is);$$

for $f \in \mathcal{S}'$ with support in $\{x \geq 0\}$, it is defined in the right half-plane complex plane ($\Re s \geq 0$). Formally, we have

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x) dx$$

If $f \notin \mathcal{S}'(R)$, but $e^{-\mu x} f \in \mathcal{S}'(\mathbb{R})$ for some $\mu > 0$, then we can define $\mathcal{L}[f]$ in the right-half plane $\Re\ s \geq \mu$. We note that by inverting the Fourier-transform we obtain

$$e^{-\mu x} f(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}[f](\mu + ik),$$

or equivalently,

$$f(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{sx} \mathcal{L}[f](s) ds$$

In using the above formula, we must ensure that the resulting expression vanishes for $x < 0$, since this was our basic assumption. Typically, one shows this by closing the contour of integration by a half-circle to the right; e^{sx} decays rapidly in the right half plane. For this argument to work, it is necessary to choose μ to the right of the singularities of f.

Example: Consider

$$u_t = u_{xx} \quad \text{ for } x \in (0,1), \quad t > 0 \ \text{ with } \ u(x,0) = 0, \quad u(0,t) = 1 = u(1,t) \ \text{ for } t > 0$$

Laplace transform in time leads to

$$s\mathcal{L}[u](x,s) = \mathcal{L}[u]_{xx}(x,s) \quad ; \quad \mathcal{L}[u](0,s) = \frac{1}{s} = \mathcal{L}[u](1,s)$$

The equation for the solution is

$$\mathcal{L}[u](x,s) = \frac{\cosh\left(\sqrt{s}\left(x - \frac{1}{2}\right)\right)}{s \cosh\left(\sqrt{s}/2\right)}$$

Using the inverse transform, we obtain

$$u(x,t) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{st} \frac{\cosh\left(\sqrt{s}(x - \frac{1}{2}\right)}{s \cosh\left(\sqrt{s}/2\right)} ds$$

The integral cannot be evaluated in closed form, however, through contour deformation, and change of variables $\sqrt{s} -> s_1$, it is possible to use calculus of residues (complex variable technique) and obtain a series form of solution.