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1 Wave Equation as 1st order homogeneous system of PDEs

Another approach to linear wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

is to notice that for $\mathbf{C}^2$ functions, it is equivalent to a system of 1st order equation

$$u_t = c v_x \quad , \quad v_t = c u_x \tag{2}$$

It is clear that if (2) is satisfied by (u, v), then elimination of v between the two equations clearly leads to (1). On the otherhand, if $u(x,t)$ solves (1), then clearly (u, v), with $v(x,t) = \frac{1}{c}\int_0^x u_t(y,t)dy - c\int_0^t u_x(0,\tau)d\tau$ satisfies (2).

If (u, v) satisfies (2), then we observe by taking a linear combination of the two equations that $u_1 = u + v$, $u_2 = u - v$ satisfies

$$\partial_t u_1 - c\partial_x u_1 = 0 \;\; ; \;\; \partial_t u_2 \; + \; c\partial_x u_2 = 0, \tag{3}$$

which are a decoupled system of 1st order nonlinear equation. Using method of characteristics, $u_1 = u + v$ is a constant on characteristics $x + ct = \text{constant}$. Further, $u_2 = u - v$ is a constant on characteristics $x - ct = \text{constant}$. Initial data on each of u_1, u_2 can be specified on a non-characteristic curve that intersects the corresponding characteristics transverally. We can directly conclude from this that $u_1 = \hat{f}(x + ct)$ and $u_2 = \hat{g}(x - ct)$ and hence recover the prior result $u(x,t) = f(x + ct) + g(x - ct)$, for f and g related to $\hat{f}$ and $\hat{g}$.

Remark 1 *In general for hyperbolic equations, we specify as many conditions on a curve* Γ*, adjoining a domain* U *as the number of different sets of characteristics that enter* U *through* Γ*. We illustrate this in Fig. 1 for wave equation, which as two sets of characteristics* $x + ct = \text{const}$ *and* $x - ct = \text{const}$.. Γ *is the boundary of the domain* Ω *in the* $x-t$ *plane. Note, in Fig. 1, we are seeking solutions for increasing* t*. The characteristic arrows will be reversed if we go backwards in time* t*.*

2 Heat equation and Maximum Principle

We now consider the solution $u(\mathbf{x}, t)$ to heat equation The text discusses this for 1 space dimension in section 2.3, but the generalization to arbitrary dimension is not very difficult. Consider

$$u_t = \kappa \nabla u \;\; \text{for} \;\; \mathbf{x} \in \Omega \subset \mathbb{R}^n \quad , t > 0 \tag{4}$$

We will assume Ω to be a bounded set. We will prove that both the maximum and minimum of u in $\Omega \times [0, T]$ is attained either at $t = 0$ or on $\partial\Omega$.

Physically, this makes sense. As you know, heat flows from hot to colder part of the domain. If you have a domain Ω with no internal heat source, the hottest or coldest spot can occur only

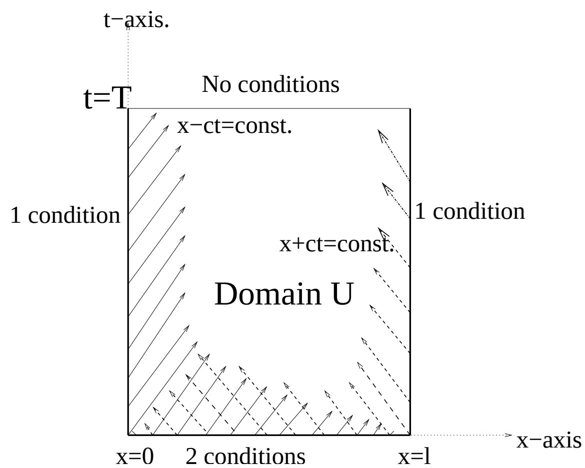

Figure 1: Characteristics entering the region U for linear wave equation

initially or on on the boundary $\partial\Omega$. A hot spot at time zero will cool off, unless heat is fed into the domain through the boundary. But heat can be fed from a boundary point to the interior only if the boundary temperature is higher that its adjoining interior point. The same argument can be made for a cold spot.

We will now prove the *weak* version of the maximum principle for *classical* solutions. We start with some definition

Definition 1

$$\Sigma = \{(\mathbf{x}, t) : t = 0 \quad \text{or } \mathbf{x} \in \partial\Omega\}$$

Also, we define for convenience

$$D = \Omega \times [0, T]$$

Theorem 2 *(Weak Maximum Principle for Heat Equation) Assume $\Omega \subset \mathbb{R}^n$ is bounded and u is a classical (strong) solution to (4), Then*

$$\max_{\bar{D}} u = \max_{\Sigma} u$$

We need the following preliminary Lemmas.

Lemma 3 *If $v(\mathbf{x}, t) \in \mathbf{C}^2(\Omega) \cap \mathbf{C}^1[0, T]$ has a maximum at an interior point $(\mathbf{x}_0, t_0) \in D$, then*

1. $\frac{\partial v}{\partial t}(\mathbf{x}_0, t_0) = 0$, $\nabla v(\mathbf{x}_0, t_0) = 0$
2. $\Delta v(\mathbf{x}_0, t_0)) \leq 0$.

Proof. The first condition is the extremum condition from multi-variable calculus for a function of $n + 1$ variables, $(\mathbf{x}, t)$. For the second condition, we note that on a parametrized curve $\mathbf{C}^2$

curve $\mathbf{X}(s)$ passing through $\mathbf{x}_0$, where $\mathbf{X}(0) = \mathbf{x}_0$, the condition for maximum as a function of variable s implies (note we are suppressing the t-dependence)

$$\frac{d^2}{ds^2} u(\mathbf{X}(s)) = \frac{d}{ds} \left(\sum_{i=1}^{n} u_{x_i} \frac{dX_i}{ds} \right) = \sum_{i,j} u_{x_i x_j} \frac{dX_j}{ds} \frac{dX_i}{ds} + \sum_{i} u_{x_i} \frac{d^2 X_i}{ds^2} \leq 0$$

Evaluating the above at $s = 0$, and using the first condition, we have the condition that

$$\sum_{i,j} u_{x_i x_j}(\mathbf{x}_0, t_0) X_i'(0) X_j'(0) \leq 0$$

Since $\mathbf{X}_i'(0)$ and $\mathbf{X}_j'(0)$ can be chosen in arbitrary fashion, it follows that the matrix (usually called Hessian) H with elements $\left\{ u_{x_i x_j}(\mathbf{x}_0, t_0) \right\}$ is semi-negative definite and therefore has no positive eigenvalues. Thus the trace of this matrix H, which is the same as the sum of the eigenvalues, is non positive, implying

$$\Delta u \leq 0$$

□

Proof of Theorem 2

Choose an auxiliary function $v_\epsilon(\mathbf{x}, t) = u(\mathbf{x}, t) + \epsilon e^{-t}$ where $\epsilon > 0$ and $u(\mathbf{x}, t)$ is a classical solution to the heat equation. We now claim that v cannot have an interior maximun $(\mathbf{x}_0, t_0) \notin \Sigma$. Assume, it does. Then, using (4), at any point

$$\frac{\partial v_\epsilon}{\partial t} - \kappa \Delta v_\epsilon = -\epsilon e^{-t} + u_t - \kappa \Delta u = -\epsilon e^{-t} < 0$$

However, using Lemma 3, it follows that at $(\mathbf{x}_0, t_0)$

$$\frac{\partial v_\epsilon}{\partial t} - \kappa \Delta v_\epsilon \geq 0$$

which is clearly contradictory. Hence $v_\epsilon(\mathbf{x}, t)$ cannot have an interior maximum. In particular, this means that

$$\max_{\bar{D}} v_\epsilon = \max_{\Sigma} v_\epsilon \leq \max_{\Sigma} u + \epsilon e^T$$

This is true for any ϵ. Letting $\epsilon \to 0$, Theorem 2 follows and the proof is complete.

Corollary 4 *(Weak Minimum Principle) For a classical solution $u(\mathbf{x}, t)$ satisfying heat equation (4), the minimum of u is also achieved on Σ.*

PROOF. We apply Theorem 2 to the quantity $-u$ which satisfies heat equation as well. □

Remark 2 *There is a stronger version of the maximum principle as well: if $(\mathbf{x}_0, t_0)$ is an interior point where $u(\mathbf{x}_0, t)$ attains a maximum M, then $u = M$ everywhere. Thus, any non-constant solution cannot have a maximum value attained at an interior point. Note that the weak maximum principle only rules out an interior maximum bigger than the maximum attained on Σ. The proof is more involved and will not be done in this course.*

Remark 3 *Looking at the proof of Theorem 2, we realize that we only used* $-u_t + \Delta u \geq 0$. *So maximum principle holds whenever this is true. Indeed, maximum principle will hold for for classical solution to the nonlinear PDE* $-u_t + \Delta u = u^2$ *In a similar vein, if* $-u_t + \Delta u \leq 0$, *then minimum principle holds.*

Remark 4 *Maximum principle holds generally for linear parabolic PDEs, heat equation is a special case.*

The maximum principle can be used to prove the uniqueness of solution to (4), with auxiliary initial and boundary conditions

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}) \quad , \quad u(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \partial\Omega \tag{5}$$

for given ϕ and g.

Theorem 5 *Any classical solution to (4) satisfying (5) is unique and at any time* $t \in [0, T]$, *depends continuously on* g *and* ϕ *in the* sup *norm over* Ω *for any* $t \in [0, T]$.

Proof.

Assume $u_1(\mathbf{x}, t)$ and $u_2(\mathbf{x}, t)$ are two classical solutions to (4) satisfying (5), with $g = g_1$ and $\phi = \phi_1$ in one case and $g = g_2$, $\phi = \phi_2$ in the other. Then, realizing that $v = u_1 - u_2$ also satisfies heat equation (4) and auxiliary conditions (5),w ith $\phi = \phi_1 - \phi_2$ and $g = g_1 - g_2$, it follows from maximum and minimum principle that

$$\sup_{\mathbf{x}\in\Omega} |u_1(\mathbf{x}, t) - u_2(\mathbf{x}, t)| \leq \sup_{\mathbf{x}\in\Omega} |\phi_1(\mathbf{x}) - \phi_2(\mathbf{x})| + \sup_{t\in[0,T], \mathbf{x}\in\partial\Omega} |g_1(\mathbf{x}, t) - g_2(\mathbf{x}, t)|$$

Continuous dependence on initial conditions follow. In the special case, when $g_1 = g_2$, $\phi_1 = \phi_2$, uniqueness of solution to (4) satisfying the same auxiliary conditions (5) follows. □

2.1 Energy type arguments for uniqueness

There is also another approach on proving uniqueness similar to the 'energy' arguments we saw in class for the wave equation. This is based on noting that the heat equation with homogeneous boundary condition $u = 0$ on $\partial\Omega$ implies

$$\frac{d}{dt} \int_\Omega \frac{1}{2} u^2 d\mathbf{x} = \int_\Omega u u_t d\mathbf{x} = \kappa \int_\Omega u \Delta u = \kappa \int_\Omega \nabla \cdot (u \nabla u) d\mathbf{x} - \kappa \int |\nabla u|^2 (\mathbf{x}, t) d\mathbf{x} = -\kappa \int |\nabla u|^2 (\mathbf{x}, t) d\mathbf{x} \leq 0 \tag{6}$$

Therefore,

$$\int_\Omega u^2(\mathbf{x}, t) d\mathbf{x} \leq \int_\Omega u^2(\mathbf{x}, 0) d\mathbf{x} \tag{7}$$

So, if $u(\mathbf{x}, 0) = 0$, it follows $u(\mathbf{x}, t) = 0$ for $t > 0$. However, the energy type arguments require *apriori* that application of divergence theorem in the last step in (7) is justified. In particular, we need $\frac{\partial u}{\partial n}$ to exist on $\partial\Omega$; a somewhat stronger condition than needed with *maximum* principle application.

3 Explicit representation for solution on $\mathbb{R}$

Consider heat equation in 1-D over the entire real line:

$$u_t = \kappa u_{xx} \qquad -\infty < x < \infty \;\; ; \;\; 0 < t < \infty \tag{8}$$

with initial condition

$$u(x,0) = \phi(x) \tag{9}$$

This is the simplest case where explicit solution is possible.

We note a few facts

1. If $u(x,t)$ is a solution to (8), so is its translate $u(x-y,t)$.
2. Any x or t derivative of a solution $u(x,t)$ is also a solution.
3. A linear combination of solution is again a solution, from linearity of the the operator $\partial_t - \kappa\partial_{xx}$.
4. An integral of solutions is again a solution. Thus if $S(x,t)$ is a solution, then so is

   $$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy$$

   for any function $g(y)$ for which integral and its x and t derivative makes sense.
5. If $u(x,t)$ is a solution, so is the *dilated* function $u(\sqrt{a}x, at)$ for any $a > 0$. This can be proved by directly substituting into (8).

We now look for specially simple solutions to (8) with hopes of building up more general solutions using the above facts, including a solution that satisfies initial condition (9).

First, we seek a solution to (8) $Q(x,t)$ that satisfies initial condition

$$Q(x,0) = 1 \;\text{ for }\; x \;>\; 0 \qquad Q(x,0) = 0 \;\text{ for }\; x < 0 \tag{10}$$

Since there is no scale in this problem, it follows that if $Q(x,t)$ solves the initial value problem, so will $Q(\sqrt{a}x, at)$ for any constant a. But, the solution to the initial value problem is unique. This suggests that

$$Q(x,t) = g(p) \;\text{ where }\; p = \frac{x}{\sqrt{4\kappa t}} \tag{11}$$

since p remains invariant under the transformation $(x,t) \to (\sqrt{a}x, at)$. If we substitute (11) back into (8), we obtain

$$0 = Q_t - \kappa Q_{xx} = \frac{1}{t}\left[-\frac{p}{2}g'(p) - \frac{1}{4}g''(p)\right] \tag{12}$$

implying $g'' + 2pg' = 0$. Therefore,

$$Q(x,t) = g(p) = c_1 \int e^{-p^2} dp + c_2 \tag{13}$$

Now, we find c_1, c_2 consistent with initial conditions. For $x < 0$, we note that $t \to 0^+$, $p \to -\infty$ and in that limit, initial conditions on Q imply $g(p) \to 0$. Further for $x > 0$, as $t \to 0^+$, $p \to +\infty$, and in that limit, initial condition implies $g(p) \to 1$. Therefore, we can write

$$g(p) = c_1 \int_{-\infty}^{p} e^{-s^2} ds \quad \text{with} \quad c_1 \int_{-\infty}^{\infty} e^{-s^2} ds = 1 \quad \text{, implying} \quad c_1 = \frac{1}{\sqrt{\pi}}$$

Therefore,

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\kappa t}}} e^{-s^2} ds = \frac{1}{2} \left\{ 1 + \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{4\kappa t}}} e^{-s^2} ds \right\} = \frac{1}{2} \left[1 + erf \left(\frac{x}{\sqrt{4\kappa t}} \right) \right] \tag{14}$$

where erf is called the error function and defined as above. Having found $Q(x,t)$, we define for $t > 0$,

$$S(x,t) = Q_x(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp \left[-\frac{x^2}{4\kappa t} \right] \tag{15}$$

and define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y) dy = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp \left[-\frac{(x-y)^2}{4\kappa t} \right] \phi(y) dy \tag{16}$$

From property 3 above, $u(x,t)$ is another solution to (8).

We will now prove that $u(x,t)$ is the unique solution that satisfies both (8) and (9).

Theorem 6 *Assuming initial condition $\phi \in \mathbf{C}^0(\mathbf{R})$ and that ϕ is bounded at ∞. Then the unique solution satisfying both (8) and (9) is given by (16).*

PROOF.

First, we note from expression (15), the following properties of $S(x,t)$:

$$\lim_{t \to 0^+} S(x-y,t) = 0 \text{ for } x \neq y \tag{17}$$

This approach is uniform for any closed interval in y that does not contain x. Further, since for any $t > 0$

$$\int_{-\infty}^{\infty} S(x-y,t) dy = \int_{-\infty}^{\infty} Q_x(x-y) dy = -\int_{-\infty}^{\infty} Q_y(x-y) dy = Q(+\infty) - Q(-\infty) = 1 \tag{18}$$

it follows from using (17) and (18) that for any $\delta > 0$, which is independent of t,

$$\lim_{t \to 0^+} \int_{x-\delta}^{x+\delta} S(x-y,t) dy = 1 \tag{19}$$

Choose $\epsilon > 0$. Then we choose δ small enough so that $|\phi(x) - \phi(y)| < \epsilon$ for $y \in (x-\delta, x+\delta)$. Then, we rewrite

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y) dy = \left\{ \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} + \int_{x-\delta}^{x+\delta} \right\} S(x-y,t)\phi(y) dy \tag{20}$$

The contribution from the first two integral vanishes as $t \to 0^+$ since

$$|\left\{\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right\} S(x-y,t)\phi(y)dy| \leq \|\phi\|_\infty |\left\{\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right\} S(x-y,t)dy| \to 0$$

We are left with

$$\int_{x-\delta}^{x+\delta} S(x-y,t)\phi(y)dy = \phi(x)\int_{x-\delta}^{x+\delta} S(x-y,t)dy + \int_{x-\delta}^{x+\delta} S(x-y,t)[\phi(y)-\phi(x)]$$

Note as $t \to 0^+$, the first term on the right tends to $\phi(x)$, while the second term

$$|\int_{x-\delta}^{x+\delta} S(x-y,t)[\phi(y)-\phi(x)]| \leq \epsilon \int_{x-\delta}^{x+\delta} S(x-y,t)dy \to \epsilon$$

Therefore, combining the three previous equation, we obtain

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \phi(x)$$

Thus $u(x,t)$, given by (16) satisfies the given initial condition. Further from expression (16), it is easily proved that we can differentiate in x and t inside the integral and since $S(x,t)$ satisfies heat equation, so does $u(x,t)$ From uniqueness, $u(x,t)$ is the solution to heat equation in $\mathbb{R}$ satisfying given initial condition.

□

Figure 1 shows a plot of $S(x,t)$ against x. Note that it is a Gaussian shape solution to heat equation (8) and becomes more and more peaked as $t \to 0^+$, while the area underneath it is always 1. The limiting value of $S(x,t)$ as $t \to 0^+$ is zero except at $x = 0$, where it is singular initially. This is therefore called the solution due a *point source* of *unit mass*, since $\int_{-\infty}^{\infty} S(x,t)dx = 1$. Thus, if (16) is approximated as a Riemann sum

$$u(x,t) = \int_{y=-\infty}^{\infty} \phi(y)S(x-y,t)dy \approx \sum_{j=-\infty}^{\infty} \phi(jh)S(x-jh,t)h, \tag{21}$$

we may interprete $u(x,t)$ as the temperature at time t due to an initial linear superposition of *point sources* at $y_j = jh$, with *mass* $\phi(y_j)(\delta y_j)$, for j ranging from $-\infty$ to ∞.

Another physical interpretation of $S(x-y,t)$ is in terms of Brownian motion. Assume a particle is randomly moving on a line (instead of in 3-D), with equal probability of moving to the right or left. Then $\int_a^b S(x-y,t)dy$ is the probability of finding the particle in the interval (a,b), when it initially at the position x. S is in this case is the probability density function, and its evolution is known to be governed by heat equation.

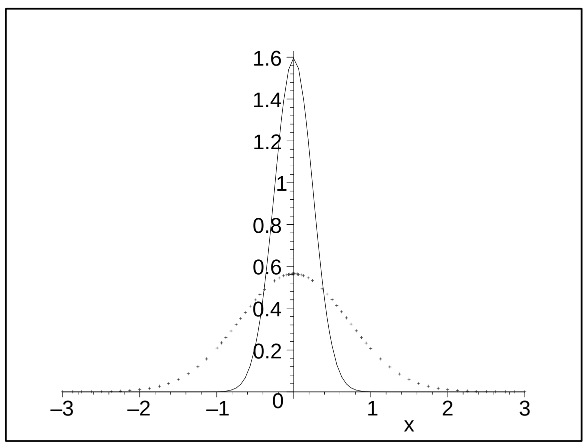

Figure 2: $S(x,t)$ versus x, for $t = 1/64$ (solid), $t = 1/8$ (dashed)