# Week 5 Lectures, Math 716, Tanveer

# 1 Separation of variable method

The method of separation of variable is a suitable technique for determining solutions to linear PDEs, usually with constant coefficients, when the domain is bounded in at least one of the independent variables. We illustrate this procedure for 1-D wave equation and 2-D heat equation for Dirichlet, Neumann and Robin boundary conditions, though the idea is equally applicable for diffusion equation and Laplace's equation, and other constant coefficient equations.

The idea of separation of variable is first to seek simple solution to the PDE in the form of a product, each term in the product depending on only one independent variable. Solutions are then constrained by boundary conditions. This results in a countably infinite set of solutions. A linear superposition of such solution is also a solution, because of the linearity of the problem. As we shall find later, such linear superposition is capable of describing all reasonable initial conditions.

## 1.1 Dirichlet Problem

Consider

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad 0 < x < l \tag{1}$$

$$u(0,t) = 0 = u(l,t) \tag{2}$$

with initial condition

$$u(x,0) = \phi(x) \quad ; \quad u_t(x,0) = \psi(x) \tag{3}$$

Recall, there is a representation of this solution by doing odd extension about $x = 0$ to the interval $(-l, 0)$, and then periodically extending this problem (with period $2l$), and then using D'Alembert representation of solution, as you will see in last week notes. Uniqueness follows from application of energy method, as we saw earlier. Here, we are seeking a different form of the same solution.

We seek simple solution to (1), ignoring initial conditions (3) for now, in a product form $u(x,t) = X(x)T(t)$ Plugging it into (1) and dividing it the resulting equation by $c^2 X(x)T(t)$, we obtain

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} \tag{4}$$

If we set $t = 1$, for instance, then the left side of (4) is a number independent of $x$. Hence the right side of (4) cannot depend on $x$ at all. Similarly, if we set $x$ to some specific value, the right side of (4) is a constant; hence the left side (4) must be independent of $t$. Therefore, we conclude both left and right side of each side of (4) is some constant $\lambda$. Therefore,

$$T''(t) + \lambda c^2 \lambda T(t) = 0 \tag{5}$$

and

$$X''(x) + \lambda X(x) = 0 \tag{6}$$

The boundary conditions (2) imply that $X(0) = 0$ and $X(l) = 0$.

If $\lambda < 0$, then the solution to (6) that satisfies $X(0) = 0$ is given by

$$X(x) = C \sinh\left(\sqrt{-\lambda} x\right) \tag{7}$$

Since the function sinh does not vanish anywhere except the origin, any nontrivial solution in the form (7) is incapable of satisfying $X(l) = 0$. Therefore, we must discard the possibility of $\lambda < 0$.

If $\lambda = 0$, then the solution to (6) that satisfies $X(0) = 0$ is

$$X(x) = Cx, \tag{8}$$

But, this is not capable of satisfying $X(l) = 0$, unless $C = 0$, which corresponds to the trivial solution. Therefore, we conclude that $\lambda \neq 0$.

We are only left with the possibility $\lambda = \beta^2 > 0$. Then, solution to (6) satisfying $X(0) = 0$ is given by

$$X(x) = C \sin \beta x \tag{9}$$

In order for the solution to be nontrivial, *i.e.* $C \neq 0$, and yet $X(l) = 0$, we must have $\sin \beta l = 0$. Therefore $\beta l = n\pi$, for integer $n \neq 0$. We note that $\sin\left(\frac{-n\pi x}{l}\right) = -\sin\left(\frac{n\pi x}{l}\right)$. This implies that $n < 0$ does not generate an independent set of functions from what we obtain for $n > 0$. Thus, we can restrict to integer $n \geq 1$, and obtain

$$X(x) = C \sin\left(\frac{n\pi x}{l}\right) \equiv C X_n(x) \tag{10}$$

Corresponding to $\beta = \frac{n\pi}{l} \equiv \beta_n$, with $n > 1$, we may solve (5), with $\lambda = \beta_n^2$, to obtain

$$T(t) = A_n \cos \beta_n ct + B_n \sin \beta_n ct \equiv T_n(t) \tag{11}$$

where $A_n$ and $B_n$ are arbitary constants. Therefore, we obtain a countably infinite set of separable solutions to to (1) satisfying boundary conditions (2), indexed by integer $n$, and in the form $u(x,t) = X_n(x)T_n(t)$. A more general solution to (1), satisfying Dirichlet boundary condition (2) is of the form

$$u(x,t) = \sum_{n \geq 1} X_n(x) T_n(t) = \sum_{n \geq 1} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \tag{12}$$

assuming that this sum converges. We note that this solution (12) satisfies initial condition

$$u(x,0) = \sum_{n \geq 1} A_n \sin \frac{n\pi x}{l} \tag{13}$$

$$u_t(x,0) = \sum_{n \geq 1} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l} \tag{14}$$

Therefore, if $\phi(x)$ and $\psi(x)$ in (3) is in the form of (13) and (14), the separation of variable method will have provided solution to IVP (1)-(3). We will later discover from the theory of Fourier Series, that this restriction on $\phi$ and $\psi$ is rather mild–that *all* initial conditions of physical interest can be accomodated with solution (12) by suitably choosing $A_n$ and $B_n$, depending on $\phi$ and $\psi$.

## 1.2 Neumann Condition

The same method works for Neumann problem as well. The text illustrates this for 1-D wave equation in section 4.2. We illustrate here for 2-D heat equation in a rectangular domain in $\mathbf{x} = (x, y)$. So, our problem of interest is

$$u_t - \kappa(u_{xx} + u_{yy}) = 0 \quad \text{for} \quad 0 < x < l, \; 0 < y < l \;, t > 0 \tag{15}$$

satisfying Neumann Boundary condition

$$u_x(0, y, t) = 0 = u_x(l, y, t) \quad , \quad u_y(x, 0, t) = 0 = u_y(x, l, t) \tag{16}$$

and initial condition

$$u(x, y, 0) = \phi(x, y) \tag{17}$$

Using energy method (see Homework 3 problem), we can prove that any solution satisfying (1)-(3) is unique. Here, we are showing existence.

As in the last subsection, first, we seek simple solution to (15) satisfying only (16) in the product form $u(x, y, t) = X(x)Y(y)T(t)$. Plugging into (15) and dividing the result by $X(x)Y(y)T(t)$, we obtain

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{T'(t)}{\kappa T(t)} \tag{18}$$

The left side depends only on $x$, while the right depends on $(y, t)$. It follows that each side of the equation has to be some constant $\lambda$. Therefore,

$$-\frac{X''(x)}{X(x)} = \lambda \;, \quad \text{implying} \quad X''(x) + \lambda X(x) = 0 \tag{19}$$

The first set of boundary conditions in (16) imply that $X'(0) = 0$ and $X'(l) = 0$. However, if $\lambda < 0$, the solution satisfying $X'(0) = 0$ is given by $X(x) = C \cosh \sqrt{-\lambda} x$ which cannot satisfy $X'(l) = 0$ for nonzero $C$, since sinh is nonzero for nonzero argument. Therefore, we are left with only possibility of $\lambda = \beta^2 \geq 0$. In that case, a nontrivial solution to (19) satisfying $X'(0) = 0$ is

$$X(x) = \cos \beta x \tag{20}$$

In order to satisfy $X'(l) = 0$, we must have $\sin \beta l = 0$, therefore

$$\sqrt{\lambda} = \beta = \frac{n\pi}{l} \equiv \sqrt{\lambda_n} \tag{21}$$

and non-trivial solution for $X(x)$ is restricted to

$$X(x) = \cos \frac{n\pi x}{l} \equiv X_n(x) \quad \text{for } n \geq 0 \tag{22}$$

Note that $n < 0$ is excluded from the above set, since cos is an even function and replacing $n$ by $-n$ does not generate any new set of solutions. However, unlike the Dirichlet problem, (21) allows the possibility of $n = 0$.

Having determined $X(x)$, we now determine $Y(y)$. For that purpose, it is convenient to rewrite (18) as

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} + \frac{T'(t)}{\kappa T(t)} \tag{23}$$

and conclude that a function of $y$ cannot be equal to a function of $x$ and $t$ unless each is some constant $\gamma$. Therefore,

$$-\frac{Y''(y)}{Y(y)} = \gamma \quad , \quad \text{implying} \quad Y''(y) + \gamma Y(y) = 0 \tag{24}$$

The second set of boundary conditions in (16) implies that

$$Y'(0) = 0 = Y'(l) \tag{25}$$

The determination of a nontrivial $Y(y)$ mirrors the procedure to determine $X(x)$. It is clear therefore, that $\gamma \geq 0$ and is restricted by the requirement

$$\sqrt{\gamma} = \frac{m\pi}{l} \equiv \sqrt{\gamma_m} \tag{26}$$

for integer $m \geq 0$, and $Y$ is restricted to

$$Y(y) = \cos\frac{m\pi x}{l} \equiv Y_m(y) \tag{27}$$

Using (23), and relations (22) and (27), we obtain

$$T'(t) + \kappa\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)T(t) = 0 \tag{28}$$

Therefore, we must have $T(t)$ of the form

$$T(t) = A_{m,n}\exp\left[-\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)\kappa t\right] \equiv T_{m,n}(t) \tag{29}$$

for some constants $A_{m,n}$, which can depend on integers $m, n$. Therefore, using linear superposition of solutions of the form $X_n(x)Y_m(y)T_{mn}(t)$, we obtain a more general solution satisfying (15) and (16)

$$u(x,y,t) = \sum_{m\geq 0, n\geq 0} A_{m,n}\exp\left[-\left(\frac{n^2\pi^2}{l^2} + \frac{m^2\pi^2}{l^2}\right)\kappa t\right]\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} \tag{30}$$

This satisfies initial condition

$$u(x,y,0) = \sum_{m\geq 0, n\geq 0} A_{m,n}\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} \tag{31}$$

Therefore, if $\phi(x,y)$ in (17) is expressible in the form (31), we will have a representation of the solution to the Neumann problem, which is known to be unique from using energy method.. From Fourier series theory, which we will soon review, this is not much of a restriction on $\phi(x,y)$. Very general functions $\phi$ in the domain $(0,l) \times (0,l)$ have the representation (31) for suitably chosen $A_{m,n}$, which of course depends on $\phi$.

## 1.3 Robin Boundary Condition

We continue discussion of method of separation of variable for the case of Robin boundary condition.

We will illustrate this for the 1-D wave equation (1), though there is a parallel for every step for heat equation (see Text, section 4.3). Our boundary conditions

$$u_x(0,t) - a_0 u(0,t) = 0 = u_x(l,t) + a_l u(l,t) \tag{32}$$

for some constant $a_0$, $a_l$.

This is a special case of the more general Neumann condition in $n$-dimension of the form

$$\frac{\partial u}{\partial n} + a(\mathbf{x}) u = 0 \quad \text{for} \quad \mathbf{x} \in \partial\Omega$$

It is not difficult to prove (Homework problem) that if we define

$$\mathcal{E} = \int_\Omega \frac{1}{2} \left( (\nabla u)^2 + \frac{1}{c^2} u_t^2 \right) d\mathbf{x} + \frac{1}{2} \int_{\partial\Omega} a u^2 \tag{33}$$

for $n$-dimensional wave equation with the above Robin boundary condition, then the energy $\mathcal{E}$ is a constant in time. This can be used to prove uniqueness of solution to the Robin problem when $a(\mathbf{x}) > 0$.

We now return to the showing the existence of soluton to 1-D wave equation with Robin condition (32) with separation of variable technique.

The initial conditions will be the same as before for the Dirichlet problem and is given by (3). As before when we try simple solutions of the form $X(x)T(t)$, we obtain

$$X''(x) + \lambda X(x) = 0 \tag{34}$$

$$T''(x) + \lambda c^2 T(t) = 0 \tag{35}$$

Consider first the case, when $\lambda = \beta^2 > 0$. Then, solution to (34) is of the form

$$X(x) = C \cos \beta x + D \sin \beta x \tag{36}$$

so that

$$X'(x) \pm a X(x) = (\beta D \pm aC) \cos \beta x + (-\beta C \pm aD) \sin \beta x \tag{37}$$

At the left end, $x = 0$, we require that

$$0 = X'(0) - a_0 X(0) = \beta D - a_0 C \tag{38}$$

So, we can solve for $D$ in terms of $C$. At the right end $x = l$, we have

$$0 = (\beta D + a_l C) \cos \beta l + (-\beta C + a_l D) \sin \beta l \tag{39}$$

equations (38) and (39) imply the matrix relation

$$\begin{pmatrix} -a_0 & \beta \\ a_l \cos \beta l - \beta \sin \beta l & \beta \cos \beta l + a_l \sin \beta l \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{40}$$

Condition for an non-trivial solution implies that the determinant of the coefficient matrix should be 0, implying

$$(\beta^2 - a_0 a_l)\tan\beta l = (a_0 + a_l)\beta \tag{41}$$

unless $\cos\beta l = 0$, which is consistent with (38) and (39) only when $\beta = \sqrt{a_0 a_l}$. Any root $\beta > 0$ of the transcendental equation (41) would give us an eigenvalue $\lambda = \beta^2$. Determining relation between $C$ and $D$ in (40), it follows that the corresponding non-trivial solution $X(x)$, called an "eigen-function", is

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right) \tag{42}$$

for nonzero $C$.

Finding $\beta$ explicitly from the transcendental equation (41) is not possible. However, it is convenient to rewrite it in the form

$$\tan\alpha = \frac{l(a_0 + a_1)\alpha}{\alpha^2 - l^2 a_0 a_l}, \quad \text{where} \quad \alpha = l\beta \tag{43}$$

#### 1.3.1 Case 1: $a_0, a_l > 0$ (Radiating boundary conditions)

We now graphically plot each side of (43), as shown in Figure 1, for the case $a_0 > 0$, $a_l > 0$, physically corresponding to radiation at both ends. The intersection of the two graphs determine eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ as shown in the Figure.

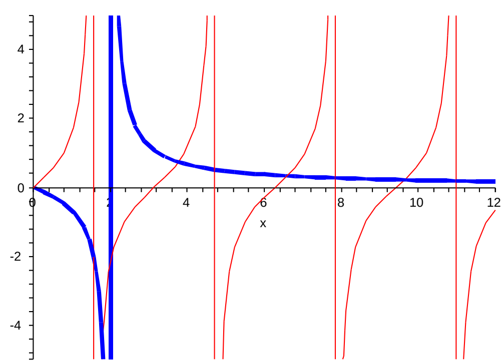


Figure 1: Plot of left and right side of (43) as function of $\alpha$. The $\alpha$ values of the $n$-th intersection point, starting from $n = 0$, determine eigenvalues $\lambda_n = \beta_n^2 = \frac{\alpha_n^2}{l^2}$.

Since the $n$-th intersection point has $\alpha \in (n\pi, (n+1)\pi)$ and for large $n$ approaches $n\pi$, it follows

$$\frac{n^2\pi^2}{l^2} < \lambda_n < \frac{(n+1)^2\pi^2}{l^2}, \tag{44}$$

$$\lim_{n\to\infty} \lambda_n - \frac{n^2\pi^2}{l^2} = 0 \tag{45}$$

The large $n$-limit corresponds to setting $a_0 = a_l = 0$, *i.e.* we get the same eigen values as for the Neumann problem.

#### 1.3.2 Case 2: $a_0 < 0, a_l > 0$, but $a_0 + a_l > 0$

Physically, this corresponds to absorption at $x = 0$, radiation at $x = l$,but *more radiation than absorption*. In this case we plot both left and right sides of (43), as shown in Figures 2 or 3, depending on the relative sizes of $a_0$ and $a_l$. The intersection points determine a discrete set of $\alpha_n$, which are the $\alpha$ values of the intersection point. This determine eigenvalues $\lambda_n = \beta_n^2 = \frac{\alpha_n^2}{l^2}$. Note that once again relations (44) and (45) hold. However, in the case corresponding to Figure 3, there is no eigenvalue $\lambda_0$ in the interval $\left(0, \frac{\pi^2}{l^2}\right)$.

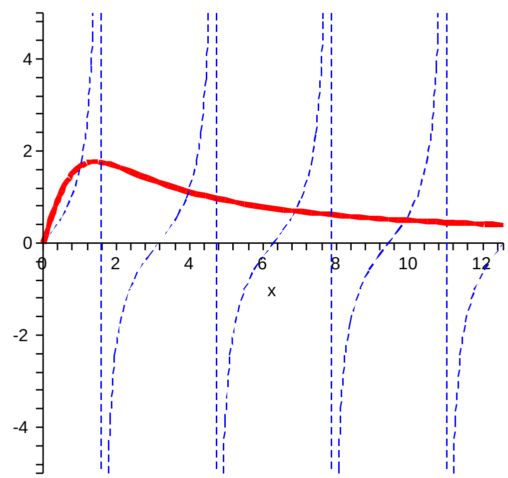


Figure 2: Plot of left and right side of (43) as function of $\alpha$, when $a_0 < 0$. $a_l > 0$, $a_0 + a_l > 0$, when $a_0 + a_l > -la_0a_l$.

There is an eigenvalue in the interval $\left(0, \frac{\pi^2}{l^2}\right)$ only if the rational curve crosses the *first* branch of the tangent curve.Since the rational curve has a single maximum, this corssing can happen only if the slope of the rational curve is greater than the slope of the tangent curve at the origin. This leads to the condition $a_0 + a_l > -la_0a_l$. This is the condition that there is an eigenvalue $\lambda_0 \in \left(0, \frac{\pi^2}{4l^2}\right)$.

**Non-positive eigenvalues:** If $\lambda = -\gamma^2 < 0$, then the solution to (34) is

$$X(x) = C \cosh \gamma x + D \sinh \gamma x. \tag{46}$$

Satisfying the Robin boundary conditions at $x = 0$ and $x = l$, in the same way as before, we

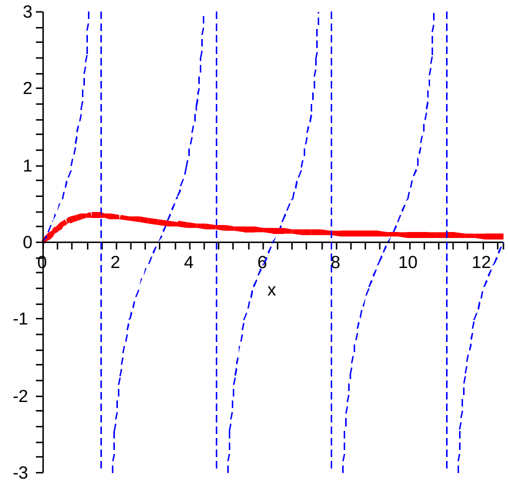


Figure 3: Plot of left and right side of (43) as function of $\alpha$, when $a_0 < 0$. $a_l > 0$, $a_0 + a_l > 0$ and $a_0 + a_l < -la_0a_l$.

find that in this case the eigenvalue equation is

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l} \tag{47}$$

So, we look for intersection of the two graphs on the left and right of (47); if there is such an intersection, the corresponding eigenfunction is

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x \tag{48}$$

Several different cases are possible.

First case, is when both $a_0, a_l > 0$, i.e. there is radition at both ends. In that case, the two curves cannot intersect, since one is positive and the other is negative, and there can be no $\gamma \neq 0$ satisfiying (47).

The second case is $a_0 < 0$ and $a_l > 0$, with $a_0 + a_l > 0$, *i.e.* more radiation than absorption. Additionally, if we have

$$a_0 + a_l < -la_0a_l$$

then the slope of the tanh curve at the origin is $l$ and it exceeds the slope of the rational curve, which is $-\frac{a_0+a_l}{a_0a_l}$. In this case, there is an intersection (see Figure 4); otherwise there is no intersection.

It is to be noted that for this case the missing eigenvalue $\lambda_0$ has now shifted to a negative eigenvalue.

As an exercise, you can show that a zero eigenvalue are possible if and only $a_0 + a_l = -a_0a_ll$.

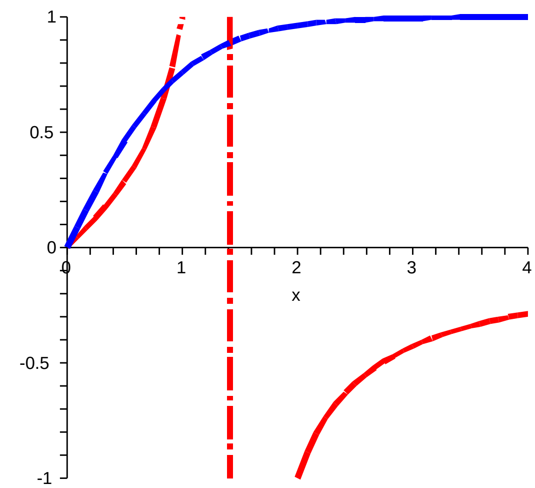


Figure 4: Plot of left and right side of (47) as function of $\gamma$, when $a_0 < 0$. $a_l > 0$, $a_0 + a_l > 0$ and $a_0 + a_l < -l a_0 a_l$.

### 1.4 Summary for Robin BC for 1-D Wave equation

A general solution obtained from separation of variable is of the form

$$u(x,t) = \sum_n T_n(t) X_n(x) \tag{49}$$

where for $\lambda_n > 0$

$$T_n(t) = A_n \cos\left\{\sqrt{\lambda_n} ct\right\} + B_n \sin\left\{\sqrt{\lambda_n} ct\right\} \tag{50}$$

and for $\lambda_n < 0$ (from (35)

$$T_n(t) = A_n \cosh\left[\sqrt{-\lambda_n} ct\right] + B_n \sinh\left[\sqrt{-\lambda_n} ct\right] \tag{51}$$

where eigenvalue $\lambda_n$ is determined as described graphically in the last subsection, and corresponding eigen fuctions are determined by (42) for $\lambda_n > 0$ and (48) for $\lambda_n < 0$. The initial conditions satisfied by (49) is

$$u(x,0) = \sum_n A_n X_n(x) \quad ; \quad u_t(x,0) = \sum_n \sqrt{|\lambda_n|} c B_n X_n(x) \tag{52}$$

We will learn later from general Sturm-Liouville theory that the set of eigen functions $\{X_n\}_n$ is complete, meaning that (44), with appropriate choice of $A_n$ and $B_n$ is able to represent any $\phi$ and $\psi$, with some minimal restrictions.

# 2 Separation of variable in Circular Geometry

Separation of variable is not restricted to rectangular geometry. The method of solution provides solution to either Laplace, heat or wave equation and other related equations in circular (in 2-D) and spherical geometries (in 3-D). We illustrate this for Laplace's equation in a circle with Dirichlet boundary condition.

## 2.1 Laplace's equation in a circle

Consider the following 2-D problem:

$$\Delta u = 0 \quad ; \quad \text{for} \quad |\mathbf{x}| < 1 \tag{53}$$

with Dirichlet boundary condition on $|\mathbf{x}| = 1$. It is convenient in this case to introduce polar coordinates $(r, \theta)$. Using expression for Laplacian in polar coordinates, $u = u(r, \theta)$ satisfies

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{for} \quad r < 1 \tag{54}$$

The Dirichlet boundary condition is

$$u(1, \theta) = f(\theta) \tag{55}$$

We seek a separation of variable solution to (54) in the form

$$u(r, \theta) = R(r)\Theta(\theta) \tag{56}$$

Plugging into (54) and dividing the result by $R(r)\Theta(\theta)/r^2$, we obtain

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2 \left( \frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} \right) \tag{57}$$

Now, since left and right sides of (57) are functions of $\theta$ and $r$ respectively, it follows that each side is some constant $\lambda$. Therefore, $\Theta(\theta)$, $R(r)$ satisfy:

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \tag{58}$$

$$R''(r) + \frac{1}{r} R'(r) = \frac{\lambda}{r^2} R(r) \tag{59}$$

Now, since the solution $u(r, \theta)$ sought is univalued in the disk of radius 1, it follows that $u(r, \theta + 2\pi) = u(r, \theta)$. Hence, we must have

$$\Theta(\theta + 2\pi) = \Theta(\theta) \tag{60}$$

Now, consider the case $\lambda < 0$. Then the solution to (58) is of the form

$$\Theta(\theta) = A \exp\left(-\sqrt{-\lambda}\theta\right) + B \exp\left(\sqrt{-\lambda}\theta\right) \tag{61}$$

But this does not satisfy (60) regardless of the value of $A$ and $B$, except for the trivial case $(A, B) = (0, 0)$. Therefore, we rule this out.

For $\lambda = 0$, we have

$$\Theta(\theta) = A\theta + a_0 \tag{62}$$

This does not satisfy (60), unless $A = 0$. However, note that it does not rule out a nonzero $a_0$.

Now, comes the case of $\lambda = \mu^2 > 0$. Then, the solution to (58) is

$$\Theta(\theta) = a \cos \mu\theta + b \sin \mu\theta \tag{63}$$

Then condition (60) for any $\theta$ implies

$$\mu = m \quad > \; 0 \;\text{ an integer implying }\; \lambda = \lambda_m \equiv m^2 \tag{64}$$

Therefore,

$$\Theta(\theta) = \Theta_m(\theta) \equiv a_m \cos m\theta + b_m \sin m\theta \tag{65}$$

Since $m = 0$ in this formula includes the case of $\lambda = 0$, (65) provides *all* the eigen functions $\Theta_m$ for $\lambda = \lambda_m = m^2$ for integer $m \geq 0$. Having determined $\lambda$, we go back to (59) to obtain

$$R''(r) + \frac{1}{r}R'(r) - \frac{m^2}{r^2}R(r) = 0 \tag{66}$$

Noting that the equation is homogeneous in $r$, we look for solutions of the form $R(r) = r^\alpha$ and plug into (66), to discover that nontrival solutions are possible when $\alpha = \pm m$, *i.e* $R(r)$ is a linear combination of $r^{-m}$ or $r^m$ for $m > 0$. In the special case $m = 0$, we find $R(r) = c_0 + d_0 \log r$ for constants $c_0$ and $d_0$. However, since the solution $u(r, \theta)$ we are seeking is a classical solution without any singularities for $r < 1$, $R(r)$ must be well-behaved in particular at $r = 0$. This rules out $r^{-m}$ term for $m > 0$ and log $r$ for $m = 0$. Thus,

$$R(r) = R_m(r) \equiv r^m \;\text{ for integer } m \geq 0 \tag{67}$$

Taking a linear combination, we obtain for $r < 1$

$$u(r, \theta) = a_0 + \sum_{m=1}^{\infty} \left(a_m r^m \cos m\theta + b_m r^m \sin \theta\right) \tag{68}$$

Assuming that this series converges, the solution is capable of satisfying Dirichlet boundary condition $u(1, \theta) = f(\theta)$ if

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos m\theta + b_m \sin m\theta\right) \tag{69}$$

As we shall see from the theory of Fourier Series, under mild restrictions, quite general functions $f(\theta)$ possess such an expansion, which is called its Fourier Series.

Note also, an alternate representation of the solution (65) in the form

$$u(r, \theta) = a_0 + \sum_{m=1}^{\infty} c_m \cos(m\theta - \delta_m) \tag{70}$$

where $a_m = c_m \cos\delta_m$ and $b_m = c_m \sin\delta_m$. and this can be written as

$$u(r,\theta) = \Re\left\{\sum_{m=0}^{\infty} C_m z^m\right\}, \tag{71}$$

where $z = re^{i\theta}$, $\Re C_0 = a_0$ and $C_m = c_m e^{-i\delta_m}$. The quanitity on the right of (71) is the real part of an analytic function of $z = x + iy = re^{i\theta}$ and this representation is not surpising since any solution to Laplace's equation in 2-D, with $(x, y)$ chosen as independent variables, is always either the real or imaginary part of an analytic function of $x + iy$.