Week 7 Lectures, Math 716, Tanveer

1 Introduction

Recall we discussed completeness of Fourier Series. This is relevant for constant coefficient partial differential equations in simple rectangular geometries or sometimes in circular geometry as well. Separation of variable procedure leads to a series representation of solution. The undetermined constants are Fourier Coefficients of the series evaluated initially or on certain segments of the boundary.

As an example, recall solution to $\nabla u = 0$ for $x^2 + y^2 < 1$ was in the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta \right)$$

If we use boundary condition $u(1, \theta) = f(\theta)$, then the constants a_n, b_n are simply the Fourier coefficients of $f(\theta)$.

In more complicated situations, separation of variable is not possible and even if it is possible, it does not lead to a simple Fourier Series. For instance, in one of your homework problems involving wave equation in a circular geometry, Bessel function arose and one needed to use some properties of Bessel functions.

We seek to construct a more general theory on completeness of series representation based on eigenfunctions of some differential operators with homogenous boundary conditions. Note that the Fourier Sine Series $\left\{ \sin \frac{n\pi x}{l} \right\}_{n=1}^{\infty}$ is a special example in 1-D as it arises from eigenfunction of the operator $-\frac{d^2}{dx^2}$ satisfying

$$-\frac{d^2 u}{dx^2} = \lambda u \quad \text{for} \quad 0 < x < l \quad ; \quad \text{with BC} \quad u(0) = 0 = u(l)$$

2 General Eigen Value Problem:

We introduce $\mathcal{L}_2(\Omega)$ inner-product

$$(f, g) = \int_{\Omega} f(\mathbf{x}) \left[g(\mathbf{x}) \right]^* d\mathbf{x} \tag{1}$$

Definition 1 *An operator $A : \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$ is symmetric (or self-adjoint) if for any u, v in the domain of A, $(v, Au) = (Av, u)$*

Definition 2 *A symmetric operator is positive if $(u, Au) > 0$ for any nonzero $u \in Domain(A)$. A symmetric operator is semi-postive if $(u, Au) \geq 0$ for any $u \in Domain(A)$.*

Lemma 3 *If $A : \mathcal{L}_2(\Omega) \to \mathcal{L}_2(\Omega)$ is symmetric (self-adjoint), then its eigenvalues are real. Further, eigenvectors corresponding to two distinct eigenvalues are orthogonal, with respect to the inner-product defined above. Further, if A is positive, any eigenvalue $\lambda > 0$ and for A semi-positive, $\lambda \geq 0$.*

Proof. If u is an eigen function of A corresponding to eigenvalue λ, then it follows from definition of inner-product

$$\lambda(u,u) = (Au,u) = (u,Au) = \lambda^*(u,u)$$

Hence $\lambda = \lambda^*$ since $(u,u) = \|u\|^2 \neq 0$ for an eigenfunction Further, if u and v are eigenfunctions corresponding to unequal eigenvalues λ and μ it follows that

$$\lambda(u,v) = (Au,v) = (u,Av) = (u,\nu v) = \nu^*(u,v) = \nu(u,v)$$

Therefore, $(\lambda - \nu)(u,v) = 0$ implying $(u,v) = 0$, *i.e.* orthogonal. If A is positive,

$$\lambda(u,u) = (Au,u) > 0$$

Therefore, $\lambda > 0$. For semi-positive A, the above clearly gives $\lambda \geq 0$ □

Remark 1 *There is no loss of generality assuming orthogonality of eigenvectors $(u,v) = 0$ even when corresponding eigenvalues $\mu = \nu$. This is because a Gram-Schmidt orthogonalization procedure can be employed.*

Remark 2 *We will now consider the operator $-\Delta$ in a domain $\Omega \subset \mathbb{R}^n$, with homogenous Dirichlet, Neumann or Robin boundary conditions. We will assume $\partial\Omega$ is piecewise $\mathbf{C}^1$. Further, there is no loss of generality in assuming all eigen-functions to be real, since the eigenvalues are real, as the following Theorem concludes.*

Lemma 4 *The operator $-\Delta$ with homogeneous Dirichlet, Neumann or Robin Boundary condition (i.e. $\frac{\partial u}{\partial n} + a(\mathbf{x})u = 0$ on $\partial\Omega$, with $a \in \mathbf{PC}^0(\partial\Omega)$) is symmetric. Further, for Dirichlet or Robin Boundary condition for $a(\mathbf{x}) > 0$ (outward radiation) the operator is positive. If $a(\mathbf{x}) \geq 0$ in Robin Boundary condition (includes Neumann as a special case), the operator is semi-postive. The corresponding eigenvalues are real and positive in the former and non-negative in the latter case.*

Proof. We will assume $f, g \in \mathbf{C}^2(\Omega) \cap \mathbf{PC}^1(\bar{\Omega})$, which is the domain of $-\Delta$ satisfying the boundary conditions given. It is known from analysis that more general f, g, with only square integrability of the function and its first-derivative, can be approximated arbitrarily closely in the $\mathcal{L}^2(\Omega)$ sense by such functions. Then,

$$\begin{aligned}(-\Delta f, g) = -\int_\Omega \Delta f(x)[g(x)]^* d\mathbf{x} &= -\int_\Omega \nabla \cdot [\nabla f g^*] d\mathbf{x} + \int_\Omega \nabla f \cdot \nabla g^* d\mathbf{x} \\ &= -\int_{\partial\Omega} \left[\frac{\partial f}{\partial n} g^* - \frac{\partial g^*}{\partial n} f\right] d\mathbf{x} - \int_\Omega f \Delta g^* d\mathbf{x} = (f, -\Delta g)\end{aligned}$$

since

$$\frac{\partial f}{\partial n} g^* - \frac{\partial g^*}{\partial n} f = 0$$

whenever homogeneous Dirichlet, Neumann or Robin Condition is imposed on f, g[1]. Therefore, $(-\Delta f, g) = (f, -\Delta g)$ and we have a symmetric (self-adjoint) operator. On the otherhand, note from above calculation, that if we choose $f = g$, then for Dirichlet conditions

$$(-\Delta f, f) = (\nabla f, \nabla f) > 0$$

whenever $f \neq 0$, since a nonzero constant f is ruled out by $f = 0$ on $\partial\Omega$. Further, for Robin boundary conditions for $a(\mathbf{x}) > 0$,

$$(-\Delta f, f) = (\nabla f, \nabla f) + \int_{\partial\Omega} a(\mathbf{x}) |f(\mathbf{x})|^2 d\mathbf{x} > 0$$

whenever $f \neq 0$. However, if $a(\mathbf{x}) \geq 0$, then the above gives $(-\Delta f, f) \geq 0$ and the operator is semi-positive. □

Remark 3 *Note that* $f =$ Constant *is possible for* $a(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in \partial\Omega$. *However, if* $a(\mathbf{x}) \in \mathbf{C}^0(\partial\Omega)$ *and a is not identically zero, then even for Robin Boundary condition, we have* $-\Delta$ *to be a positive operator.*

Remark 4 *There are more general symmetric differential operators than Laplacian itself. For instance, the eigenvalue problem of the Laplacian is a special case of*

$$-\nabla \cdot (p \nabla u) + qu = \lambda m u \quad \text{for} \quad \mathbf{x} \in \Omega$$

where $q, m \in \mathbf{C}^0(\Omega)$, *while* $p \in \mathbf{C}^1(\Omega)$ *and* $m(\mathbf{x}) > 0$, $p(\mathbf{x}) > 0$ *and we can assume one of Dirichlet, Neumann, Robin Boundary conditions (or other such boundary conditions) that make the operator symmetric. In 1-D, these are referred to as the Sturm-Liouville operators.*

3 Minimum principles for the Eigenvalues of Laplacian in Dirichlet problem

Consider the eigenvalue problem for the Dirichlet problem:

$$-\Delta u = \lambda u \quad \text{for} \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \quad \mathbf{u} = 0 \quad \text{on} \ \partial\Omega \tag{2}$$

where Ω is an arbitary bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ piecewise smooth. We will assume $u \in \mathbf{C}^2(\Omega) \cap \mathbf{PC}^1(\bar{\Omega})$. We will order the eigenvalues, which we know from last section to be real and positive, as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq .. \leq \lambda_n \leq .. \tag{3}$$

where repeated eigenvalues are indexed by a different integer. The corresponding eigenvectors will be assumed to be real, without any loss of generality.

We will relate the eigenvalues to the following minimization problem:
(MP) Determine $u \in \mathbf{C}^2$, *with* $u = 0$ *on* $\partial\Omega$ *so that it minimizes*

$$m = \inf_{w \in \mathbf{C}^2} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w = 0 \text{ on } \partial\Omega, \quad w \neq 0 \right\}$$

[1]Note in the Robin case, we are using $a(\mathbf{x})$ is assumed real

where w is real valued.

While it is clear that $m \geq 0$, what is not clear is whether the m is actually attained for any function $w \in \mathbf{C}^2$. This requires a more sophisticated theory involving Sobolev space $H_0^1(\Omega)$ (material for more advanced PDE theory Math 835-836) than we are ready to discuss. We will assume that the MP problem indeed has a solution.

Theorem 5 *(Minimum Principle for First Eigenvalue) Assume that $u(\mathbf{x})$ is a solution to the MP problem above. Then $m = \lambda_1$, the smallest eigenvalue of $-\Delta$ and u is the corresponding eigenfunction satisfying $u = 0$ on $\partial\Omega$.*

PROOF. We take any real *trial function* $w \in \mathbf{C}^2$, with $w = 0$ on $\partial\Omega$. Then, from the above minimization criteria

$$m = \frac{\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}|}{|u(\mathbf{x})|^2 d\mathbf{x}} \leq \frac{\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x}|}{|w(\mathbf{x})|^2 d\mathbf{x}}$$

In particular if we take $w(\mathbf{x}) = u(\mathbf{x}) + \epsilon v(\mathbf{x})$, then

$$f(\epsilon) \equiv = \frac{(\nabla u + \epsilon \nabla v, \nabla u + \epsilon \nabla v)}{(u + \epsilon v, u + \epsilon v)}$$

has a minimum value at $\epsilon = 0$. Expanding out, we get

$$f(\epsilon) = \frac{(\nabla u, \nabla u) + 2\epsilon(\nabla u, \nabla v) + \epsilon^2(\nabla v, \nabla v)}{(u, u) + 2\epsilon(u, v) + \epsilon^2(v, v)}$$

On straight forward differentiation with respect to ϵ, condition for minimum becomes

$$0 = f'(0) = \frac{2\|u\|^2(\nabla u, \nabla v) - 2\|\nabla u\|^2(u, v)}{\|u\|^4}$$

So, it follows that

$$(\nabla u, \nabla v) = \frac{\|\nabla u\|^2}{\|u\|^2}(u, v) = m(u, v)$$

Therefore, from integration by parts and using $v = 0$ on $\partial\Omega$,

$$(\Delta u + mu, v) = 0$$

for all v. Therefore, $\Delta u + mu = 0$ and m is an eigenvalue of $-\Delta$ with Dirichlet BC. Since m is the smallest such value, $m = \lambda_1$, and $u(\mathbf{x})$ which solves the MP problem is actually the corresponding eigenfunction. □

Remark 5 *If we solve the MP problem without imposing the boundary condition $w = 0$ on $\partial\Omega$, then the corresponding minimum m will actually be the smallest eigenvalue for $-\Delta$ for the Neumann problem. The proof is very similar to above, except that at the last stage, integration by parts of $(\nabla u, \nabla v) = m(u, v)$ leads to*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v d\mathbf{x} + (\Delta u + mu, v) = 0$$

Since this is true for any $v(\mathbf{x})$, both $\Delta u + mu = 0$ in Ω and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Thus u is an eigenfunction of $-\Delta$ with homogeneous Neumann condition.

Remark 6 *One can show that for the Robin problem, the relevant minimization that leads to eigenvalues and eigenfunctions of $-\Delta$ is to minimize*

$$\frac{(\nabla w, \nabla w) + \int_{\partial\Omega} a w^2 d\mathbf{x}}{(w, w)}$$

for nonzero $w \in \mathbf{C}^2$.

Theorem 6 *(Minimum principle for n-eigenvalue for Dirichlet Problem) Suppose λ_1, λ_2, ... λ_{n-1} are all known with corresponding eigenvectors v_1, v_2,...v_{n-1}. Then*

$$\lambda_n = \inf_{w \in \mathbf{C}^2} \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w \neq 0, w = 0 \text{ on } \partial\Omega \right\} \tag{4}$$

when w constrained by additional requirement

$$0 = (w, v_1) = (w, v_2) = (w, v_3).. = (w, v_{n-1}), \tag{5}$$

where we assume further the inf *is achieved for some $w = u$.*

Proof. As before let $u \in \mathbf{C}^2$ that assumes the constrained minimum value. We take $w = u + \epsilon v$ for some real vaued v, with $(w, v_1) = (w, v_2) = (w, v_3) = ... = 0$ for any ϵ. This means that both u and v are orthogonal to the set $\{v_1, v_2, ..v_{n-1}\}$. As in the proof of last theorem, condition of minimization leads to

$$(\Delta u + \tilde{m} u, v) = 0 \tag{6}$$

where $\tilde{m}$ is the minimum value of the quotient on the right of (4) with constraint (5). Further, we note that for $j = 1, 2, ..n - 1$,

$$(\Delta u + \tilde{m} u, v_j) = (u, \Delta v_j) + \tilde{m}(u, v_j) = (\tilde{m} - \lambda_j)(u, v_j) = 0 \tag{7}$$

since u is orthogonal to v_j. Now, take an arbitrary $\tilde{v} \in \mathbf{C}^2$. Since $\tilde{v} \in \mathcal{L}_2(\Omega)$, we can use projection to write

$$\tilde{v} = \frac{(\tilde{v}, v_1) v_1}{\|v_1\|^2} + \frac{(\tilde{v}, v_2) v_2}{\|v_2\|^2} + .. + \frac{(\tilde{v}, v_{n-1}) v_{n-1}}{\|v_{n-1}\|^2} + v, \tag{8}$$

where $(v, v_j) = 0$ for $j = 1$,..,$n - 1$. Therefore, Using (8), (6) and (7), we obtain

$$(\Delta u + \tilde{m} u, \tilde{v}) = (\Delta u + \tilde{m} u, v) = 0$$

for arbitrary $\tilde{v}$ as above. Therefore, it follows that

$$\Delta u + \tilde{m} u = 0$$

and $\tilde{m}$ is an eigenvalue of $-\Delta$ with Dirichlet boundary condition. Since λ_1, λ_2, ..λ_{n-1} were obtained by minimization of the same quantity as on the right of (4) with fewer constraints in (5), it follows that $\tilde{m} \geq \lambda_{n-1}$ It follows that $\lambda_n = \tilde{m}$. □

Remark 7 *We also have the same minimum principle for the n-th eigenvalue* $\tilde{\lambda}_n$ *of* $-\Delta$ *for the Neumann problem, except that the space of trial functions* $w(\mathbf{x})$ *is not constrained by the requirement* $w = 0$ *on* $\partial\Omega$*. Hence the following inequality involving eigenvalues of the Neumann and Dirichlet problem follows:*

$$\tilde{\lambda}_n \leq \lambda_n$$

Remark 8 *We can construct similar minimum principle for the Robin Problem, except that we need to minimize*

$$\frac{\|\nabla w\|^2 + \int_{\partial\Omega} aw^2 d\mathbf{x}}{\|w\|^2}$$

4 Asymptotics for large n for eigenvalues

We seek to understand the behavior of λ_n as $n \to \infty$. We start with two dimensional domain.

4.1 Rectangular domains

In the case the domain $\Omega = \{(x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b\} \subset \mathbb{R}^2$ the discrete eigen values of $-\Delta$ for Dirichlet Problem are given by

$$\lambda = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \quad \text{for} \quad l, m \geq 1$$

with corresponding eigenfunctions $\sin \frac{l\pi x_1}{a} \sin \frac{m\pi x_2}{b}$. For Neumann problem, we have the same set of eigenvalues, but it includes $l = 0$ and $m = 0$. The corresponding eigen functions are $\cos \frac{l\pi x_1}{a} \cos \frac{m\pi x_2}{b}$.

It is awkward to associate each (l, m) to n in counting the eigenvalues and yet more difficult determine the asymptotic formula for $n-th$ eigenvalue for large n. For this purpose, we introduce the *enumeration function*

$$N(\lambda) = \text{ number of eigenvalues that do not exceed } \lambda$$

When we order λ_n then $N(\lambda)$ is the number of integer lattices (l, m) which are contained in the first quadrant of the ellipse

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} \leq \frac{\lambda}{\pi^2} \quad \text{for } l, m > 0$$

Therefore, $N(\lambda)$ is at most the area of this quarter ellipse:

$$N(\lambda) \leq \frac{\lambda ab}{4\pi}$$

For large λ, $N(\lambda)$ differs from this area at most by a multiple of the length of the perimeter, which scales as $\sqrt{\lambda}$. Therefore,

$$\frac{\lambda ab}{4\pi} - C\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda ab}{4\pi}$$

for some constant C. Substituting $N(\lambda) = n$, and $\lambda = \lambda_n$, we obtain

$$\frac{\lambda_n ab}{4\pi} - C\sqrt{\lambda_n} \leq n \leq \frac{\lambda_n ab}{4\pi}$$

Therefore,

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab} = \frac{4\pi}{\text{Rectangle Area}} \tag{9}$$

For Neumann condition, the only difference is that each of l and m are allowed to be zero as well. However, this only accounts for a constant multiple of the rectangle perimeter. Hence, in that case the corresponding eigenvalue $\tilde{\lambda}_n$ still statisfy the condition.

$$\lim_{n\to\infty} \frac{\tilde{\lambda}_n}{n} = \frac{4\pi}{ab} = \frac{4\pi}{\text{Rectangle Area}} \tag{10}$$

We can repeat the same argument for a rectangular region in any number of dimension. For instance in three dimensions, we get

$$\lim_{n\to\infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{\text{Volume of Rectangular Box}} \tag{11}$$

4.2 Maximin Principle

We now return to discussion of eigenvalues for a more general domain. For that purpose we need maximin principle as stated below.

Definition 7 *In the theorems below, a trial function y will refer to a function in $\mathbf{C}^2(\Omega)\cap\mathbf{C}^1(\bar{\Omega})$, A test function w is also in the same space, but is constrained by conditions of orthogonality and boundary condition as appropriate.*

Theorem 8 *(Maximin principle for Dirichlet Problem) Consider $n-1$ trial functions $\{y_j\}_{j=1}^{n-1}$ for $n \geq 2$. Let*

$$\lambda_{n^*} = \inf_w \frac{\|\nabla w\|^2}{\|w\|^2}$$

among all test functions w orthogonal to $\{y_j\}_{j=1}^{n-1}$ with $w = 0$ on $\partial\Omega$. Then

$$\lambda_n = \sup_{y_1, y_2, ..y_{n-1}} \lambda_{n^*}$$

over all choices of $n-1$ trial functions.

Proof. Fix an arbitrary choice of y_1, y_2, $..y_{n-1}$. Pick $w = \sum_{j=1}^n c_j v_j$, where v_1, $..v_n$ are the first n eigenfunctions of $-\Delta$, chosen to be an orthonormal set w.l.o.g. Orthogonality condition on w implies:

$$0 = \left(\sum_{j=1}^n c_j v_j, y_k\right) = \sum_{j=1}^n (v_j, y_k) c_j \quad \text{for} \quad k = 1, 2, ..n-1 \tag{12}$$

This constitutes a homogeneous system of $n-1$ equations with n unknowns; a nonzero $(c_1, c_2, ..c_n)$, and therefore a nonzero w, can be found to satisfy the orthogonality condition.

Since from definition,

$$\lambda_{n^*} \leq \frac{\|\nabla w\|^2}{\|w\|^2} = \frac{\sum_{j,k} c_j c_k(-\Delta v_j, v_k)}{\sum_{j,k} c_j c_k(v_j, v_k)} = \frac{\sum_{j=1}^n \lambda_j c_j^2}{\sum_{j=1}^n c_j^2} \leq \lambda_n,$$

it follows $\sup_{y_1,..y_{n-1}} \lambda_{n^*} \leq \lambda_n$. However, if we make the particular choice $y_1 = v_1$, $y_2 = v_2$, $y_{n-1} = v_{n-1}$, then a solution to (12) is given by $(c_1, c_2, ..., c_n) = (0, 0, 0, ..., 1)$ and in that case, computation above shows $\lambda_n^* = \lambda_n$. □

Remark 9 *The same maximin principle holds for Neumann BC, with test functions not constrained by boundary conditions.*

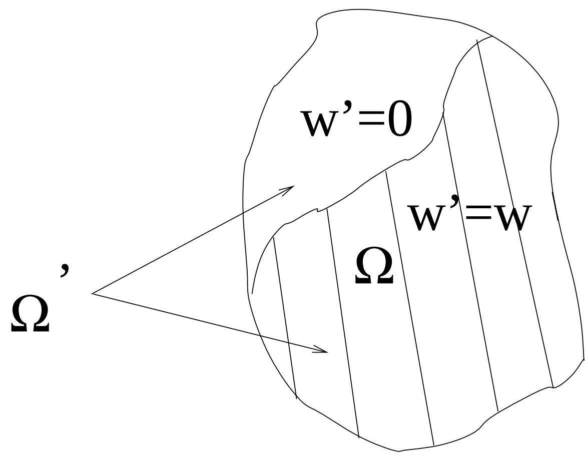

Figure 1: Choice of *test* function w' in enlarged domain Ω'

Theorem 9 *Consider the eigenvalue of the operator $-\Delta$ either for the Dirichlet or Neumann problem. If the domain is enlarged, then each eigenvalue is decreased.*

Proof. Let Ω be the original domain, and Ω' be the enlarged domain. In the Dirichlet case consider the expression,

$$\lambda_n = \sup_{y_1, y_2, ..y_{n-1}} \lambda_{n^*}$$

If $w(\mathbf{x})$ is any *test function* in Ω orthogonal to a given set of $n-1$ *trial functions*, we define *test* function $w'(\mathbf{x})$ in Ω' so that $w'(\mathbf{x}) = w(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and zero otherwise[2] (See Fig. 1). Every such

[2]Stictly speaking, we cannot choose w' to be a *test* function since it is not in $\mathbf{C}^2$. However, for any function which, together with its derivative is in $\mathcal{L}_2(\Omega)$, that includes our w', it is known from Sobolev space theory that there exists a $\mathbf{C}^\infty(\Omega)$ function which will approximate $\frac{\|\nabla w'\|^2}{\|w\|^2}$ as closely as we like. So we ignore this gap in our "proof".

test function in Ω corresponds to a *test* function in Ω'. However, the class of *test* functions Ω' is larger as an arbitrary w' need not vanish on $\Omega' \setminus \Omega$. Hence for given $\{y_1, y_2, ..y_{n-1}\}$, $\lambda_n^* \geq (\lambda_n')^*$. Since the restriction of of $\{y_j\}_{j=1}^{n-1}$ to Ω are legitimate *trial* functions in Ω, it follows that $\lambda_n \geq \lambda_n'$.

For the Neumann problem the same reasoning applies, since the only difference is that a *test* functions w is not required to choose any boundary condition.

□

Consider an arbitrary domain composed of a union of nonintersecting subdomains: $\Omega = \Omega_1 \cup \Omega_2 \cup .. \cup \Omega_m$. For the Dirichlet problem, $-\Delta$ in the domain Ω has eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

while the corresponding Neumann Eigenvalues are denoted by

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \tilde{\lambda}_3 \leq ...$$

We order all the collection of Dirichlet eigenvalues of every subdomain Ω_j, $j = 1, ...m$ as follows:

$$\mu_1 \leq \mu_2 \leq \mu_3 \leq ...$$

and also the corresponding Neumann Eigenvalues in an increasing sequence:

$$\tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \tilde{\mu}_3 \leq ...$$

Theorem 10

$$\tilde{\mu}_n \leq \tilde{\lambda}_n \leq \lambda_n \leq \mu_n$$

Proof. For domain Ω, we know that the Neumann eigenvalues are smaller than the corresponding Dirichlet eigenvalues as minimization is over the wider class of *test* functions w, which need not be zero on the boundary. Therefore, $\tilde{\lambda}_n \leq \lambda_n$.

Now, we compare the eigenvalues λ_n of the Dirichlet problem in Ω with the n-th value μ_n in the collection of eigenvalues of Dirichlet problems in all subdomains. The set of all *trial* functions y in Ω contain by restriction the set of all *trial* functions in any subdomain Ω_j, $j = 1, 2..., m$. The set $\mathcal{S}_\lambda$ of all *test* functions w orthogonal to trial functions $\{y_j\}_{j=1}^{n-1}$ in Ω contain $\mathcal{S}_\mu$, the set of all *test* functions that are identically zero except exactly in one subdomain. Therefore, from the maximin principle,

$$\mu_n = \sup_{y_1, y_2, ..y_{n-1}} \inf_{w \in \mathcal{S}_\mu} \frac{\|\nabla w\|^2}{\|w\|^2} \geq \sup_{y_1, y_2, ..y_{n-1}} \inf_{w \in \mathcal{S}_\lambda} \frac{\|\nabla w\|^2}{\|w\|^2} = \lambda_n$$

Now choose *trial* functions $\{y_1, y_2, ..y_{n-1}\}$ in Ω so that there exists corresponding orthogonal *test* function u for the Neumann problem with support limited to exactly one subdomain Ω_{j^*} so that

$$\frac{\|\nabla u\|^2}{\|u\|^2} = \tilde{\mu}_n$$

Since u is orthogonal to *trial* functions, in Ω_{j^*}, we may choose $\{y_k\}_{k=1}^{n-1}$ to be identically zero outside chosen to be zero outside Ω_{j^*}. Define $\mathcal{S}_{\tilde{\mu}}$ to be the set of functions in Ω whose restriction to each Ω_j is a *test* function for Ω_j [3]. Orthonormality of $\mathcal{S}_{\tilde{\mu}}$ to $\{y_k\}_{k=1}^{n-1}$ in each subdomain Ω_j for $j \neq j^*$ follows since $y_k = 0$ in this subdomain. For $w \in \mathcal{S}_{\tilde{\lambda}}$, a general test function for Neumann problem in Ω orthogonal to $\{y_k\}_{k=1}^{n-1}$,

$$\|\nabla w\|^2 = \sum_{j=1}^{m} \int_{\Omega_j} |\nabla w|^2 dx \geq \tilde{\mu}_n \sum_{j=1}^{m} \int_{\Omega_j} |w|^2 dx = \tilde{\mu}_n \|w\|^2$$

Therefore,

$$\inf_{w \in \mathcal{S}_{\tilde{\mu}}} \frac{\|\nabla w\|^2}{\|w\|^2} \geq \tilde{\mu}_n$$

Since $\mathcal{S}_{\tilde{\lambda}} \subset \mathcal{S}_{\tilde{\mu}}$, the latter allowing larger class of possibly dicontinuous functions across $\partial\Omega_j$, we obtain

$$\inf_{w \in \mathcal{S}_{\tilde{\lambda}}} \frac{\|\nabla w\|^2}{\|w\|^2} \geq \inf_{w \in \mathcal{S}_{\tilde{\mu}}} \frac{\|\nabla w\|^2}{\|w\|^2} \geq \tilde{\mu}_n$$

Therefore,

$$\tilde{\lambda}_n = \sup_{y_1, y_1, .. y_{n-1}} \inf_{w \in \mathcal{S}_{\tilde{\lambda}}} \geq \tilde{\mu}_n$$

□

The above allows us to prove the following theorem:

Theorem 11 *For a two-dimensional domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth $\partial\Omega$, the eigenvalues of $-\Delta$ for either Dirichlet or Neumann conditions satisfy*

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{A(\Omega)}$$

where $A(\Omega)$ is the area of the domain Ω For a three-dimensional domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary, the eigenvalues of $-\Delta$ for either Dirichlet or Neumann conditions satisfy

$$\lim_{n \to \infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{V(\Omega)}$$

where $V(\Omega)$ is the volume of Ω.

Proof.

We simply give a sketch of the proof. An arbitary plane domain Ω can be approximated as a union of rectangles, just as in the evaluation of double integrals. So, we now carry out the argument for Ω when it is a union of rectangles. Now for a domain which is a union of rectangles, we define

$$M(\lambda) = \text{the number of } \; \mu_1, \mu_2, ... \text{ that do not exceed } \lambda$$

where μ_j as defined in the previous theorem. Then adding up the integer lattice points which are located within Ω. we get

$$\lim_{\lambda \to \infty} \frac{M(\lambda)}{\lambda} = \sum_p \frac{A(\Omega_p)}{4\pi} = \frac{A(\Omega)}{4\pi}$$

[3]Note a function in $\mathcal{S}_{\tilde{\mu}}$ is generally discontinuous across $\partial\Omega_j$ since Neumann conditions do not require test functions to be zero at the boundary

Therefore taking $M(\mu_n) = n$, and taking the reciprocal limit, we get,

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{4\pi}{A(\Omega)}$$

Similarly for the Neumann problem, we get

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n}{n} = \frac{4\pi}{A(\Omega)}$$

Therefore, using last Theorem, we complete the proof in 2-D when Ω is a union of rectangles. The 3-D proof is similar, except that we approximate the arbitary domain by a union of rectangular boxes.

□

5 Completeness of Eigenfunctions

Theorem 12 *For the Dirichlet boundary conditions, the eigenfunctions* $\{v_j\}_{j=1}^{\infty}$ *are complete in the* $\mathcal{L}_2(\Omega)$ *sense, i.e. for* $c_j = \frac{(f,v_j)}{\|v_j\|^2}$,

$$\lim_{N\to\infty} \|f - \sum_{j=1}^{N-1} c_j v_j\|^2 \to 0$$

The same is true for the Neumann condition.

Proof. We will prove only in the special case of $f \in \mathbf{C}^2(\Omega)$ since it is known from analysis that more general function, which together with its first derivative is in $\mathcal{L}_2$, can be approximated arbitrarily close in a norm involving the $\mathcal{L}_2$ norm of the function and its first derivative (so called $\mathcal{H}^1$ norm). Define

$$r_N(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{N-1} c_j v_j(\mathbf{x})$$

From orthogonality, we have

$$(r_N, v_n) = (f - \sum_{j=1}^{N-1} c_j v_j, v_n) = (f, v_n) - c_n\|v_n\|^2 = 0 \quad \text{for} \quad n = 1, 2, ..N-1$$

Thus, the remainder r_N is a trial function that satisfies the constraint for the minimization problem to find the n-eigen-value. Therefore,

$$\lambda_N = \inf_w \frac{\|\nabla w\|^2}{\|w\|^2} \leq \frac{\|\nabla r_N\|^2}{\|r_N\|^2}$$

We note that

$$\|\nabla r_N\|^2 = \|\nabla f - \sum_{j=1}^{N-1} c_j \nabla v_j\|^2 = (\nabla f, \nabla f) - 2\sum_{j=1}^{N-1} c_j(\nabla f, \nabla v_n) + \sum_{j,i=1}^{N-1} c_j c_i(\nabla v_j, \nabla v_i)$$

However,

$$(\nabla v_j, \nabla v_i) = \int_\Omega (\nabla v_j) \cdot (\nabla v_i) d\mathbf{x} = -\int_\Omega v_i \Delta v_j d\mathbf{x} = \lambda_j (v_i, v_j) = 0 \quad \text{for } j \neq i$$

Similarly

$$(\nabla f, \nabla v_j) = -(f, \Delta v_j) = \lambda_j (f, v_j) = \lambda_j c_j \|v_j\|^2$$

Therefore,

$$\|\nabla r_N\|^2 = \|\nabla f\|^2 - \sum_{j=1}^{N-1} \lambda_j c_j^2 \|v_j\|^2$$

Therefore, we have

$$\|\nabla r_N\|^2 \leq \|\nabla f\|^2$$

However, from the characterization of the eigenvalue λ_N in terms of a minimal quotient, we have

$$\lambda_N \leq \frac{\|\nabla r_N\|^2}{\|r_N\|^2}$$

Therefore,

$$\|r_N\|^2 \leq \frac{\|\nabla r_N\|^2}{\lambda_N} \leq \frac{\|\nabla f\|^2}{\lambda_N}$$

Since we know $\lambda_N \to \infty$ as $N \to \infty$, we obtain the proof of completeness. There is essentially no difference in the proof for the Neumann problem. □

Remark 10 *The completeness proof presented here is equally valid for eigenfunctions for the Dirichlet problem of a symmetric differential operator of the type*

$$-\nabla \cdot (p \nabla u) + qu = \lambda mu \quad \text{for} \quad \mathbf{x} \in \Omega$$

where $q(\mathbf{x})$ and $m(\mathbf{x})$ are in $\mathbf{C}^0(\Omega)$, while $p \in \mathbf{C}^1(\Omega)$ and $m(\mathbf{x}) > 0$, $p(\mathbf{x}) > 0$. It is also valid for Neumann or Robin boundary conditions as long as the corresponding linear differential operator A is symmetric (self-adjoint) and positive:

$$(Au, v) = (u, Av) \quad ; \quad (u, Au) > 0 \quad \text{for any} \quad u \neq 0$$

Remark 11 *In 1-D contexts, such problems are referred to as Sturm-Liouville boundary value problems.*

Remark 12 *Completeness allows representation of solutions to initial value problems for both heat and wave equations. Consider for instance solving*

$$u_t = \kappa \Delta u \quad \text{for } \mathbf{x} \in \Omega, u(\mathbf{x}, 0) = \phi(\mathbf{x}), \quad u = 0 \quad \text{for } \mathbf{x} \in \partial\Omega$$

If $\{v_n\}_{n=1}^\infty$ are the eigenfunctions of $-\Delta$ with homogeneous Dirichlet conditions, then it is clear that

$$u(\mathbf{x}, t) = \sum_{n=1}^\infty A_n e^{-\kappa \lambda_n t} v_n(\mathbf{x})$$

is a solution to the heat equation for $t > 0$ *and that it is capable of satisfying any initial conditions in* $\mathcal{L}_2(\Omega)$ *since*

$$u(\mathbf{x}, 0) = \sum_{n=1}^{\infty} A_n v_n(\mathbf{x}) = \phi(\mathbf{x})$$

We simply need to choose $A_n = \frac{(\phi, v_n)}{\|v_n\|^2}$*. We note also for instance that for large* t*, solution is dominated by the first eigen function component, we get*

$$u(\mathbf{x}, t) \sim A_1 \exp\left[-\lambda_1 \kappa t\right] v_1(\mathbf{x})$$