Divergent series, Borel Summation and applications to fluid dynamics

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Analytic functions and Convergent Series

Recall: $f(z)$ **is said to have a derivative at** z**, if**

$f'(z) \equiv \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ **exists independent of** $arg\ h$**.**

Recall f **is** *analytic* **at** z **if** f' **exists in an open set containing** z

Also, f **is analytic in a domain** $\mathcal{D}$ **if it is analytic at each point in** $\mathcal{D}$**.**

If f **is analytic at** z_0**, then it has a convergent Taylor expansion:**

$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ **in some neighborhood of** z

Conversely, a convergent series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ **defines an analytic function.**

1-1 correspondence between convergent series and analytic fns.

Isomorphism under usual algebraic operations.

Series solution to differential equation

Series representation useful in differential equation, eg:

$$\frac{d^2y}{dz^2} - zy = 0$$

Seek solution $y = \sum_{n=0}^{\infty} a_n z^n$, plug into ODE and equate like powers of x, to obtain recurrence relation for $n \geq 1$:

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \ , \ \ \text{with} \ \ a_2 = 0$$

General Theory gives both linearly independent solutions in the series form; also gives infinite radius of convergence in this case.

Method can be extended some nonlinear ODEs as well or 1st order system of PDEs with analytic coefficients and initial data on a non-characteristic curve.

Example of naturally occurring divergent series

Suppose we tried $y = \sum_{n=0}^{\infty} a_n z^n$ as solution to $z^2 \frac{dy}{dz} + y = z$.

Recurrence relation:

$$a_{n+1} = -n a_n \quad \text{for} \quad n \geq 1, \quad \text{with} \quad a_0 = 0, \quad a_1 = 1$$

gives

$$y = -\sum_{n=1}^{\infty} (-1)^n (n-1)! z^n \qquad (1)$$

With $z = 1/x$, obtain $\frac{dy}{dx} - y = -\frac{1}{x}$. Unique solution y satisfying $y \longrightarrow 0$ as $x \longrightarrow +\infty$:

$$y(x) = -e^x \int_{\infty}^{x} \frac{e^{-t}}{t} dt \qquad (2)$$

Q: In what sense is (1) a representation of solution (2) ?

"Divergent Series is the Devil's invention" (Abel)

Asymptotic Power Series in Poincare's sense

Definition:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{a_n}{x^n} \quad , \quad \text{as } x \to +\infty$$

implies for any $M \geq 1$,

$$\lim_{x \to +\infty} \frac{\left[[f(x) - \sum_{n=1}^{M} a_n x^{-n} \right]}{x^{-M}} = 0$$

$$\text{or} \qquad f(x) - \sum_{n=1}^{M} a_n x^{-n} = o\left(x^{-M}\right)$$

Example: Integration by parts shows that as $x \to +\infty$.

$$f(x) = e^x \int_{\infty}^{x} \frac{e^{-t}}{t} dt \sim \sum_{n=1}^{\infty} \frac{(-1)^n (n-1)!}{x^n}$$

Divergent Series in PDEs

Consider for instance the heat equation for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$:

$$u_t = u_{xx} \ \ , \ \ u(x,0) = u_0(x) \ \text{ with } \ u_0 \text{ analytic}$$

Try a formal series $u(x,t) = \sum_{n=0}^{\infty} t^n u_n(x)$ to obtain:

$$u_{n+1} = \frac{1}{(n+1)} u_n'' \ \text{ get } \ u(x,t) = \sum_{n=0}^{\infty} t^n \frac{u_0^{(2n)}(x)}{n!}$$

Divergent for generic analytic initial condition u_0

Asymptotic series and functions

If $f(x)$ has an asymptotic power series representation as $x \to \infty$, then it is unique.

Not all functions have an asymptotic power series. Sometimes more general representations needed:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Further asymptotic series in the sense of Poincare does not provide unique correspondence between asymptotic series and actual functions, example: Let $g(x) = f(x) + e^{-x}$. Then if

$$f(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}, \;\; \text{then } g(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}$$

Borel Transform and Borel Sum of divergent series

Borel Transform:

$$\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{(k-1)!} \quad \text{for} \quad k \geq 1$$

$$\mathcal{B}[\tilde{y}] = \mathcal{B}\left[\sum_{k=1}^{\infty} a_k x^{-k}\right](p) = \sum_{k=1}^{\infty} \frac{a_k}{(k-1)!} p^{k-1}$$

If the following conditions hold:

1. **The Borel Transform $Y(p) \equiv \mathcal{B}[\tilde{y}](p)$ analytic for $p \geq 0$,**
2. **$e^{-\alpha p} Y(p)$ absolutely integrable in $(0, \infty)$ for some $\alpha > 0$**

Then, $L\mathcal{B}[\tilde{y}] = [LY](x) \equiv \int_0^\infty e^{-px} Y(p) dx$ exists for $\mathrm{Re}\, x > \alpha$ and called Borel sum of $\tilde{y}$. When defined, Borel sum is an isomorphism between classes of divergent series and functions they correspond to, under usual algebraic operations.

Borel Sum of formal ODE solutions

For the ODE, $\frac{dy}{dx} - y = -\frac{1}{x}$, we obtained $\tilde{y}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}$

$$Y(p) = \mathcal{B}[\tilde{y}](p) = \sum_{n=0}^{\infty} (-1)^n p^n = \frac{1}{1+p}$$

$$L\mathcal{B}\tilde{y} = LY = \int_0^{\infty} \frac{e^{-px}}{1+p} dx = e^x \int_{\infty}^{x} \frac{e^{-t}}{t} dt$$

Reliance on explicit series for Borel Sum too much to ask

Borel sum of nonlinear ODE solution

Instead, directly apply L^{-1} to equation; for instance

$$y' - y = -\frac{1}{x} + y^2; \quad \text{with} \lim_{x\to\infty} y = 0$$

Defining $[u * v][p] = \int_0^p u(s)v(p-s)ds$, $Y(p) = [L^{-1}y](p)$ satisfies:

$$-pY(p) - Y(p) = -1 + Y * Y \quad \text{implying}\, Y(p) = \frac{1}{1+p} - \frac{Y*Y}{1+p} \tag{3}$$

For functions Y analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ bounded, can be shown that (3) has unique solution for sufficiently large α.

Implies ODE solution $y(x) = \int_0^\infty Y(p)e^{-px}dp$ for $Re\, x > \alpha$

Above results special case of general ODE results (O. Costin, 98)

3-D Navier-Stokes (NS) fluid dynamics problem

$$v_t + (v \cdot \nabla) v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ is the fluid velocity and $p \in \mathbb{R}$ pressure at $x = (x_1, x_2, x_3) \in \Omega$ at time $t \geq 0$. Further, the operator $(v \cdot \nabla) = \sum_{j=1}^{3} v_j \partial_{x_j}$, ν = nondimensional visocity (inverse Reynolds number)

The problem supplemented by initial and boundary conditions:

$$v(x, 0) = v_0(x) \;\; (IC) \,, \; v = 0 \text{ on } \partial\Omega \text{ for stationary solid boundary}$$

We take $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3[0, 2\pi]$; no-slip boundary condition avoided, but assume in the former case $\|v_0\|_{L^2(\mathbb{R}^3)} < \infty$.

Millenium problem: Given smooth v_0 and f, prove or disprove that there exists smooth 3-D NS solution v for all $t > 0$. Note: global solution known in 2-D.

NS - a fluid flow model; importance of blow-up

$$v_t + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f \quad ; \quad \nabla \cdot v = 0,$$

Navier-Stokes equation models incompressible fluid flow.

$v_t + (v \cdot \nabla)v \equiv \frac{Dv}{Dt}$ represents fluid particle acceleration. The right side (force/mass) can be written: $\nabla \cdot \mathrm{T} + f$, where T: a tensor of rank 2, called stress with

$$\mathbf{T}_{jl} = -p\delta_{j,l} + \frac{\nu}{2}\left[\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right]$$

The second term on the right is viscous stress approximated to linear order in ∇v. Invalid for large $\|\nabla v\|$ or for non-Newtonian fluid (toothpaste, blood)

Incompressibility not valid if v comparable to sound velocity

If NS exhibited blow up, the model itself becomes invalid; terms not included in NS approximation potentially important.

Classical results for 3-D NS problem

Smooth solutions known only for $t \in [0, T]$ for small enough T

Classical approaches to global existence of PDEs rely on "energy" bounds, *i.e.* bounds on positive definite functionals of v.

It is known that Kinetic energy is $\frac{1}{2}\|v(.,t)\|^2_{L^2(\Omega)}$ is controlled in time. Bound not enough to ensure smooth solution for any t.

Also, known that cumulatitive dissipation $\nu \int_0^T \|\nabla v(.,t)|^2_{L^2(\Omega)} dt < \infty$, but no pointwise control over $\|\nabla v(.,t)\|^2_{L^2(\Omega)}$ known.

Leray (1932a-34) found "weak solutions" to 3-D NS, but these are in spaces of function where v and ∇v can blow up. Uniqueness still an issue.

There are known sufficient conditions (Beale-Kato-Majda, Constantin-Fefferman, Serrin, Prodi, Sverak,..) that guarantee smooth global solutions in time. Not known if these are satisfied.

An alternate Borel based approach

Formal expansion of N-S solution possible for small t:

$v(x,t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$.

Above can be Borel summed for analytic v_0 and f (O. Costin & S.T) into an actual solution in the form:

$v(x,t) = v_0(x) + \int_0^{\infty} e^{-p/t} U(x,p) dp$. **This form transcends assumptions on analyticity of v_0 and f or of t small**

Advantageous sometimes to use generalized Laplace Transform:

$$v(x,t) = v_0(x) + \int_0^{\infty} U(x,q) e^{-q/t^n} dq$$

3-D NS equivalent to integral equation whose solution known *a priori* **for $q \in [0, \infty)$. Global NS problem equivalent to finding optimal α so that $e^{-\alpha q} \| U(., q) \|$ is integrable in $(0, \infty)$. Some results along this line (O. Costin, G. Luo & S.T), check my website)**