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Rigorous Results in Steady Finger Selection in Viscous Fingering

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Abstract

This paper concerns existence of steadily translating finger solution in a Hele-Shaw cell for small but non-zero surface tension (ϵ^2). Though there are numerous numerical and formal asymptotic results for this problem, we know of no mathematically rigorous results that address the selection problem. We rigorously conclude that for relative finger width λ in the range $[\frac{1}{2}, \lambda_m]$, with $\lambda_m - \frac{1}{2}$ small, analytic symmetric finger solutions exist in the asymptotic limit of surface tension $\epsilon^2 \to 0$ if and only if the Stokes constant for a relatively simple nonlinear differential equation is zero. This Stokes constant S depends on the parameter $a \equiv \frac{2\lambda-1}{(1-\lambda)}\epsilon^{-\frac{4}{3}}$ and earlier calculations by a number of authors have shown this to be zero for a discrete set of values of a.

The methodology consists of proving existence and uniqueness of analytic solutions for a weak half-strip problem for any λ in a compact subset of (0, 1). The weak problem is shown equivalent to the original finger problem in the function space considered, provided we invoke a symmetry condition. Next, we consider behavior of solution in a neighborhood of an appropriate complex turning point for the restricted case $\lambda \in [\frac{1}{2}, \lambda_m]$, for some $\lambda_m > \frac{1}{2}$. This turning point account for exponentially small terms in ϵ, as $\epsilon \to 0^+$ that generally violate symmetry condition. We prove that the the symmetry condition is satisfied for small ϵ when the parameter a is constrained appropriately.

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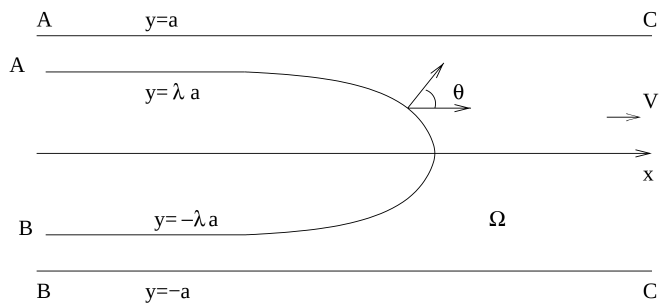

Fig. 1.1. Steady finger and flow in the lateral plane

1. Introduction

1.1. Background Work

A Hele-shaw cell is a pair of long parallel plates of width a, separated by a small gap b, with $b \ << \ a$. The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw has been studied a lot since 1950's. Many review papers have appeared in the literature.Bensimon et al (1986), Saffman (1986), Homsy (1987), Kessler et al (1988), Pelce (1988) summarized the literature as of the 1980's. A more recent review by Tanveer (2000) focusses on the singular effect of surface tension.

A fluid of viscosity μ is being pushed by a less viscous fluid of negligible viscosity, the more viscous fluid is displaced at infinity with velocity V. Usually a single steady finger eventually forms and propagates at constant velocity U with relative finger width λ (Figure 1.1).

The gap-averaged velocity (u, v) in the (x, y)-plane satisfies Darcy's law:

$$(u, v) = -\frac{b^2}{12\mu}\nabla p = \nabla \phi; \tag{1.1}$$

where p is the pressure and ϕ is the velocity potential. From incompressibillity,

$$\Delta \phi = 0, \text{ in flow domain } \Omega; \tag{1.2}$$

On the side walls:

$$v = 0, \text{ at } y = \pm a. \tag{1.3}$$

In the far field:

$$(u, v) = V\hat{x} + O(1), \text{ as } x \to +\infty; \tag{1.4}$$

where $\hat{x}$ is unit vector on x-axis. Ignoring thin film (Reinelt 1987,Park & Homsy 1985, Tanveer 1990) or contact angle complications (Weinstein et al 1990), which in some cases leads to significant effects, the pressure condition is:

$$\phi = \frac{b^2 T}{12\mu}\frac{1}{\rho}; \tag{1.5}$$

where ρ is the radius of curvature, T is the surface tension. The kinematic condition for a steady finger is:

$$\frac{\partial \phi}{\partial n} = U \cos\theta; \tag{1.6}$$

where θ is the angle between the interface nomal and the positive x -axis(see Fig. 1.1). There are a number of different but equivalent formulations, starting with the one by Mclean and Saffman (1981). Here we use the formulation by Tanveer (2000) which is valid for fingers that are *a priori* symmetric about the channel centerline. Numerical as well as formal results suggest that only symmetric finger solutions are possible (Tanveer 1987, Combescot & Dombre 1988). We can set $a = 1$ and $V = 1$ as this corresponds to nondimensionalizing all lengths by a and velocities by V.

By integrating (1.2) in the domain Ω, the finger width λ is related to U as follows:

$$\lambda = \frac{1}{U}; \tag{1.7}$$

In a frame moving with the steady symmetric finger, the condition (1.6) transforms, without loss of generality, into

$$\psi = 0; \tag{1.8}$$

where ψ is the stream function (harmonic conjugate of ϕ) so that $W = \phi + i\psi$ is an analytic function of $z = x + iy$. The nondimensional pressure condition (1.5) in the moving frame becomes

$$\phi + \frac{1}{\lambda} x = \frac{\mathcal{B}}{\rho}. \tag{1.9}$$

where $\mathcal{B} = \frac{b^2 T}{12\mu V a^2}$. On the side walls, (1.3) implies that

$$\psi = \pm \left[\frac{1}{\lambda} - 1\right] \text{ on } y = -1 \text{ and } y = 1 \text{ respectively} \tag{1.10}$$

while the far field condition as $z \to +\infty$ in the finger frame is

$$W \sim -\left[\frac{1}{\lambda} - 1\right] z + O(1); \tag{1.11}$$

We consider the conformal map $z(\xi)$ of the cut upper-half-ξ-plane, as shown in Figure 1.2, to the flow domain Ω in Figure 1.1. The real ξ axis corresponds to the finger boundary, with $\xi = -\infty, +\infty$ corresponding to the finger tails at $z = -\infty + i\lambda, z = -\infty - \lambda i$ respectively. The two sides of the cut correspond to the two side walls respectively. It is easily seen that the complex potential $W(\xi)$ is given by:

$$W(\xi) = \frac{1-\lambda}{\pi\lambda} \ln(\xi^2 + 1) + c \tag{1.12}$$

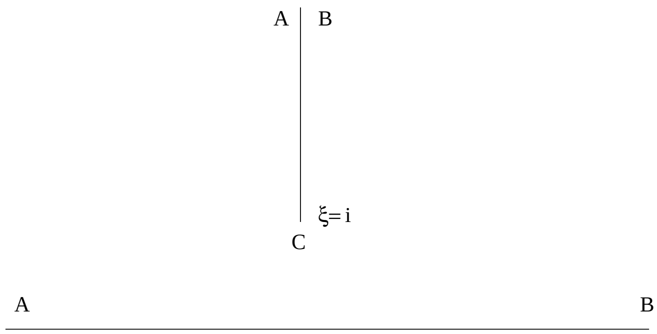

Fig. 1.2. Upper half ξ-plane with a cut on imaginary axis from $\xi = i$ to $i\infty$

where c is some real constant. We define $F(\xi)$ so that

$$z(\xi) = -\frac{1}{\pi}\ln(\xi - i) - \frac{1-2\lambda}{\pi}\ln(\xi + i) - i\lambda - \frac{2-2\lambda}{\pi}F(\xi); \qquad (1.13)$$

It follows that F, as defined above, is analytic in the entire upper half ξ-plane (Tanveer 2000). The pressure condition (1.9) translates into requiring that on the real ξ axis:

$$Re\ F = \frac{\epsilon^2}{|F' + H|} Im \left[\frac{F'' + H'}{F' + H}\right]; \qquad (1.14)$$

where

$$H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1}, \text{with } \gamma = \frac{\lambda}{1-\lambda}, \epsilon^2 = \frac{\pi^2 \lambda \mathcal{B}}{4(1-\lambda)^2}; \qquad (1.15)$$

For zero surface tension $\epsilon = 0$, it follows that $F = 0$; this corresponds to what is usually referred to in the literature as Saffman and Taylor solutions (Zhuravlev (1956), Saffman & Taylor (1958)). This form a family of exact solutions for symmetric fingers with arbitrary width of finger $\lambda \in (0,\ 1)$. The Saffman-Taylor solutions, in our formulation, correspond to the conformal map

$$z_0(\xi) = -\frac{1}{\pi}\ln(\xi - i) - \frac{1-2\lambda}{\pi}\ln(\xi + i) - i\lambda; \qquad (1.16)$$

This is easily seen to be univalent since the boundary corrrespondence is one to one. In particular, the finger shape can be explicitly described by Re z as a function of Im z. If ϵ and an appropriate norm of F and F' as introduced later are sufficiently small, then it can be shown that $z(\xi)$ is also univalent (see Theorem 1.5 in the sequel).

For non-zero tension, no exact solutions exist. Mclean and Saffman's (1981) numerical work suggested that solutions exist only for some isolated values of finger width. They also found a formal perturbation series solution in powers of surface tension ϵ^2; however no restriction on λ was obtained from the formal calculation. Romero(1982) and Vanden-Broeck(1983) determined a discrete set of branches of steady solutions. So there was an apparent contradiction between numerics and perturbation solution of Mclean and Saffman. Analytical work by Kruskal and Segur (1985,1991) on the simpler problem of geometric model of crystal growth paved the way for resolving the discrepancy between numerics and perturbation for the Saffman-Taylor finger problem. Kessler and Levine (1985) were the first to suggest that exponentially small terms missing in the Mclean-Saffman perturbation expansion is the reason for discrepancy with numerical calculations. Their numerical calculations supported this contention, though accurate calculations are not possible for indefinitely small surface tension. However, the conclusions above were confirmed by formal analytical calculations of Combescot *et al* (1986,1988), Shraiman (1986), Hong and Langer(1986), Tanveer (1987a), Dorsey and Martin (1987), and Chapman (1999). Subsequent numerical and formal asymptotic calculations (Kessler & Levine (1987), Tanveer (1987b)) suggest that all but one branch are linearly unstable.

Though the formal results on exponential asymtotics are generally accepted, not all conclusions based on these have been universally accepted (DeGregoria & Schwartz (1987), Xu (1991)). This makes the need for rigorous results more compelling.

It is to be noted that the problem is not readily accessible to real domain methods. The problem with $\epsilon = 0$ is a structurally unstable system, as can be concluded from the results of this paper. The solution set for $\epsilon \to 0^+$ is different from that for $\epsilon = 0$. This makes control of terms difficult in the real domain. The importance of a term in selecting λ is determined by its relative size in a neigborhood of the turning point in some complex domain (See discussions in Tanveer 2000). Small changes in the equation as measured in any real Sobolev norm may result in large changes in the solution set. It is to be noted that these comments are not applicable if ϵ stays away from zero. Indeed, there is a proof (Su, 2001) of existence of at least one solution for a two-sided Hele-Shaw type model equation (though not quite physically appropriate for actual Hele-Shaw flow), when surface tension is fixed and nonzero.

1.2. Notations

Definition 1.1. Let $\mathcal{R}$ be an open connected (see Figure 1.3) region on complex ξ plane bounded by lines r_u and r_l defined as follows:

$$r_u = r_{u,1} \cup r_{u,2} \cup r_{u,3} \cup r_{u,4} \cup r_{u,5} \cup r_{u,6}$$

$$r_l = \{\xi : \xi = -bi + re^{-i(\varphi_0+\mu)}\} \cup \{\xi : \xi = -bi + re^{i(\pi+\varphi_0+\mu)}\}$$

where $0 < b < Min\ (1, \gamma),\ \alpha > 0, 2\alpha < b,\ 0 < \varphi_0,\ \mu\ <\ \frac{\pi}{2}$ with $\varphi_0 + \mu < \frac{\pi}{2}$ and

$$r_{u,1} = \{\xi : \xi = (\alpha - \nu)i - R + re^{i(\pi-\varphi_0)}, 0 \le r < \infty\}\ ,$$

$$r_{u,2} = \{\xi : Im\ \xi = \alpha - \nu, -R \le Re\ \xi \le -\nu\},$$

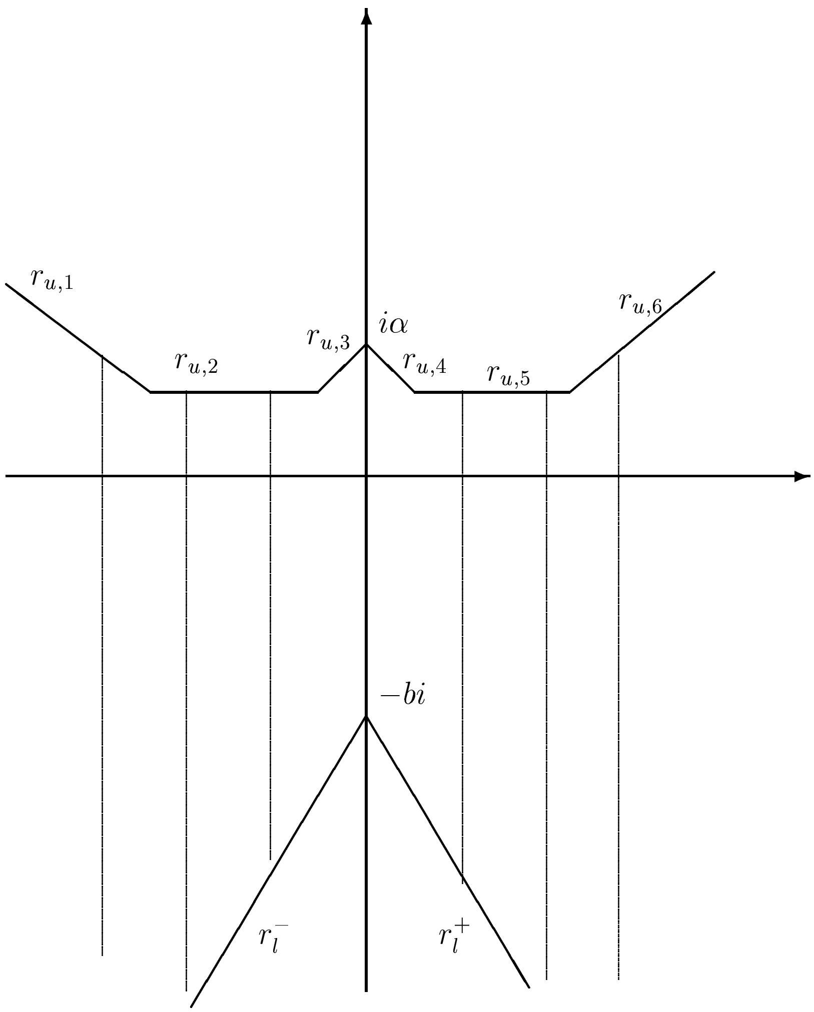

Fig. 1.3. Region $\mathcal{R}$ in complex ξ plane.

$$r_{u,3} = \{\xi : \xi = \alpha i + re^{-3\pi i/4}, 0 \le r \le \sqrt{2}\nu\},$$

$$r_{u,4} = \{\xi : \xi = \alpha i + re^{-\pi i/4}\}, 0 \le r \le \sqrt{2}\nu\}$$

$$r_{u,5} = \{\xi : Im\ \xi = \alpha - \nu, \nu \le Re\ \xi \le R\}\ ,$$

$$r_{u,6} = \{\xi : \xi = (\alpha - \nu)i + R + re^{i\varphi_0}, 0 \le r < \infty\}$$

where $R > 0$ is large enough and $0 < \nu < \alpha$ is small enough so that Property 2 and Property 3 in section 2 hold.

Remark 1.1. $1-b$, α and φ_0 are chosen independent of ϵ. There are some restrictions imposed on their values in order that certain properties (Property 1-5 in the sequel) are ensured.

Definition 1.2. $\mathcal{R}^- = \mathcal{R} \cap \{Re\ \xi\ <\ 0\}$; $\mathcal{R}^+ = \mathcal{R} \cap \{Re\ \xi\ >\ 0\}$

Let $0 < \tau < 1$ be independent of ϵ. Its precise choice will not be important. For $k = 0, 1, 2$, we introduce spaces of functions:

Definition 1.3. For $k = 0, 1, 2$, define

$$\mathbf{A}_k = \{F : F(\xi) \text{ analytic in } \mathcal{R} \text{ and continuous in } \overline{\mathcal{R}},$$
$$\text{with } \sup_{\xi \in \overline{\mathcal{R}}} |(\xi - 2i)^{k+\tau} F(\xi)| < \infty\}$$
$$\|F\|_k := \sup_{\xi \in \overline{\mathcal{R}}} |(\xi - 2i)^{k+\tau} F(\xi)|$$

In the above definition, replacing $\mathcal{R}$ with $\mathcal{R}^-$, we get:

Definition 1.4. For $k = 0, 1, 2$,

$$\mathbf{A}_k^- = \{F : F(\xi) \text{ is analytic in } \mathcal{R}^- \text{ and continuous in its closure},$$
$$\text{with } \sup_{\xi \in \mathcal{R}^-} |(\xi - 2i)^{k+\tau} F(\xi)| < \infty\},$$
$$\|F\|_k^- := \sup_{\xi \in \overline{\mathcal{R}^-}} |(\xi - 2i)^{k+\tau} F(\xi)|$$

Remark 1.2. $\mathbf{A}_k$ are Banach spaces and $\mathbf{A} \equiv \mathbf{A}_0 \supset \mathbf{A}_1 \supset \mathbf{A}_2$. If $F \in \mathbf{A}$, then F satisfies property:

$$F(\xi) \sim O(\xi^{-\tau}), \text{ as } |\xi| \to \infty, \xi \in \mathcal{R}. \tag{1.17}$$

Definition 1.5. Let $\mathcal{D}$ be any connected set in the complex ξ-plane; we introduce norms: $\|F\|_{k,\mathcal{D}} := \sup_{\xi \in \mathcal{D}} |(\xi - 2i)^{k+\tau} F(\xi)|, k = 0, 1, 2.$

Definition 1.6. Let $\delta, \delta_1 > 0$ be two constants, define spaces:

$$\mathbf{A}_{0,\delta} = \{f : f \in \mathbf{A}, \|f\|_0 \le \delta\};\ \mathbf{A}_{1,\delta_1} = \{g : g \in \mathbf{A}_1, \|g\|_1 \le \delta_1\};$$

Lemma 1.3. *Let $z(\xi), z_0(\xi)$ be as defined in (1.13) and (1.16). For fixed $\lambda \in (0, 1)$ and sufficiently small enough δ, δ_1,if $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, then $z(\xi)$ is one to one for ξ real.*

Proof. Suppose that $z(\xi_1) = z(\xi_2)$, where ξ_1 and ξ_2 are real and $\xi_2 \ge \xi_1$. From (1.16), we choose $L > 0$ independent of ϵ but large enough so that

$$|Im\ z_0(\xi) + \lambda| < \frac{\lambda}{3}, \text{ for } \xi \in (L, \infty);$$
$$|Im\ z_0(\xi) - \lambda| < \frac{\lambda}{3}, \text{ for } \xi \in (-\infty, -L);$$

For small enough δ, since $|F(\xi)| \leq |\xi - 2i|^{-\tau}\delta$,

$$|Im\ z(\xi) + \lambda| < \frac{\lambda}{2}, \text{for } \xi \in (L, \infty);$$
$$|Im\ z(\xi) - \lambda| < \frac{\lambda}{2}, \text{for } \xi \in (-\infty, -L);$$

This means it is not possible for $\xi_1 \in (-\infty, -L)$ and $\xi_2 \in (L, \infty)$.
Case (ii): $(\xi_1, \xi_2) \subset [-L, L]$, by mean value theorem, there is $\xi_3 \in [-L, L]$ so that Im $z'(\xi_3) = 0$. However $Im\ z'(\xi_3) = -\frac{2-2\lambda}{\pi}\left[\frac{\gamma}{\xi_3^2+1} + Im\ F'(\xi_3)\right]$. For sufficiently small δ_1, $Im\ z'(\xi_3) < 0$ for $\xi_3 \in [-L, L]$. Therefore, we rule out $(\xi_1, \xi_2) \subset [-L, L]$.
Case (iii): $\xi_1 \in (-\infty, -L)$, $\xi_2 \in (-\infty, -L/2)$. From mean value theorem, there is $\xi_3 \in (-\infty, -L/2)$ so that Re $z'(\xi_3) = 0$. Using the fact that $Re\ z_0(\xi) = \frac{(1-\lambda)}{\pi}\ln(\xi^2+1)$, we have for $\xi \in (-\infty, -L/2)$,

$$\begin{aligned}
|Re\ z'(\xi)| &= \left|\frac{2(1-\lambda)\xi}{\pi(1+\xi^2)} - \frac{(2-2\lambda)}{\pi}Re\ F'(\xi)\right| \\
&\geq \frac{2(1-\lambda)}{\pi|\xi-2i|}\left[\frac{|\xi|\ |\xi-2i|}{1+|\xi|^2} - \frac{\|F'\|_1}{|\xi-2i|^\tau}\right] \\
&\geq \frac{2(1-\lambda)}{\pi|\xi-2i|}\left(C - K\|F'\|_1\right) > 0.
\end{aligned}$$

for δ_1 small enough, since C, K depends only on L, τ. So we rule out this case as well. Using symmetry we rule out $\xi_2 \in (L, \infty), \xi_1 \in (L/2, \infty)$.
Case (iv): $\xi_1 \in (-\infty, -L), \xi_2 \in (-L/2, L/2)$. Note for small δ,

$$\begin{aligned}
&|Re\ z(\xi_1) - Re\ z(\xi_2)| \\
&= \left|\frac{(1-\lambda)}{\pi}\ln\frac{(1+\xi_1^2)}{(1+\xi_2^2)} - \frac{(2-2\lambda)}{\pi}\left(Re\ F(\xi_1) - Re\ F(\xi_2)\right)\right| \\
&\geq \frac{(1-\lambda)}{\pi}\left[\ln(1+L^2) - \ln(1+L^2/4) - C\delta\right] > 0.
\end{aligned}$$

So we rule out this case out as well. By symmetry, we can rule out $\xi_2 \in (L, \infty), \xi_1 \in (-L/2, L/2)$. Therefore we have ruled out all possibilities except $\xi_1 = \xi_2$.

Lemma 1.4. *For λ in a compact subset of (0, 1), for all sufficiently small δ_1, if $F' \in \mathbf{A}_{1,\delta_1}$, then the mapping $z(\xi)$ is one to one on either side of the branch cut $\{Re\ \xi = 0\ ;\ Im\ \xi > 1\}$.*

Proof. We note that for $\xi = i\eta$, $\eta > 1$, from expression for $z(\xi)$,

$$\frac{d}{d\eta}Re\ z(i\eta) = -\frac{1}{\pi}\left(\frac{1}{\eta-1} + \frac{1-2\lambda}{\eta+1}\right) + (2-2\lambda)\ Re\ [iF'(i\eta)]$$

which is clearly nonzero for any $\eta \in (1, \infty)$ for sufficiently small δ_1, since $|Re\ [iF'(i\eta)]| = O(\frac{\delta_1}{\eta})$ Thus, $Re\ z(i\eta)$ is monotonic for $\eta > 1$. So, on either side of the imaginary ξ-axis branch-cut starting at $\xi = i$, $z(\xi)$ maps the imaginary axis segment in a one to one manner onto each of the Hele-Shaw side-walls at $Im\ z = \pm 1$

Theorem 1.5. *For λ in a compact subset of (0, 1), for sufficiently small enough δ and δ_1, if $F \in \mathbf{A}_{0,\delta}$ and $F' \in \mathbf{A}_{1,\delta}$, then the mapping $z(\xi)$ given by (1.13) is univalent.*

Proof. From (1.13), it follows that $z(\xi)$ is analytic inside the cut-upper-half ξ-plane. Lemmas 1.3 and 1.4 show that $z(\xi)$ also maps from the boundaries of the cut-upper-half ξ plane in a one to one manner to $\partial\Omega$ for sufficiently small δ and δ_1. It follows that $z(\xi)$ is univalent.

Following Tanveer(2000), the problem of determining a smooth steady symmetric finger is equivalent to finding function analytic F in the upper-half ξ plane, which is $\mathbf{C^2}$ in its closure, *i.e.* continuous derivatives upto the second on Im $\xi = 0$ and is required to satisfy the following conditions:
Condition (i): $F(\xi)$ satisfies (1.14) on the real ξ axis.
Condition (ii):

$$F(\xi), \xi F'(\xi) \to 0 \text{ as } \xi \to \pm\infty; \tag{1.18}$$

Condition (iii)(symmetry condition):

$$\text{Re } F(-\xi) = \text{Re } F(\xi)\ ,\ \text{Im } F(-\xi) = -\text{Im } F(\xi) \text{ for } \ \xi \text{ real;} \tag{1.19}$$

Remark 1.6. An additional condition about univalence of z was stated in Tanveer (2000), which is not necessary in view of Theorem 1.5. Also, it is easily seen that if $F(\xi)$ satisfies (1.14), so does $[F(-\xi^*)]^*$. Hence, it is enough to require that equation (1.14) be satisfied for a $\mathbf{C}^2(-\infty, 0]$ function F for Re $\xi \leq 0$, provided the symmetry condition (1.19) is also satisfied.

Remark 1.7. If $F \in \mathbf{A}_0$,and satisfies symmetry condition (iii) on the real axis, then it follows from successive Taylor expansions of $F(\xi)$ at a point on the imaginary ξ axis segment $-b < Im\ \xi < \alpha$, starting at $\xi = 0$ that Im $F(\xi) = 0$. From Schwartz reflection principle, $F(\xi) = [F(-\xi^*)]^*$, so $\|F\|_0^- = \|F\|_0$. Conversely, if $F \in \mathbf{A}_0^-$ and satisfies $Im\ F(\xi) = 0$ for the imaginary ξ axis segment $-b < Im\ \xi < \alpha$, then $F(\xi) = [F(-\xi^*)]^*$ extends F to the right half of $\mathcal{R}$ and $\|F\|_0^- = \|F\|_0$ and (1.19) is then automatically satisfied.

Finger problem: *The problem tackled in this paper will be to find function F analytic in $\{Im\ \xi > 0\} \cup \mathcal{R}$ so that $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$ for some τ fixed in (0, 1), δ, δ_1 small but independent of ϵ, so that conditions (i) and (iii) on the real axis are satisfied.*

Remark 1.8. Solution to the finger problem, as defined above, implies all conditions (i)-(iii) are satisfied. The converse has not been proved in this paper. However, in a companion paper (Tanveer & Xie, '02), it has been proved that if any classical finger solution F exist at all, *i.e.* F is analytic in the upper-half ξ-plane and $\mathbf{C}^2$ in its closure and satisfies condition (i)-(iii), then it belongs to $\mathbf{A}_0$, provided we assume F decays algebraically on the real axis as $\xi \rightarrow \pm\infty$ and that $|F|$ is appropriately small (consistent with a Saffman-Taylor like solution). The algebraic decay rate is suggested by McLean & Saffman (1981) formal expansion near the finger tail. Further, it has been proved (Tanveer & Xie, '02) that such classical solutions cannot exist for fixed $\lambda \in (0, \frac{1}{2})$ for sufficiently small ϵ.

1.3. Main Results

The formal strategy of calculation of finger width (Combescot et al 1986,1988, Tanveer 1987,Dorsey & Martin 1987) involves analytic continuation of equation in an inner neighborhood of turning points in the complex plane and ignoring integral contribution and other terms that are formally small. In Combescot et al (1986,1988) formulation, by ignoring nonlocal integral contributions and other formally higher-order corrections, the problem of determining a smooth steady finger to determining eigenvalue a so that $G(y)$ is a solution to

$$y\frac{d}{dy}\left(y\frac{dG}{dy}\right) + G^{-2} = y^2 + 2^{-1/3}a; \tag{1.20}$$

satisfying asymptotic matching condition:

$$G(y) \sim y^{-1}, \text{ as } y \to \infty \text{ for } \arg y \in (-\pi, 0]; \tag{1.21}$$

and in addition satisfying:

$$Im\ G = 0, \text{ for large enough } y \text{ on the positive real axis.} \tag{1.22}$$

The finger width λ is related to a through

$$a = \frac{2\lambda - 1}{1 - \lambda}\epsilon^{-4/3}; \tag{1.23}$$

Combescot et al (1988) computed numerically many terms of the formal divergent asymptotic expansion for large y

$$G(y) \sim \frac{1}{y} - \frac{\bar{\delta}_1}{2y^3} + \frac{1}{2y^4} + \frac{3\bar{\delta}_1^2}{5y^5} - \frac{3\bar{\delta}_1}{y^6} + \cdots$$

where $\bar{\delta}_1 = 2^{-1/3}a$. Based on the form of the expansion, they were able to deduce the nature of the singularity in the Borel plane, and conclude that for large y on the positive real axis

$$Im\ G(y) \sim \tilde{S}(a) y^{-3/4} e^{-\frac{2^{3/2}}{3}y^{3/2}}; \tag{1.24}$$

where the Stokes multiplier $\tilde{S}(a)$ was determined numerically as a function of a. Numerical calculation revealed $\tilde{S}(a) = 0$ for a discrete set of values of a. These results are consistent with Tanveer (1987), Dorsey & Martin (1987) and Chapman (1999). The first few of these discrete set of eigen values $\{a_n\}$, in order of increasing size, were found to be $\{1.0278, 3.7168, 7.0934, \cdots\}$. Asymptotic determination of a was also made for $a >> 1$. Combescot *et al* (1988) assume that

$$\tilde{S}(a) = 0 \text{ iff } Im\ G(y) = 0 \text{ for large } y \text{ on the positive axis .} \tag{1.25}$$

the latter condition being necessary for the existence of a symmetric finger. It is not clear *apriori* that zeroing out the leading order exponentially small term necessarily means that all other exponentially small terms should vanish.

In this paper, among the many results, it will be rigorously proved that the leading order form of $Im\ G(y)$ in (1.24) is indeed true and that assumption (1.25) is indeed justified. However, we will not compute the Stokes constants $\tilde{S}(a)$, but rely on Combescot et al 's numerical computation to assume that there exist a discrete set $\{a_n\}$ so that $\tilde{S}(a_n) = 0, \tilde{S}'(a_n) \neq 0$. It is to be noted that recent theoretical development in exponential asymptotics for general nonlinear ordinary differential equations (Costin, 1998) makes it possible to rigorously confirm these calculations to within small error bounds, though this analysis is yet to be carried out for this problem.

Through a transformation:

$$y = \frac{3^{2/3}}{2}\eta^{2/3}, G(y) = \frac{2}{3^{2/3}\eta^{2/3}}(1 + \psi(\eta)); \tag{1.26}$$

$\psi(\eta)$ satisfies:

$$\frac{d^2\psi}{d\eta^2} - \frac{1}{3\eta}\frac{d\psi}{d\eta} - \psi = -\frac{4}{9\eta^2} - \frac{4}{9\eta^2}\psi + \frac{2a}{3^{4/3}\eta^{4/3}} - \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n+1)\psi^n \tag{1.27}$$

and matching conditions as $|\eta| \to \infty$:

$$\psi(\eta, a) \sim \frac{2a}{3^{4/3}\eta^{4/3}}, 0 \leq \arg\eta < \frac{5\pi}{8}; \tag{1.28}$$

Note that

$$Im\ \psi(\eta) = Im\ (yG(y)) \sim \tilde{S}(a)y^{1/4}e^{-\frac{2^{3/2}}{3}y^{3/2}} \sim \frac{3^{1/6}}{2^{1/4}}\tilde{S}(a)\eta^{1/6}e^{-\eta};$$

One of the main theorems proved in this paper is

Theorem 1.9. *(1)There exists sufficiently large enough $\rho_0 > 0$ such that (1.27) with condition (1.28) has unique analytic solution $\psi_0(\eta, a)$ in the region $|\eta| > \rho_0, \arg\eta \in [0, \frac{5\pi}{8}]$.*
(2)Further on the real η axis as $\eta \to \infty$

$$Im\ \psi_0(\eta, a) \sim \frac{3^{1/6}}{2^{1/4}}\tilde{S}(a)\eta^{1/6}e^{-\eta}; \tag{1.29}$$

(3)Further $Im\ \psi_0(\eta, a) = 0$ for real η and $\eta > \rho_0$ iff $\tilde{S}(a) = 0$.

The proof of Theorem 1.9 is given in Section 4.

Remark 1.10. Theorem 1.9 is only a small part of what will be necessary. In order to construct rigorous theory for this problem, we will also need to control integral and other formally small terms that were neglected in previous studies of Combescot et al (1986,1988),Tanveer (1987), Dorsey & Martin (1987) and Chapman (1999).

The primary result of this paper for the finger problem is the following:

Theorem 1.11. *In a range $\frac{1}{2} \leq \lambda \leq \lambda_m$, δ, δ_1 and $\lambda_m - \frac{1}{2}$ small (but independent of ϵ) so that (2.24) holds, the following statements hold for all sufficiently small ϵ:*
(1) For each $\beta_0 \in \{a_n\}$ for which the Stokes constant $\tilde{S}(\beta_0) = 0$ in Theorem 1.9, if $\tilde{S}'(\beta_0) \neq 0$, there exists an analytic function $\beta(\epsilon^{2/3})$ with $\lim_{\epsilon \to 0} \beta(\epsilon^{2/3}) = \beta_0$ so that if

$$\frac{2\lambda - 1}{1 - \lambda} = \epsilon^{4/3} \beta(\epsilon^{2/3}), \tag{1.30}$$

then there exists a solution of the Finger problem $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$. Hence for small ϵ,

$$\frac{2\lambda - 1}{1 - \lambda} = \epsilon^{4/3} \left(\beta_0 + \beta_1 \epsilon^{2/3} + \beta_2 \epsilon^{4/3} + \cdots \right); \tag{1.31}$$

(2) For λ for which $\tilde{S}(a) \neq 0$, where $a \equiv \frac{2\lambda - 1}{\epsilon^{4/3}(1-\lambda)}$, there can be no solution F for small ϵ to Finger problem when $F \in \mathbf{A}_{0,\delta}$, $F' \in \mathbf{A}_{1,\delta_1}$.

The proof of this theorem is not given until the end of §4, after many preliminary results. Our solution strategy consists of two steps:

(a) Relaxing the symmetry condition $F(\xi) = [F(-\xi^*)]^*$ on the imaginary axis interval $-b < Im\ \xi < \alpha$, (*i.e.* $Im\ F = 0$ is relaxed on that $Im\ \xi$-axis segment), we prove the existence and uniqueness of solutions to an appropriate problem in the half strip $\mathcal{R}^-$ (defined as Weak Problem) for any λ belonging to a compact subset of $(0, 1)$, for all sufficiently small ϵ. There is no restriction on λ otherwise.

(b) The symmetry condition is then invoked to determine restriction on λ that will guarantee existence of solution to the Finger problem. In this part, we restrict our analysis to $\lambda \in [\frac{1}{2}, \lambda_m]$.

However, in going through the step (a), we encounter singularities of integrals of the type $(P) \int_{-\infty}^{\infty} \frac{G(t)}{t-\xi} dt$ since continuity of $G(t)$ at $t = 0$ is not guaranteed when symmetry is relaxed. This makes proof difficult. We circumvent this difficulty by proving the equivalence of the finger problem to a problem where integral quantities are of the type $\int_r \frac{G_1(t)}{t-\xi} dt$, where the path r is outside the domain $\mathcal{R}$ and $G_1(t)$ can be calculated on r using F on the domain $\mathcal{R}$. The singularities in this case are outside $\mathcal{R}$ and do not affect the regularity argument for F in $\mathcal{R}$. In Section 2, we prove equivalence of the finger problem to a set of two problems (*Problem 1 and Problem 2*) in a complex strip domain. Problem 1 is to solve an integro-differential equation for F in a Banach space $\mathbf{A}_0$. By deforming the contour of integration for the integral term in Problem 1, we obtain Problem 2. By relaxing symmetry condition on an $Im\ \xi$ axis segment, we derive a Weak Problem in the left half strip $\mathcal{R}^-$.

In Section 3, by contraction argument, the existence and uniqueness of analytic solutions to the Weak problem are proved for λ in a compact subset of $(0, 1)$ for all sufficiently small ϵ. In Section 4, we carry out step (b) in our solution strategy. By introducing suitably scaled dependent and independent variables in a neighborhood of a turning point, we formulate the inner problem. For the leading order inner-equation, the form of exponentially small terms are obtained and Theorem 1.9 proved. For the full problem, using implicit function theorem, it is argued that for small ϵ, there exists a discrete set of analytic functions $\beta(\epsilon^{2/3})$ so that if $a = \beta(\epsilon^{2/3})$, then Im $F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$. This implies that symmetry condition (1.19) is satisfied; hence Theorem 1.11 follows.

2. Formulation of Equivalent Problems

If the symmetry condition (iii) were relaxed just at $\xi = 0$ in the original finger problem, and we were to try to prove existence of solution F for arbitary λ in a domain $\mathcal{R}^-$, (using symmetry condition (iii) to define F on $\mathcal{R}^+$), we face a severe problem in controlling nonlocal terms. The integrands involved in calculation of $I(\xi)$ as defined below are not continous; hence $I(\xi)$ is singular at $\xi = 0$ on the boundary of $\mathcal{R}^-$.

We circumvent this difficulty by showing that the original finger problem is equivalent to Problem 1 and then a Problem 2. The latter only involves integration paths outside the domain $\mathcal{R}$ and can be calculated simply from F in $\mathcal{R}$. Additionally, when symmetry condition in the form $Im\ F = 0$ on the imaginary ξ-axis segment $(-ib, i\alpha)$ is relaxed in Problem 2, the singularity introduced from the nonlocal terms is at a nonzero distance from the domain $\mathcal{R}$ and does not interfere with controlling F within domain $\mathcal{R}^-$ of the weak problem.

In this particular section, we are going to formulate Problem 1 and Problem 2, which will be proved to be equivalent to Finger Problem defined in Section 1. We then formulate a Weak problem in terms of an integro-differential equation in $\mathcal{R}^-$, but relax the the symmetry condition $Im\ F = 0$ on the imaginary ξ-axis segment $(-ib, i\alpha)$, which follows from (1.19) (see Remark 1.7). Problem 2, unlike Problem 1, involves nonlocal quantities (I_2 for instance) with integration paths outside domain $\mathcal{R}$. Hence, when symmetry is relaxed for F in the weak problem, singularities from nonlocal contributions are at nonzero distances from the domain $\mathcal{R}^-$ of F. In this section, as well as the next, we will restrict λ to a compact subset of $(0, 1)$ so that all the constants appearing in all estimates are independent of λ.

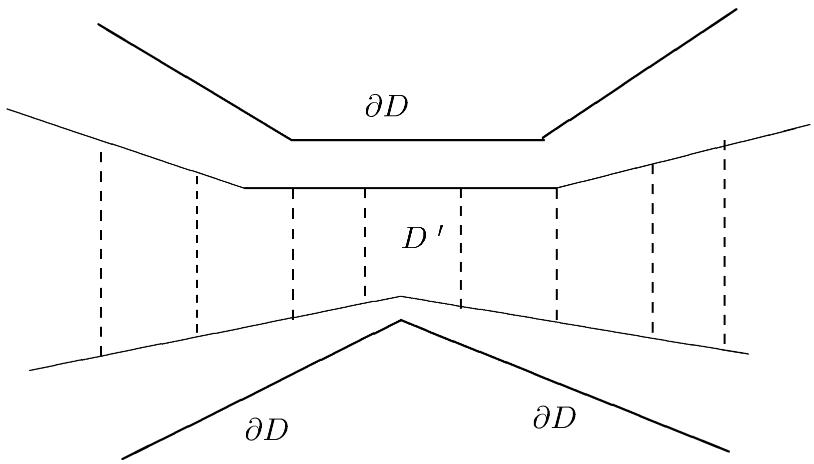

Fig. 2.1. Angular subset $\mathcal{D}'$ of $\mathcal{D}$.

2.1. Formulation of Problem 1

Definition 2.1. Let $\mathcal{D}$ be an open connected set in complex plane with one or more straight line boundaries. $\mathcal{D}'$ is defined as an angular subset of $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$, $dist(\mathcal{D}', \partial\mathcal{D}) > 0$ and $\mathcal{D}'$ has straight line boundaries that make a nonzero angle with respect to $\partial\mathcal{D}$ asymptotically at large distances from the origin (see figure 2.1). This means that if $z' \in \mathcal{D}'$ and $z \in \partial\mathcal{D}$,then $dist(z, \partial\mathcal{D}') \geq C|z| \sin\theta_0$, as $|z| \to \infty$; $dist(z', \partial\mathcal{D}) \geq C|z'| \sin\theta_0$, as $|z'| \to \infty$,where C is some positive constant and $0 < \theta_0 \leq \pi/2$.

Remark 2.1. $\mathcal{D}'$ may just be a straight line segment..

Lemma 2.2. *If $F(\xi)$ satisfies property (1.17) in $\mathcal{D}$,then*

$$F'(\xi) \sim O(\xi^{-(1+\tau)}), \ \ as \ |\xi| \to \infty, \xi \in \mathcal{D}'.$$

$$F''(\xi) \sim O(\xi^{-(2+\tau)}), \ \ as \ |\xi| \to \infty, \xi \in \mathcal{D}'.$$

where $\mathcal{D}'$ is any angular subset of $\mathcal{D}$

Proof. This is standard. The proof can easily be adapted from Olver (1974) (Theorem 1.4.2).

Definition 2.2. Define

$$\bar{H}(\xi) = [H(\xi^*)]^* = \frac{\xi - i\gamma}{\xi^2 + 1}; \tag{2.1}$$

$$\bar{F}(\xi) = [F(\xi^*)]^*; \tag{2.2}$$

Remark 2.3. If F is analytic in domain $\mathcal{D}$ containing real axis with property (1.17), then $\bar{F}$ is analytic in $\mathcal{D}^*$ and $\bar{F}(\xi) = F^*(\xi)$ for ξ real and $\bar{F}(\xi) \sim O(\xi^{-\tau})$, as $|\xi| \to \infty, \xi \in \mathcal{D}^*$; $\bar{F}'(\xi) \sim O(\xi^{-1-\tau})$, $\bar{F}''(\xi) \sim O(\xi^{-2-\tau})$ as $|\xi| \to \infty, \xi \in \mathcal{D}'^*$, where $\mathcal{D}'$ is any angular subset of $\mathcal{D}$ and superscript * denotes conjugate domain obtained by reflecting about the real axis.

Definition 2.3. Let $\mathcal{D}$ be a connected set; for any two functions f, g with second derivative existing in $\mathcal{D}$ and small enough $\|f'\|_{1,\mathcal{D}}$ and $\|g'\|_{1,\mathcal{D}}$ so that $f' + H \neq 0, g' + \bar{H} \neq 0$ in $\mathcal{D}$, we define operator G so that

$$G(f,g)[t] := \frac{1}{(f'(t)+H(t))^{1/2}(g'(t)+\bar{H}(t))^{1/2}} \times \left[\frac{f''(t)+H'(t)}{f'(t)+H(t)} - \frac{g''(t)+\bar{H}'(t)}{g'(t)+\bar{H}(t)}\right] \tag{2.3}$$

Remark 2.4. If $F \in \mathbf{A}$ and $F' + H \neq 0$ in $\mathcal{R}$,then $G(F, \bar{F})(t)$ is analytic in $\mathcal{R} \cap \mathcal{R}^*$, since in that case $\bar{F}' + \bar{H} \neq 0$ in $\mathcal{R}^*$.

Lemma 2.5. *Let $\mathcal{D}$ and f, g be as in definition 2.3. If each of dist$(\mathcal{D}, -i)$, dist $(\mathcal{D}, i)$, dist$(\mathcal{D}, -i\gamma)$, and dist $(\mathcal{D}, i\gamma)$ are greater than 0 and independent of ϵ, as $\epsilon \to 0$, then we have for small enough $\|f'\|_{1,\mathcal{D}}, \|g'\|_{1,\mathcal{D}}$,*

$$\|G(f,g)\|_{0,\mathcal{D}} \leq \frac{C(1 + \|f''\|_{2,\mathcal{D}}\|g'\|_{1,\mathcal{D}} + \|f'\|_{1,\mathcal{D}}\|g''\|_{2,\mathcal{D}})}{(K_1 - \|f'\|_{1,\mathcal{D}})^{3/2}(K_1 - \|g'\|_{1,\mathcal{D}})^{3/2}} + \frac{C(\|f''\|_{2,\mathcal{D}} + \|g''\|_{2,\mathcal{D}} + \|f'\|_{1,\mathcal{D}} + \|g'\|_{1,\mathcal{D}})}{(K_1 - \|f'\|_{1,\mathcal{D}})^{3/2}(K_1 - \|g'\|_{1,\mathcal{D}})^{3/2}} \tag{2.4}$$

where

$$\begin{aligned} 0 < H_m &\leq \inf_{\mathcal{D}} \{|(t-2i)H(t)|, |(t-2i)\bar{H}(t)|\}; \\ K &\geq \sup_{\mathcal{D}} |t-2i|^{-\tau} > 0; \\ K_1 &= \frac{H_m}{K} > 0; \end{aligned} \tag{2.5}$$

Constants C, K and K_1 are independent of ϵ and λ, since λ is in a compact subset of $(0, 1)$.

Proof. Without ambiguity, the norms $\|\cdot\|$ denoted in this proof refer to $\|\cdot\|_{.,\mathcal{D}}$, where sup is over the domain $\mathcal{D}$. From (2.3):

$$\begin{aligned} G(f,g) = &\frac{[H'(t)\bar{H}(t) - H(t)\bar{H}'(t)]}{(f'(t)+H(t))^{3/2}(g'(t)+\bar{H}(t))^{3/2}} \\ &+ \frac{(f''(t)g'(t) - f'(t)g''(t))}{(f'(t)+H(t))^{3/2}(g'(t)+\bar{H}(t))^{3/2}} \\ &+ \frac{\left(f''(t)\bar{H}(t) + H'(t)g'(t) - H(t)g''(t) - f'(t)\bar{H}'(t)\right)}{(f'(t)+H(t))^{3/2}(g'(t)+\bar{H}(t))^{3/2}} \end{aligned} \tag{2.6}$$

Using (1.15), (2.1):

$$\sup\{|t-2i|^4|H'(t)\bar{H}(t)-H(t)\bar{H}'(t)|\le C$$
$$\sup|(t-2i)H|\le C, \sup|(t-2i)^2H'|\le C$$
$$\sup|(t-2i)\bar{H}|\le C, \sup|(t-2i)^2\bar{H}'|\le C$$

where C is made independent of ϵ and λ for λ in a fixed compact subset of (0, 1). We also have

$$\begin{aligned}|(f'(t)+H(t))|^{3/2} &\ge \left[H_m|t-2i|^{-1}-\|f'\|_1|t-2i|^{-1-\tau}\right]^{3/2}\\ &\ge (H_m-K\|f'\|_1)^{3/2}|t-2i|^{-3/2}\\ &\ge C(K_1-\|f'\|_1)^{3/2}|t-2i|^{-3/2};\\ |(g'(t)+\bar{H}(t))|^{3/2} &\ge \left[H_m|t-2i|^{-1}-\|g'\|_1|t-2i|^{-1-\tau}\right]^{3/2}\\ &\ge (H_m-K\|g'\|_1)^{3/2}|t-2i|^{-3/2}\\ &\ge C(K_1-\|g'\|_1)^{3/2}|t-2i|^{-3/2};\end{aligned} \tag{2.7}$$

where C is made independent of ϵ and λ. (2.4) follows immediately from (2.6) and (2.7).

Lemma 2.6. *If $F\in\mathbf{A}$, then $G(F,\bar{F})(t)=O(t^{-\tau})$, as $|t|\to\infty, t$ in any angular subdomain of $\mathcal{R}\cap\mathcal{R}^*$.*

Proof. From Remark 1.2, Lemma 2.2, Remark 2.3 and Lemma 2.5, with $\mathcal{D}$ being an angular subdomain of $\mathcal{R}\cap\mathcal{R}^*$, the proof follows.

Remark 2.7. Let H_m, K, K_1 be as in (2.5) of Lemma 2.5 with $\mathcal{D}=\mathcal{R}$, then H_m, K, K_1 can be chosen to be independent of ϵ and λ since λ is in a fixed compact subset of (0, 1). If $F'\in\mathbf{A}_{1,\delta_1}, \delta_1<\frac{K_1}{2}$ then,

$$|\xi-2i||F'+H|\ge(H_m-|\xi-2i|^{-\tau}\|F'\|_1)\ge C(K_1-\|F'\|_1)>C\frac{K_1}{2}>0; \tag{2.8}$$

so $F'+H\ne 0$ holds in $\mathcal{R}$. From now on we put further restriction $\delta_1<\frac{K_1}{2}$.

Remark 2.8. Let $F'\in\mathbf{A}_{1,\delta_1}$, by Remark 2.7, we have $\bar{F}'+\bar{H}\ne 0$ in $\mathcal{R}^*$, so $G(F,\bar{F})$ is analytic in $\mathcal{R}\cap\mathcal{R}^*$.

Definition 2.4. Let $F\in\mathbf{A}, F'\in\mathbf{A}_{1,\delta_1}$, define operator I so that

$$I(F)[\xi]=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(F,\bar{F})(t)dt}{t-\xi}\ \text{for}\ \ Im\ \xi\ <\ 0, \tag{2.9}$$

Lemma 2.9. *Let $F \in \mathbf{A}, F' \in \mathbf{A}_{1,\delta_1}$ and F be analytic in $Im\ \xi > 0$ as well. If F satisfies equation (1.14) on the real axis . Then F satisfies :*

$$\epsilon^2 F''(\xi) + L(\xi)F(\xi) = N(F, I, \bar{F})(\xi), \ \textit{for}\ \xi \in \{Im\ \xi < 0\} \cap \mathcal{R}; \qquad (2.10)$$

where

$$L(\xi) = -iH^{3/2}(\xi)\bar{H}^{1/2}(\xi) = -\frac{i\sqrt{\gamma^2+\xi^2}(\xi+i\gamma)}{(\xi^2+1)^2} \qquad (2.11)$$

and the operator N is defined as

$$\begin{aligned} N(F, I, \bar{F}) &= \epsilon^2 \left(\frac{\bar{H}'H}{\bar{H}} - H' \right) - i\epsilon^2 (F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2} I(F) \\ &+ iF \ \left[(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2} \right] \\ &+ \epsilon^2 \left[(\bar{F}''+\bar{H}') \frac{F'+H}{\bar{F}'+\bar{H}} - \frac{\bar{H}'H}{\bar{H}} \right]; \end{aligned} \qquad (2.12)$$

Proof. Since F is analytic in $\mathcal{R} \cup \{Im\ \xi > 0\}$ and satisfies equation (1.14), by using Poisson formula, we have for $Im\ \xi > 0$:

$$\begin{aligned} F(\xi) &= \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{(t-\xi)} \frac{1}{|F'(t)+H(t)|} \mathrm{Im} \ \left[\frac{F''(t)+H'(t)}{F'(t)+H(t)} \right] \\ &= -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F,\bar{F})(t)dt}{t-\xi} \end{aligned} \qquad (2.13)$$

Using Plemelj Formula (see for eg. Carrier, Krook & Pearson, 1966), we analytically extend above equation to the lower half plane to obtain

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i} G(F, \bar{F})(\xi), \quad \text{for} \ \ \mathrm{Im}\ \xi < 0; \qquad (2.14)$$

Multiplying the above by $i(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2}$ leads to (2.10), once definition 2.3 is used.

Definition 2.5. Let $\alpha_0 > 0$ be a fixed number independent of ϵ so that $b > \alpha_0 > 2\alpha$ (thus $-\alpha_0 i \in \mathcal{R}$). Define two rays:

$$r_0^+ = \{\xi : \xi = \alpha_0 i + \rho e^{i(\varphi_0 + \frac{1}{2}\mu)}, 0 < \rho < \infty\};$$

$$r_0^- = \{\xi : \xi = \alpha_0 i + \rho e^{i(\pi - \varphi_0 - \frac{1}{2}\mu)}, 0 < \rho < \infty\};$$

Let r_0 be the directed contour along $r_0^- \cup r_0^+$ from left to right (See Fig. 2.3).

Definition 2.6. If $F \in \mathbf{A}$, define $F_-(\xi)$ for ξ below r_0:

$$F_-(\xi) = -\frac{1}{2\pi i} \int_{r_0} \frac{\bar{F}(t)}{t-\xi} dt; \qquad (2.15)$$

Remark 2.10. Since $r_0 \subset \mathcal{R}^*$ and $\bar{F}$ satisfies (1.17) in $\mathcal{R}^*$, the above integral is well defined. It is obvious that $F_-(\xi)$ is analytic below r_0. Also, if $F \in \mathbf{A}_0^-$ only and the relation $[F(-t^*)]^* = F(t)$ were invoked to define F in $\mathcal{R}^+$, then it is possible to use the symmetry between contours r_0^+ and r_0^- to write

$$\begin{aligned} F_-(\xi) &= -\frac{1}{2\pi i}\left[\int_{r_0^-} \frac{\bar{F}(t)dt}{t-\xi} + \int_{r_0^+} \frac{[\bar{F}(-t^*)]^* dt}{t-\xi}\right] \\ &= \; -\frac{1}{2\pi i}\int_{r_0^-}\left[\frac{\bar{F}(t)dt}{t-\xi} - \frac{[\bar{F}(t)]^*(dt)^*}{t^*+\xi}\right] \end{aligned} \qquad (2.16)$$

This alternate expression is equivalent to (2.15) when symmetry condition $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$ holds; however, (2.16) defines an analytic function F_- below r_0 even when symmetry condition is relaxed, as it is for the Weak Problem in §3. In such cases, $F_-(\xi)$ has a singularity at $\xi = i\alpha_0$ since $Im\ F(t)$ is not continuous at $t = i\alpha_0$. But this singularity is outside $\mathcal{R}$. We also notice that F_-, as defined by (2.16) satisfies symmetry condition even when $F \in \mathbf{A}^-$ does not.

Lemma 2.11. *If $F \in \mathbf{A}$ and F is also analytic in $Im\ \xi > 0$, then $\bar{F}(\xi) = F_-(\xi)$, for $\xi \in \mathcal{R}$.*

Proof. Since F is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$, then $\bar{F}$ is analytic in $\mathcal{R}^* \cup \{\text{Im } \xi < 0\}$. We use property (1.17) and the Cauchy Integral formula to obtain

$$F_-(\xi) = -\frac{1}{2\pi i}\int_{r_0} \frac{\bar{F}(t)}{t-\xi} dt = \bar{F}(\xi), \text{ for } \xi \in \mathcal{R};$$

Lemma 2.12. *Let $\Gamma = \{t, t = \xi_0 + \rho e^{i\varphi}, 0 \le \rho < \infty\}$ be a ray, $2i$ not in Γ . $\mathcal{D}$ is some connected set such that $dist(2i, \mathcal{D}) > 0$ and*

$$dist(\xi, \Gamma) \ge m|\xi - \xi_0| > 0; \text{ for } \xi \in \mathcal{D}; \qquad (2.17)$$

$$dist(t, \mathcal{D}) \ge m|t - \xi_0|; \text{ for } t \in \Gamma; \qquad (2.18)$$

$$dist(t, 2i) \ge m|t - \xi_0|; \text{ for } t \in \Gamma; \qquad (2.19)$$

for some constant $m > 0$ independent of ϵ (see Fig. 2.2). Assume g to be a continuous function on Γ with $\|g\|_{0,\Gamma} < \infty$, then for $k = 0, 1, 2,$

$$\sup_{\mathcal{D}} \left|(\xi - 2i)^{k+\tau} \int_\Gamma \frac{g(t)}{(t-\xi)^{k+1}} dt\right| \le C\|g\|_{0,\Gamma}; \qquad (2.20)$$

where constant C depends on φ, m and τ only .

Proof.

$$\left|(\xi - 2i)^{k+\tau} \int_\Gamma \frac{g(t)}{(t-\xi)^{k+1}} dt\right| \le \|g\|_{0,\Gamma} |\xi - 2i|^{k+\tau} \int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau}; \qquad (2.21)$$

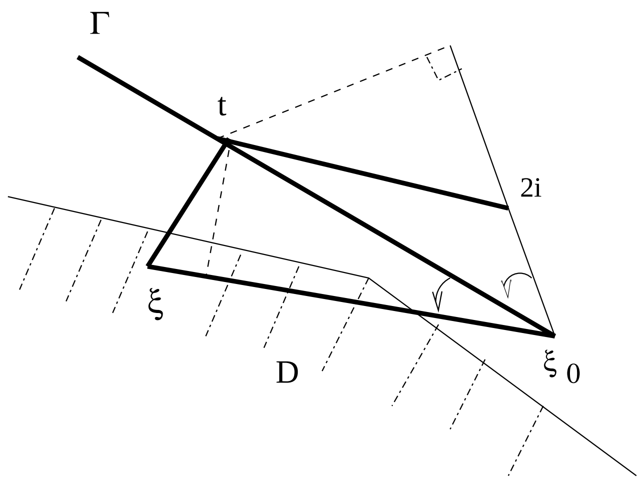

Fig. 2.2. Relevant to Lemma 2.12 and Remark 2.13

On Γ, $t - \xi_0 = \rho e^{i\varphi}, |dt| = d\rho$. Breaking up the integral in (2.21) into two parts:

$$\int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau} = \int_0^{|\xi-\xi_0|} \frac{d\rho}{|(t-\xi)|^{k+1}|t-2i|^\tau} + \int_{|\xi-\xi_0|}^\infty \frac{d\rho}{|(t-\xi)|^{k+1}|t-2i|^\tau}; \tag{2.22}$$

For the first integral in (2.22), we use (2.17) and (2.19) and for the second, we use (2.18) and (2.19) to obtain (on scaling ρ by $|\xi - \xi_0|$):

$$\int_\Gamma \frac{|dt|}{|(t-\xi)|^{k+1}|t-2i|^\tau} = \frac{C}{|\xi-\xi_0|^{k+\tau}} \left(\int_0^1 \frac{d\rho}{\rho^\tau} + \int_1^\infty \frac{d\rho}{\rho^{k+\tau+1}} \right);$$

Remark 2.13. Note if Γ is in $\mathcal{D}'_c$, an angular subset of $\mathcal{D}_c$ (complement of $\mathcal{D}$),then (2.17) and (2.18) hold (see Figure 2.2). Also, note (2.19) is valid for any Γ in $\mathcal{R}$.

Definition 2.7.

$$\Omega_0 = \left\{ \xi : \xi \text{ is below } \{\xi = \frac{1}{2}\alpha_0 i + \rho e^{i(\pi - \varphi_0 - \frac{1}{3}\mu)}\} \cup \{\xi = \frac{1}{2}\alpha_0 i + \rho e^{i(\varphi_0 + \frac{1}{3}\mu)}\} \right\}. \tag{2.23}$$

Remark 2.14. $\mathcal{R} \cup \{\text{Im } \xi < 0\}$ is an angular subset of Ω_0, and Ω_0 is itself an angular subset of the region $\{\xi \text{ below } r_0\}$.

Lemma 2.15. *Let $F \in \mathbf{A}^-$, then*

$$\sup_{\xi\in\Omega_0} |(\xi - 2i)^{k+\tau} F_-^{(k)}(\xi)| \le K_2 \|F\|_0^-; \; k = 0, 1, 2.$$

where constant K_2 depends only on φ_0 and α_0.

Proof. From remarks 2.13-2.14, conditions (2.17)-(2.19) hold with $\Gamma = r_0$ and $\mathcal{D} = \Omega_0$. Using $\|\bar{F}\|_{0,r_0^-} \le \|F\|_0^-$,(2.16) and applying Lemma 2.12, with $g = \bar{F}$, we obtain the proof.

Remark 2.16. From now on, we choose $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$ with additional restriction,

$$\delta < \frac{K_3}{2} \equiv \frac{H_m}{2KK_2}, \delta_1 < \frac{K_1}{2}; \tag{2.24}$$

where H_m, K are as in (2.5) with $\mathcal{D} = \mathcal{R}$, while K_2 is defined as in Lemma 2.15. This ensures

$$|\xi - 2i||F'_- + \bar{H}| \ge (H_m - K\|F'_-\|_1) \ge (H_m - KK_2\|F\|_0) \ge C\frac{K_3}{2} > 0; \tag{2.25}$$

so $F'_- + \bar{H} \ne 0$ in $\mathcal{R}$, therefore $G(F, F_-)$ is analytic in $\mathcal{R}^-$. Further, If F satisfies symmetry condition, it is clear then that $G(F, F_-) \in \mathbf{A}$

Remark 2.17. Using Lemmas 2.5 and 2.15, $G(F, F_-)(t) = O(t^{-\tau})$ as $|t| \to \infty, t$ in any angular subset of $\mathcal{R} \cap \Omega_0$ including the real axis.

Definition 2.8. If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, define operator $I_1(F)$ so that for Im $\xi < 0$:

$$I_1(F)[\xi] = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(F, F_-)(t)}{t - \xi} dt, \tag{2.26}$$

Define

$$\begin{aligned} N_1(\xi) \equiv N(F, I_1(F), F_-)(\xi) \; &= \epsilon^2 \left(\frac{\bar{H}'H}{\bar{H}} - H'\right) \\ &+ \; i \; F \; \left[(F' + H)^{3/2}(F'_- + \bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2}\right] \\ &- i\epsilon^2 (F' + H)^{3/2}(F'_- + \bar{H})^{1/2} I_1(F) \\ &+ \; \epsilon^2 \left[(F''_- + \bar{H}')\frac{F' + H}{F'_- + \bar{H}} - \frac{\bar{H}'H}{\bar{H}}\right]; \end{aligned} \tag{2.27}$$

Lemma 2.18. *If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$ and F is analytic in $Im\ \xi > 0$ and also satisfies (1.14), then in the region $\{Im\ \xi < 0\} \cap \mathcal{R}$, F satisfies*

$$\epsilon^2 \; F''(\xi) + L(\xi) \; F(\xi) \; = \; N_1(\xi); \tag{2.28}$$

Proof. Since conditions of Lemma 2.9 are satisfied, (2.10) holds. Since F is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$ with property (1.17), from Lemma 2.11,$F_- = \bar{F}$; hence $I_1(F)(\xi) = I(F)(\xi)$. Therefore (2.10) implies (2.28).

Lemma 2.19. *If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, and F satisfies equation (2.28) in $\mathcal{R} \cap \{Im\ \xi < 0\}$, then F is analytic in $\mathcal{R} \cup \{Im\ \xi > 0\}$ and satisfies (1.14) on the real ξ axis.*

Proof. Note on using expression for N_1 from (2.27), (2.28) can be rewritten as:

$$F(\xi) = \epsilon^2 I_1(F)(\xi) + \frac{\epsilon^2}{i} G(F, F_-)[\xi], \text{for Im} \quad \xi < 0; \qquad (2.29)$$

where operator G is defined by (2.3). Analytically continuing the above equation to the upper half plane, we have:

$$F(\xi) = -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, F_-)(t) dt}{t - \xi}, \text{ for Im } \xi > 0; \qquad (2.30)$$

so $F(\xi)$ is analytic in the upper half plane. From Lemma 2.11, $F_-(\xi) = \bar{F}(\xi)$; hence on the real ξ-axis, $F_-(\xi) = F^*(\xi)$. Equation (2.30) reduces to:

$$F(\xi) = \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - \xi} \frac{1}{|F'(t) + H(t)|^{1/2}} \text{ Im } \left[\frac{F''(t) + H'(t)}{F'(t) + H(t)} \right], \text{ for } Im\ \xi > 0;$$

On taking the limit $Im\ \xi \to 0^+$, the above implies (1.14).

Because of remark 1.7 about equivalence of condition (iii) to $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$, Lemmas 2.18 and 2.19 imply:

Theorem 2.20. *The finger problem is equivalent to*
Problem 1: *Find function $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, satisfying $Im\ F = 0$ on $\{Re\ \xi = 0\} \cup \mathcal{R}$, so that (2.28) is satisfied in $\mathcal{R} \cap \{Im\ \xi < 0\}$.*

2.2. Formulation of Problem 2

Let $\alpha_1 > 0$ be a fixed constant independent of ϵ so that $-\alpha_1 i \in \mathcal{R}$. Define two rays (see Figure 2.3):

Definition 2.9.

$$r_1^+ = \{\xi : \xi = -\alpha_1 i + \rho e^{-i\varphi_0}, 0 < \rho < \infty\};$$

$$r_1^- = \{\xi : \xi = -\alpha_1 i + \rho e^{i(\pi + \varphi_0)}, 0 < \rho < \infty\};$$

$r_1 = r_1^- \cup r_1^+$ is a directed contour from left to right (see Fig. 2.3).

Definition 2.10. Let $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$. For ξ above r_1, define

$$F_+(\xi) = \frac{1}{2\pi i} \int_{r_1} \frac{F(t) dt}{t - \xi} \qquad (2.31)$$

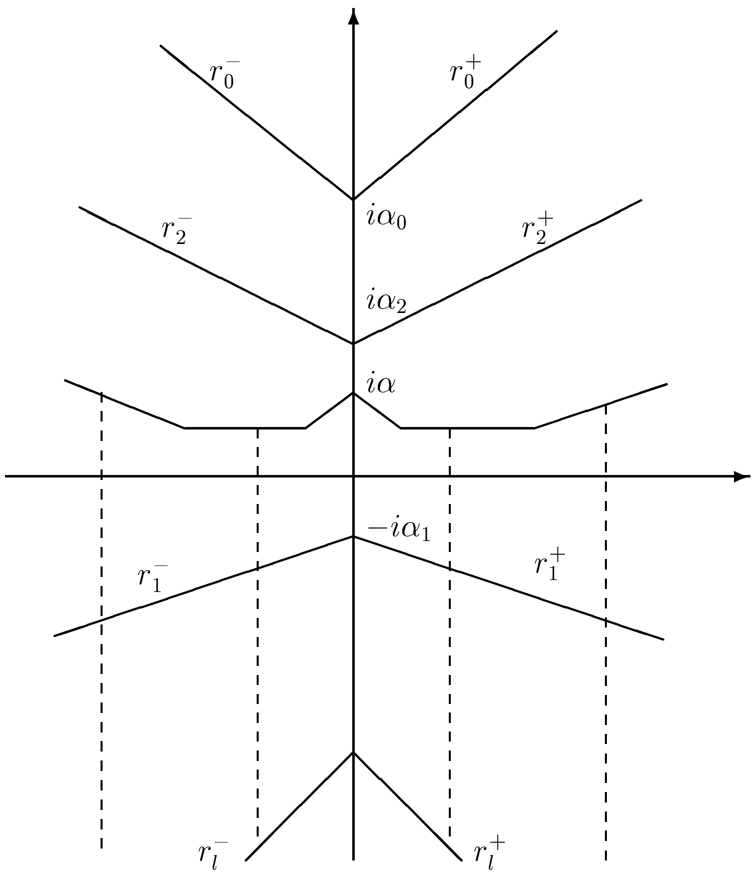

Fig. 2.3. Rays defined in Subsection 2.2.

Remark 2.21. F_+ as defined above is analytic for ξ above r_1. Also, it is to be noted that for $F \in \mathbf{A}_0^-$ only, when $F(\xi) = [F(-\xi^*)]^*$ is invoked to define F on $\mathcal{R}^+$, we can write

$$\begin{aligned} F_+(\xi) &= \frac{1}{2\pi i}\left[\int_{r_1^-} \frac{F(t)dt}{t-\xi} + \int_{r_1^+} \frac{[F(-t^*)]^* dt}{t-\xi}\right] \\ &= \frac{1}{2\pi i}\int_{r_1^-}\left[\frac{F(t)dt}{t-\xi} - \frac{[F(t)]^* dt^*}{t^*+\xi}\right] \end{aligned} \tag{2.32}$$

This expression is equivalent to (2.31) when symmetry condition: $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$ is satisfied. However, even without symmetry, (2.32) still defines an analytic function for ξ above r_1, with possible singularity at $\xi = -i\alpha_1$. It is easy to see that $F_+(\xi)$ satisfies symmetry condition on $\{Re\ \xi = 0\} \cup \mathcal{R} \cup \{Im\ \xi > -\alpha_1\}$ even when F does not. Further, if $F \in \mathbf{A}_0$

is also analytic in $\{Im\ \xi\ >\ 0\}$ then it is clear from (2.31) on closing the contour from the top that $F_+(\xi) = F(\xi)$.

Definition 2.11.

$$\Omega_1 = \left\{ \xi : \xi \text{ is above } \{\xi = -\frac{\alpha_1}{2} i + \rho e^{i(\pi + \frac{\varphi_0}{2})}\} \cup \{\xi = -\frac{\alpha_1}{2} i + \rho e^{-i\frac{\varphi_0}{2}}\} \right\}; \tag{2.33}$$

Remark 2.22. Ω_1 is an angular subset of the region $\{\xi : \xi \text{ above } r_1\}$, r_1 as in Definition 2.9.

Lemma 2.23. *Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$, then*

$$\sup_{\xi \in \Omega_1} \left| (\xi - 2i)^{k+\tau} F_+^{(k)}(\xi) \right| \leq K_2 \|F\|_0^- \text{ for } k = 0, 1, 2.$$

where $K_2 > 0$ is independent of ϵ, λ.

Proof. The proof is very similar to that of Lemma 2.15. From remarks 2.13-2.22, conditions (2.17)-(2.19) hold with $\Gamma = r_1$ and $\mathcal{D} = \Omega_1$. Using $\|\bar{F}\|_{0,r_0^-} \leq \|F\|_0^-$,(2.32) and applying Lemma 2.12, with $g = F$, we obtain the proof.

Definition 2.12. Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$. Let F_- be given by (2.16) and F_+ by (2.32). For $\xi \in \{Im\ \xi\ >\ 0\}$, define

$$\hat{F}(\xi) = -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F_+, F_-)[t]dt}{t - \xi} dt \tag{2.34}$$

Remark 2.24. Using Lemma 2.5, with $\mathcal{D} = \Omega_1 \cap \Omega_0$, $G(F_+, F_-) \in \mathbf{A}_{0,\Omega_1 \cap \Omega_0}$. It follows from Lemma 2.12, with $\Gamma = \mathbf{R}$ (real axis) and $\mathcal{D} = r_u$, where r_u is the upper-boundary of $\mathcal{R}$, that $\hat{F} \in \mathbf{A}_{0,r_u}$, $\hat{F}' \in \mathbf{A}_{1,r_u}$ and $\hat{F}'' \in \mathbf{A}_{2,r_u}$ and $Im\ \hat{F} = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R} \cap \{Im\ \xi > 0\}$. Further, for $F \in \mathbf{A}_{0,\delta}$, if F is analytic in the upper-half plane as well, then $F_+ = F$ and $F_- = \bar{F}$. On comparing (2.34) with (2.13), it follows that $\hat{F} = F$.

Lemma 2.25. *If $F \in \mathbf{A}^-_{0,\delta}$, then*

$$\sup_{\xi \in r_u} \left| (\xi - 2i)^{k+\tau} \hat{F}^{(k)}(\xi) \right| \leq \epsilon^2 K_4 \quad \text{for} \quad k = 0, 1, 2.$$

where $K_4\ >\ 0$ is independent of ϵ and λ.

Proof. We know from using Lemmas 2.5, 2.15 and 2.23 that

$$\|G(F_+, F_-)\|_{0,\mathbf{R}} \leq \frac{C(1 + \|F\|_0^- + [\|F\|_0^-]^2)}{(K_3 - \|F\|_0)^3}$$

where C is independent of λ and ϵ. Using lemma 2.12 on (2.34), with $\Gamma = \mathbf{R}$ (real axis), and $\mathcal{D} = r_u$, we complete the proof.

Definition 2.13. Let α_2 be a constant independent of ϵ so that $\alpha < \alpha_2 < \frac{1}{2}\alpha_0$. Define two rays (see Figure 2.3):

$$r_2^+ = \{\xi : \xi = \alpha_2 i + \rho e^{i\varphi_0 + \frac{1}{4}\mu}, 0 < \rho < \infty\};$$

$$r_2^- = \{\xi : \xi = \alpha_2 i + \rho e^{i(\pi - \varphi_0 - \frac{1}{4}\mu)}, 0 < \rho < \infty\};$$

r_2 is a directed contour from left to right on the path $r_2^- \cup r_2^+$.

Remark 2.26. r_2 is an angular subset of $\Omega_0 \cap \Omega_1$ and $\mathcal{R}$ is below r_2.

Definition 2.14. If $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$, let F_- be given by (2.15) and F_+ as in (2.31). Define operator I_2 so that

$$I_2(F)[\xi] = -\frac{1}{2\pi}\int_{r_2} \frac{G(F_+, F_-)(t)dt}{t - \xi}, \text{for } \xi \text{ below } r_2; \tag{2.35}$$

Remark 2.27. Because of Lemmas 2.5, 2.15 and 2.23, $G(F_+, F_-)(t) \sim O(t^{-\tau})$ as $|t| \to \infty$ and analytic for t in any angular subset of $\Omega_0 \cap \Omega_1$ that includes r_2; so $I_2(F)[\xi]$ is analytic below r_2. Also, from the symmetry of each of F_+ and F_-, it is not difficult to see that $I_2(F)[\xi]$ also satisfies symmetry condition on $\{Re\ \xi = 0\} \cap \mathcal{R}$.

Lemma 2.28. *Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$. Then $I_2(F) \in \mathbf{A}$, and $\|I_2(F)\|_0 \le K_5$; where $K_5 > 0$ is independent of ϵ and λ.*

Proof. From Lemma 2.15 and Lemma 2.23,

$$\|F_-^{(k)}\|_{k,r_2} \le K_2 \|F\|_0^- \quad , \quad \|F_+^{(k)}\|_{k,r_2} \le K_2 \|F\|_0^-$$

Applying Lemma 2.5 (with $\mathcal{D} = r_2$) and Lemma 2.12 (with $\Gamma = r_2$ and $\mathcal{D} = \mathcal{R}$) to (2.35), we complete the proof.

Definition 2.15.

$$\begin{aligned} N_2(\xi) \ &\equiv \ N(F, I_2(F), F_-)(\xi) = \epsilon^2 \left[\frac{\bar{H}' H}{\bar{H}} - H'\right] \\ &+ i\ F\ \left[(F' + H)^{3/2}(\hat{F}' + \bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2}\right] \\ &- i\epsilon^2\ (F' + H)^{3/2}(\hat{F}' + \bar{H})^{1/2} I_2(F) + \ \epsilon^2 \left[(F_-'' + \bar{H}')\frac{F' + H)}{F_-' + \bar{H}} - \frac{\bar{H}' H}{\bar{H}}\right]; \end{aligned} \tag{2.36}$$

Lemma 2.29. *If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$ and F satisfies (2.28) in $\mathcal{R} \cap \{Im\ \xi < 0\}$, then for $\xi \in \mathcal{R}$, F satisfies*

$$\epsilon^2\ F''(\xi) + L(\xi)\ F(\xi) = N_2(\xi); \tag{2.37}$$

Proof. If δ is small enough, from Lemmas 2.15 and 2.23, $|(\xi - 2i)^{1+\tau} F_-'|$ and $|(\xi - 2i)^{1+\tau} F_+'|$ and are each small in the domain $\Omega_0 \cap \Omega_1$ which contains the region between r_2 and r_1; hence $F_-' + \bar{H} \neq 0$ and $F_+' + H \neq 0$ in that domain. From Lemma 2.19, F is analytic in $\{Im\ \xi > 0\} \cup \mathcal{R}$; hence $F_+ = F$. By deforming the contour r_2 in (2.35) back to real axis, it follows $I_2(F)(\xi) = I_1(F)(\xi)$, for Im $\xi < 0$; so $N_2(\xi) = N(F, I_2(F), F_-)(\xi) = N(F, I_1(F), F_-)(\xi) = N_1(\xi)$, for Im $\xi < 0$. By analytic continuation, F satisfies (2.37) for $\xi \in \mathcal{R}$.

Definition 2.16.

$$g_1(\xi) = L^{-1/4}(\xi) \exp\{-\frac{P(\xi)}{\epsilon}\}, \qquad (2.38)$$

$$g_2(\xi) = L^{-1/4}(\xi) \exp\{\frac{P(\xi)}{\epsilon}\}, \qquad (2.39)$$

where

$$P(\xi) = i \int_{-i\gamma}^{\xi} L^{1/2}(t) dt = i \int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4}\ (\gamma + it)^{1/4}}{(1 + t^2)} dt; \qquad (2.40)$$

The following properties of the function $P(\xi)$ will be established in Appendix.

Property 1 Re $P(\xi)$ decreases monotonically along the negative Re ξ axis from $-\infty$ to 0, with $Re\ P(-\infty) = +\infty$. Also, $Re\ P(\xi)$ decreases monotonically on the $Im\ \xi$ axis segment $(-ib,\ i\alpha)$.

Property 2 There exists a constant R so that for $|\xi| \geq R$, Re $P(t)$ is increasing along any ray $r = \{t : t = \xi - se^{i\varphi}, 0 < s < \infty\}$ in $\mathcal{R}$ from ξ to $\xi - \infty e^{i\varphi}$, with φ constrained by the relation

$$-\frac{\pi}{2} + \hat{\delta} \ \leq\ arg\ \left(-\xi e^{-i\varphi}\right) \leq\ \frac{\pi}{4} - \hat{\delta}$$

for some positive $\hat{\delta}$ independent of any parameter. Further, $\frac{d}{ds}$ Re $P(t(s) \geq C_1 |t(s) - 2i|^{-1}$, where C_1 is a constant independent of ϵ and λ.

Property 3 There exists sufficiently small ν independent of ϵ, so that $\frac{d}{ds}[ReP(t(s))] \geq C > 0$ on straightline $\{t(s) = \alpha i + se^{-i\frac{3\pi}{4}}, 0 \leq s \leq \sqrt{2}\nu\}$, where C is some constant independent of ν and ϵ.

Property 4 For any $\xi \in \mathcal{R}^-$, there is a connected path $\mathcal{P}(i\alpha, \xi)$, which is a $\mathbf{C}^1$ curve (or at least piecewise $\mathbf{C}^1$)) from $t = i\alpha$ to $t = \xi$ so that Re $P(t)$ increases mononotically from αi to ξ and $\frac{d}{ds} Re\ P(t(s)) \geq \frac{C}{|t(s) - 2i|} > 0$ on $\mathcal{P}(\alpha i,\ \xi)$; where s is an arclength parametrization of each $\mathbf{C}^1$ path segment, with s increasing towards $t = \xi$, and C is a constant independent of ϵ. $Re\ P(\xi)$ attains its minimum in $\mathcal{R}^-$ at $\xi = \alpha i$.

Property 5 For any $\xi \in \mathcal{R}^-$, there is a path $\mathcal{P}(\xi, -\infty)$ which is $\mathbf{C}^1$ curve (or at least piecewise $\mathbf{C}^1$)) from $t = \xi$ to $t = -\infty$ (in $\mathcal{R}^-$) so that Re $P(t)$ is increasing from $t = \xi$ to $t = -\infty$ and $\frac{d}{ds}[ReP(t(s))] \geq \frac{C}{|t-2i|} > 0$

for $t(s) \in \mathcal{P}(\xi, -\infty)$, s being an arc-length parametrization of any $\mathbf{C}^1$ segment, which increases towards $t = -\infty$ and constant C is independent of ϵ.

$g_1(\xi), g_2(\xi)$ are the two WKB solutions of the corresponding homogenueous equation of (2.37),they satisfy the following equation exactly:

$$\epsilon^2 g''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi))g(\xi) = 0; \tag{2.41}$$

where

$$L_1(\xi) = \frac{L''(\xi)}{4L(\xi)} - \frac{5L'^2(\xi)}{16L^2(\xi)} \tag{2.42}$$

The Wronskian of g_1 and g_2 is

$$W(\xi) = g_1(\xi)g_2'(\xi) - g_2(\xi)g_1'(\xi) = \frac{2i}{\epsilon}, \tag{2.43}$$

Remark 2.30. From (2.11) and (2.42), the singularities of L_1 at $\xi = -i$ and $\xi = -\gamma i$ are double poles and $|\xi - 2i|^2 |L_1(\xi)|$ is bounded in $\mathcal{R}^-$.

Remark 2.31. Note that equation (2.37) implies

$$\mathcal{V}F(\xi) \equiv \epsilon^2 F''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi))F(\xi) = \bar{N}_2(\xi); \tag{2.44}$$

where

$$\bar{N}(F, I_2, F_-) \equiv N(F, I_2(F), F_-) + \epsilon^2 L_1 F \ ; \ \bar{N}_2(\xi) = \bar{N}(F, I_2(F), F_-)[\xi]; \tag{2.45}$$

Lemma 2.32. *If $F \in \mathbf{A}^-, F' \in \mathbf{A}_1^-$, then for $\xi \in \mathcal{R}^-$,*

$$\left| \frac{F_-'}{\bar{H}} \right| \leq K_6 \|F\|_0; \tag{2.46}$$

$$\left| \frac{F'}{H} \right| \leq K_7 \|F'\|_1 \tag{2.47}$$

where K_6, K_7 are constants that are independent of ϵ, λ.

Proof. From (2.5) and Lemma 2.15:

$$\left| \frac{F_-'}{\bar{H}} \right| \leq \frac{|\xi - 2i|^{-1-\tau} \|F_-'\|_1}{|\xi - 2i|^{-1} H_m} \leq \frac{K K_2}{H_m} \|F\|_0^- \equiv K_6 \|F\|_0^-;$$
$$\left| \frac{F'}{H} \right| \leq \frac{|\xi - 2i|^{-1-\tau} \|F'\|_1^-}{|\xi - 2i|^{-1} H_m} \leq \frac{K}{H_m} \|F'\|_1^- \equiv K_7 \|F'\|_1^-;$$

Lemma 2.33. *Let*

$$G_1(F, F_-)(t) = (F'(t) + H(t))^{3/2} (F_-'(t) + \bar{H}(t))^{1/2}; \tag{2.48}$$

If $F \in \mathbf{A}_0^-, F' \in \mathbf{A}_1^-$, then in $\mathcal{R}^-$,

$$|G_1(F, F_-)| \leq C|\xi - 2i|^{-2} \left(K_7 \|F'\|_1^- + 1\right)^{3/2} \left(K_6 \|F\|_0^- + 1\right)^{1/2}; \tag{2.49}$$

where C is independent of ϵ, λ; K_6, K_7 are as in Lemma 2.32.

Proof. From (1.15), (2.1):

$$C_1|\xi - 2i|^{-1} \le |H| \le C_2|\xi - 2i|^{-1}; \tag{2.50}$$

$$C_1|\xi - 2i|^{-1} \le |\bar{H}| \le C_2|\xi - 2i|^{-1}; \tag{2.51}$$

where C_1, C_2 can be chosen independent of ϵ, λ. So, from Lemma 2.32:

$$\begin{aligned} |G_1(F, F_-)| &= |H^{3/2}\bar{H}^{1/2}| \left|\frac{F'}{H} + 1\right|^{3/2} \left|\frac{F'_-}{\bar{H}} + 1\right|^{1/2} \\ &\le C|\xi - 2i|^{-2} \left(K_7\|F'\|_1^- + 1\right)^{3/2} \left(K_6\|F\|_0^- + 1\right)^{1/2}; \end{aligned}$$

Lemma 2.34. *Let*

$$G_2(F, F_-)(\xi) = \left[(F''_- + \bar{H}')\frac{F' + H}{F'_- + \bar{H}} - \frac{\bar{H}'H}{\bar{H}}\right](\xi); \tag{2.52}$$

If $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_1$, *then in* $\mathcal{R}^-$,

$$\begin{aligned} &|G_2(F, F_-)| \\ &\le \frac{C}{\left(K_3 - \|F\|_0^-\right)}|\xi - 2i|^{-2-\tau}\left\{\|F'\|_1^- + \|F\|_0^- + \|F\|_0^- \left(\|F'\|_1^- + 1\right)\right\}; \end{aligned} \tag{2.53}$$

where C *is independent of* ϵ, λ*;* K_3 *is as in Remark 2.16.*

Proof. Note From (1.15), (2.1),

$$\bar{H}' = -\frac{(\xi - i\gamma)^2 + (\gamma^2 - 1)}{(\xi^2 + 1)^2}; \tag{2.54}$$

$$\frac{\bar{H}'H}{\bar{H}} = -\frac{[(\xi - i\gamma)^2 + (\gamma^2 - 1)](\xi + i\gamma)}{(\xi^2 + 1)^2(\xi - i\gamma)}; \tag{2.55}$$

$$\begin{aligned} |F'_- + \bar{H}| &\ge |\bar{H}| \left|\frac{F'_-}{\bar{H}} + 1\right| \ge |\bar{H}| \left(1 - \frac{\|F'_-\|_1|\xi - 2i|^{-1-\tau}}{H_m|\xi - 2i|^{-1}}\right) \\ &\ge C|\bar{H}| \left(K_3 - \|F\|_0^-\right); \end{aligned} \tag{2.56}$$

Using Lemma 2.15,

$$\begin{aligned} |G_2(F, F_-)| &= \left|F'\frac{\bar{H}'}{F'_- + \bar{H}} - \frac{\bar{H}'H}{\bar{H}}\frac{F'_-}{F'_- + \bar{H}} + F''_-\frac{F' + H}{F'_- + \bar{H}}\right| \\ &\le \frac{C\|F'\|_1^-|\xi - 2i|^{-2-\tau}}{\left(K_3 - \|F\|_0^-\right)} + \frac{C\|F\|_0^-|\xi - 2i|^{-2-\tau}}{\left(K_3 - \|F\|_0^-\right)} \\ &+ \frac{C\|F\|_0^- \left(\|F'\|_1^- + 1\right)|\xi - 2i|^{-2-\tau}}{\left(K_3 - \|F\|_0^-\right)} \end{aligned}$$

Lemma 2.35. *Let*

$$G_3(F,F_-) = (F'+H)^{3/2}(F'_- + \bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2}; \qquad (2.57)$$

Assume that $F \in \mathbf{A}^-, F' \in \mathbf{A}_1^-$*,then for* $\xi \in \mathcal{R}^-$

$$|G_3(F,F_-)(\xi)| \le C\ |\xi - 2i|^{-2}\Big\{ \left(K_7\|F'\|_1^- + 1\right)^{3/2} \left(K_6\|F\|_0^- + 1\right)^{1/2} - 1\Big\}; \qquad (2.58)$$

where C *is independent of* ϵ, γ*.* K_6, K_7 *are as in Lemma 2.32.*

Proof. Using(2.50),(2.51) and Lemma 2.32:

$$\begin{aligned}
|G_3(F,F_-)| &\le |H^{3/2}\bar{H}^{1/2}| \left| \left(\frac{F'}{H}+1\right)^{3/2} \left(\frac{F'_-}{\bar{H}}+1\right)^{1/2} - 1 \right| \\
&\le C|\xi-2i|^{-2}\Big\{ \left(\frac{|F'|}{|H|}+1\right)^{3/2} \left(\frac{|F_-|}{|\bar{H}|}+1\right)^{1/2} - 1\Big\} \\
&\le C|\xi-2i|^{-2}\Big\{ \left(K_7\|F'\|_1^- + 1\right)^{3/2} \left(K_6\|F\|_0^- + 1\right)^{1/2} - 1\Big\};
\end{aligned}$$

Lemma 2.36. *Let* $F \in \mathbf{A}_\delta^-, F' \in \mathbf{A}_{1,\delta_1}^-$*, then* $\bar{N}_2 \in \mathbf{A}_2^-$ *with* $\|N_2\|_2^- \le C(\epsilon^2 + \delta\|F'\|_1^- + \delta\|F\|_0^-)$*.*

Proof. Note that

$$\begin{aligned}
\bar{N}_2 = \epsilon^2 &\left(\frac{\bar{H}'H}{\bar{H}} - H'\right) - i\epsilon^2 G_1(F,F_-) I_2(F) + iFG_3(F,F_-) \\
&+ \epsilon^2 G_2(F,F_-) + \epsilon^2 F L_1
\end{aligned} \qquad (2.59)$$

$$\left(\frac{\bar{H}'H}{\bar{H}} - H'\right) = -\frac{2i\gamma}{(\xi^2+1)(\xi-i\gamma)} \qquad (2.60)$$

Apply Lemma 2.28, Lemma 2.33 and Lemma 2.35 to get

$$|I_2(F)G_1(F,F_-)| \le K_5|\xi-2i|^{-2-\tau} \left(K_7\|F'\|_1^- + 1\right)^{3/2} \left(K_6\|F\|_0^- + 1\right)^{1/2};$$

$$\begin{aligned}
&|F||G_3(F,F_-)| \\
&\quad \le C\|F\|_0^- |\xi-2i|^{-2-\tau} \left[\left(K_7\|F'\|_1^- + 1\right)^{3/2} \left(K_6\|F\|_0^- + 1\right)^{1/2} - 1\right];
\end{aligned}$$

Using the expression of L_1 in (2.42),

$$|F||L_1(\xi)| \le C|\xi-2i|^{-2-\tau}\|F\|_0^-; \qquad (2.61)$$

$$\epsilon^2 \left|\frac{\bar{H}'H}{\bar{H}} - H'\right| \le C\epsilon^2|\xi-2i|^{-3};$$

From (2.59) and Lemma 2.34, the proof follows.

Lemma 2.37. *Let $N \in \mathbf{A}_2^-$, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt \in \mathbf{A}^-, \text{ and } \|f_1\|_0^- \leq C \ \|N\|_2^-;$$

where C is a constant independent of ϵ and γ.

Proof. Case 1: $|\xi| \geq R$, where R is chosen large enough to ensure Property 2 for $P(\xi)$. On path $\mathcal{P}(\xi, -\infty) = \{t : t = \xi - se^{i\phi},\ 0 < s < \infty\}$, ϕ chosen to ensure $Arg(-\xi e^{-i\phi}) \in [-\frac{3\pi}{4} + \delta, \frac{\pi}{4} - \delta]$, $Re\ (P(t) - P(\xi))$ increases monotonically with s from 0 to $+\infty$.

$$|f_1(\xi)| = \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(\xi,-\infty)} N(t) L^{-1/4}(t) \exp\{\frac{1}{\epsilon}(P(\xi) - P(t))\} dt \right|$$
$$\leq \|N\|_2^- |L^{-1/4}(\xi)|$$
$$\times \int_0^1 \frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\left|\frac{d}{ds} ReP(t(s))\right|} d\left[\exp\{\frac{1}{\epsilon}(ReP(\xi) - ReP(t(s)))\}\right]$$

Note that $|L^{-1/4}(\xi)| \leq C|\xi - 2i|^{1/2}$, $|\xi - 2i| \leq |t - 2i|$ for t on the integral range,we have

$$|Re\,(P'(t)t'(s))\,| \geq C|L^{1/2}(t)| \geq C|t - 2i|^{-1}$$
$$\frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\left|\frac{d}{ds} ReP(t(s))\right|} \leq C|\xi - 2i|^{-\frac{1}{2}-\tau};$$

So $|f_1(\xi)| \leq C\|N\|_2^- |\xi - 2i|^{-\tau}$ and the lemma follows for this case.

Case 2: $\xi \in \mathcal{R}^- \cap \{|\xi| \leq R\}$. From Property 5, there exists a path $\mathcal{P}(\xi, -\infty)$ in $\mathcal{R}^-$ from $t = \xi$ to $t = -\infty$ on which Re $P(t)$ increases monotonically from 0 to $-\infty$ so that $Re\ [P'(t(s))t'(s)] \geq \frac{C}{|t-2i|}$. Also, for $|\xi| \leq R$, $|L^{-1/4}(\xi)| \leq C$, where constant C independent of ϵ and λ. Using the same steps as in Case 1, we estimate $|f_1(\xi)| \leq C\|N_2\|_2^-$. The lemma follows since $|\xi - 2i|$ is bounded in this case.

Lemma 2.38. *Let $N \in \mathbf{A}_2^-$; then there exists $\epsilon_0 > 0$ so that for $0 < \epsilon \leq \epsilon_0$*

$$f_2(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{\alpha i}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt \in \mathbf{A}^-, \ \text{satisfying } \|f_2\|_0^- \leq \ C \ \|N(\xi)\|_2^-.$$

where C is independent of ϵ and λ.

Proof. Case 1: $|\xi| \leq 4R^2$. From Property 4, there is path $\mathcal{P}(i\alpha, \xi)$ on which Re $P(t)$ is monotonically increasing as t goes from αi to ξ.

$$|f_2(\xi)| = \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(i\alpha,\xi)} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt \right|$$
$$\leq C\|N\|_2^- |L^{-1/4}(\xi)|$$
$$\times \int_{\mathcal{P}(i\alpha,\xi)} |(t - 2i)^{-2-\tau}||L^{-1/4}(t)| \frac{\left|d[\exp\{-\frac{1}{\epsilon}(ReP(\xi) - ReP(t))\}]\right|}{\frac{d}{ds} Re\ P(t(s))}$$

when $|\xi| \leq R^2, t \in \mathcal{P}(i\alpha, \xi)$, we also have:

$$|L^{-1/4}(\xi)| \leq C; \ \frac{d}{ds} Re P(t(s)) \geq C \ > \ 0;$$

where C is independent of ϵ . So

$$|f_2(\xi)| \leq C \|N\|_2^- \int_{\mathcal{P}(i\alpha,\xi)} d[\exp\{-\frac{1}{\epsilon} Re(P(\xi) - P(t))\}]$$
$$\leq C\|N\|_2^- [1 - \exp\{-\frac{1}{\epsilon} Re(P(\xi) - P(\alpha i))\}] \leq C\|N\|_2^-;$$

where we used that Re $(P(\xi) - P(\alpha i)) \geq 0$(see Property 4).
Case 2: $\xi \in \mathcal{R}^-, |\xi| \geq 4R^2$.
We choose a convenient point ξ_1, with $|\xi_1| = 2R$ so that the straightline $\mathcal{S}$ in the t-plane connecting ξ_1 to ξ so that $|t|$ is monotonically increasing and

$$Re(P(t) - P(\xi)) \leq C_1 \int_{|\xi|}^{|t|} \frac{1}{r} dr \leq C_1 \ln(\frac{|t|}{|\xi|}); \qquad (2.62)$$

where C_1 is independent of ϵ or λ. This is possible because of the asymptotic behavior of $P'(\xi) \sim e^{-i\pi/4}\xi^{-1}$ for large $|\xi|$ in $\mathcal{R}^-$, as can be deduced from (2.40). We then choose

$$\mathcal{P}(i\alpha, \xi) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$$

where

$$\mathcal{P}_1 = \mathcal{S} \cap \left\{ \sqrt{|\xi|} \ < \ |t| \ \leq \ |\xi| \right\}$$
$$\mathcal{P}_2 = \mathcal{S} \cap \left\{ 2R \ < \ |t| \ \leq \ \sqrt{|\xi|} \right\}$$
$$\mathcal{P}_3 = \mathcal{P}(i\alpha, \xi_1);$$

where $\mathcal{P}(i\alpha, \xi_1)$ is the path connecting $t = i\alpha$ to $t = \xi_1$, as ensured in Property 4. We now rewrite:

$$f_2(\xi) = \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_1} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt$$
$$+ \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_2} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt$$
$$+ \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(i\alpha,\xi_1)} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt; \qquad (2.63)$$

We know for $t \in \mathcal{P}_1 \cup \mathcal{P}_2$,

$$C_3|t| \leq |\gamma \pm it| \leq C_2|t|, C_3|t| \leq |1 \pm it| \leq C_2|t|;$$

where C_3,C_2 depend only on R. To estimate the first integral in (2.63):

$$\begin{aligned}
&\left|\frac{2}{\epsilon}L^{-1/4}(\xi)\int_{\mathcal{P}_1} N(t)L^{-1/4}(t)\exp\{-\frac{1}{\epsilon}(P(\xi)-P(t))\}dt\right| \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)| \\
&\times\int_{\sqrt{|\xi|}}^{|\xi|}|(t-2i)^{-2-\tau}||L^{-1/4}(t)|\exp\{-\frac{1}{\epsilon}(ReP(\xi)-ReP(t))\}dt \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)|\int_{\sqrt{|\xi|}}^{|\xi|}t^{-\frac{3}{2}-\tau}\exp\{-\frac{1}{\epsilon}(ReP(\xi)-ReP(t))\}dt \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)|\int_{\sqrt{|\xi|}}^{|\xi|}t^{-\frac{3}{2}-\tau}\exp\{\frac{C_1}{\epsilon}\ln(\frac{t}{|\xi|})\}dt \\
&\leq C\|N\|_2^-||\xi-2i|^{-\tau};
\end{aligned}$$

for small enough ϵ. To estimate the second integral in (2.63):

$$\begin{aligned}
&\left|\frac{2}{\epsilon}L^{-1/4}(\xi)\int_{\mathcal{P}_2} N(t)L^{-1/4}(t)\exp\{-\frac{1}{\epsilon}(P(\xi)-P(t))\}dt\right| \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)| \\
&\times\int_{|\xi_1|}^{\sqrt{|\xi|}}|(t-2i)^{-2-\tau}|L^{-1/4}(t)|\exp\{-\frac{1}{\epsilon}(ReP(\xi)-ReP(t))\}dt \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)|\int_{|\xi_1|}^{\sqrt{|\xi|}}\exp\{-\frac{1}{\epsilon}(ReP(\xi)-ReP(t))\}dt \\
&\leq C\frac{2}{\epsilon}\|N\|_2^-|L^{-1/4}(\xi)|\int_{|\xi_1|}^{\sqrt{|\xi|}}\exp\{\frac{C_1}{\epsilon}\ln(\frac{t}{|\xi|})\}dt \\
&\leq C\|N\|_2^-|L^{-1/4}(\xi)||\xi|^{-\frac{C_1}{2\epsilon}+\frac{1}{2}} \\
&\leq C\|N\|_2^-|\xi|^{-\tau} \text{ for } \epsilon \text{ small enough}
\end{aligned}$$

The third integral in (2.63) can be bounded by $C\|N\|_2^-|\xi|^{-\tau}$ as in case 1, as can be deduced from the large ξ behavior $e^{-Re\ P(\xi)} \sim |\xi|^{-\frac{C_1}{\epsilon}}$ for sufficiently small ϵ.

Definition 2.17. Define operator $\mathcal{U}$ so that for $N \in \mathbf{A}_2^-$,

$$\mathcal{U}N[\xi] := -\frac{1}{\epsilon^2}g_1(\xi)\int_{\alpha i}^{\xi}\frac{N(t)}{W(t)}g_2(t)dt + \frac{1}{\epsilon^2}g_2(\xi)\int_{-\infty}^{\xi}\frac{N(t)}{W(t)}g_1(t)dt; \quad (2.64)$$

By taking derivative of (2.64), with respect to ξ, it is convenient to define another operator $\mathcal{U}_1$ on $\mathbf{A_2}^-$:

Definition 2.18.

$$\mathcal{U}_1 N(\xi) := -\frac{1}{\epsilon^2} h_1(\xi) g_1(\xi) \int_{\alpha i}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} h_2(\xi) g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt; \tag{2.65}$$

where

$$h_1(\xi) = -\frac{L'(\xi)}{4L(\xi)} - i\frac{L^{1/2}(\xi)}{\epsilon}\ , \quad h_2(\xi) = -\frac{L'(\xi)}{4L(\xi)} + i\frac{L^{1/2}(\xi)}{\epsilon}; \tag{2.66}$$

Lemma 2.39.

$$\sup_{\mathcal{R}} |(\xi - 2i) h_k(\xi)| \leq C\left(\frac{1}{\epsilon} + 1\right); k = 1, 2 \tag{2.67}$$

where C is a constant independent of ϵ.

Proof. The lemma follows from equations (2.66) and (2.11).

Definition 2.19.

$$\hat{\beta} = \frac{h_2(i\alpha)\hat{F}(i\alpha) - \hat{F}'(i\alpha)}{[h_2(i\alpha) - h_1(i\alpha)] g_1(i\alpha)} \tag{2.68}$$

Lemma 2.40. *Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$, $\hat{F}$ defined by (2.34), then $\hat{\beta} g_1(i\alpha)$ is real and for small enough ϵ,*

$$\|\hat{\beta} g_1\|_0^- \leq C\epsilon^2 \quad ; \quad \|\hat{\beta} h_1 g_1\|_1^- \ \leq\ C\epsilon \tag{2.69}$$

where C is independent of ϵ.

Proof. From definition, it is clear that each of $h_1(i\alpha)$ and $h_2(i\alpha)$ are purely imaginary and $O(\epsilon^{-1})$ as $\epsilon \to 0^+$. Equation (2.68) and the bounds in Lemma 2.25 immediately implies $|\hat{\beta} g_1(i\alpha)| < C\epsilon^2$. Since $\hat{F}(\xi)$ satisfies symmetry condition (see remark 2.24), $\hat{F}(i\alpha)$ is real, while $\hat{F}'(i\alpha)$, $h_1(i\alpha)$ and $h_2(i\alpha)$ are purely imaginary; it follows that $\hat{\beta} g_1(i\alpha)$ is purely real. Since $Re\ P$ is a minimum at $\xi = i\alpha$, it is easily seen that both $|\xi - 2i|^\tau |g_1(\xi)/g_1(i\alpha)|$ and $\epsilon|\xi - 2i|^{1+\tau} |h_1(\xi) g_1(\xi)/g_1(i\alpha)|$ can be bounded by constants independent of ϵ, for small enough ϵ. The lemma follows.

Lemma 2.41. *Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$, $\hat{F}$, $\mathcal{V}\hat{F}(\xi)$ be defined as in (2.34) and (2.44), respectively. Then for $Im\ \xi > 0$,*

$$\hat{F}(\xi) = -\frac{1}{\epsilon^2} g_1(\xi) \int_i^{\xi} \frac{\mathcal{V}\hat{F}(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{\mathcal{V}\hat{F}(t)}{W(t)} g_1(t) dt, \tag{2.70}$$

Proof. Using integration by parts twice in the right hand side of (2.70), we get the lemma once we use $\mathcal{V} g_1 = \mathcal{V} g_2 = 0$ and the analyticity of $\hat{F}$ for $Im\ \xi > 0$.

Lemma 2.42. *If F is a solution of* **Problem 1**, *then F satisfies the following equation for $\xi \in \mathcal{R}^-$:*

$$F(\xi) = \hat{\beta} g_1(\xi) + \mathcal{U}\bar{N}_2(\xi) \tag{2.71}$$

where $\hat{\beta}$ is given by (2.68).

Proof. Since **Problem 1** is equivalent to the original finger problem, F is analytic in the upper half plane; hence $F = F_+$, $\bar{F} = F_-$, and therefore $\hat{F} = F$ (see remark 2.24). From Lemma 2.41,

$$\begin{aligned} F(\xi) &= -\frac{1}{\epsilon^2} g_1(\xi) \int_i^\xi \frac{\mathcal{V}\hat{F}}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^\xi \frac{\mathcal{V}\hat{F}}{W(t)} g_1(t) dt \\ &= \frac{1}{\epsilon^2} g_1(\xi) \int_{\alpha i}^{i} \frac{\mathcal{V}\hat{F}}{W(t)} g_2(t) dt + \mathcal{U}\left(\mathcal{V}\hat{F}\right)(\xi) \\ &= \hat{\beta} g_1(\xi) + \mathcal{U}\left(\mathcal{V}F\right)[\xi] \ = \ \hat{\beta} g_1(\xi) + \mathcal{U}\bar{N}_2(\xi) \end{aligned}$$

The last equality follows from integration by parts and using $\mathcal{V}g_2 = 0$, $W(t) = \frac{2i}{\epsilon}$, the identity $g_1 g_2 = \frac{2i}{\epsilon(h_2 - h_1)}$ and the definition of $\hat{\beta}$ in (2.68).

Definition 2.20. Problem 2: Find function $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$ so that F satisfies symmetry condition $Im\ F = 0$ on $\mathcal{R} \cap \{Re\ \xi = 0\}$ and equation (2.71) in $\mathcal{R}^-$.

Theorem 2.43. *Let $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$. If $F(\xi)$ satisfies the symmetry condition $Im\ F = 0$ on $\mathcal{R} \cap \{Re\ \xi = 0\}$ and the integral equation (2.71) in $\mathcal{R}^-$, then for sufficiently small ϵ, δ and δ_1, F is a solution to* **Problem 1** *(and hence a solution to the original Finger Problem).*

The proof of Theorem 2.43 will be given later after several lemmas.

Definition 2.21.

$$\beta_+ = \frac{h_2(i\alpha)F_+(i\alpha) - F_+'(i\alpha)}{[h_2(i\alpha) - h_1(i\alpha)]g_1(i\alpha)} \tag{2.72}$$

Lemma 2.44. *Assume $F \in \mathbf{A}^-_{0,\delta}$, $F' \in \mathbf{A}^-_{1,\delta_1}$ and F satisfies integral equation (2.71) in $\mathcal{R}^-$ as well as the symmetry condition $Im\ F(\xi) = 0$ for $\xi \in \{Re\ \xi = 0\} \cap \mathcal{R}$. Let $U(\xi) = F(\xi) - F_+(\xi)$, then*
(1) $U(\xi)$ is analytic in $\mathcal{R} \cup \{\ Im\ \xi < 0\}$.
(2) For $\xi \in \mathcal{R}^- \cap \{Im\ \xi\ >\ 0\}$, U satisfies:

$$U(\xi) = \mathcal{U}\bar{\mathcal{M}}(\xi) + (\hat{\beta} - \beta_+)g_1(\xi) \tag{2.73}$$

where $\bar{M}(\xi) = \bar{\mathcal{M}}(U, U')[\xi] \equiv \mathcal{M}(U, U')[\xi] + \epsilon^2 L_1(\xi) U(\xi)$;

$$\begin{aligned} M(U, U')(\xi) := &-\frac{\epsilon^2 i}{2\pi} \mathcal{G}_1(\xi) \int_{-\infty}^{\infty} \frac{\mathcal{G}_1^{-1}(t) U''(t) + \mathcal{G}_4(t) U'(t)}{t - \xi} dt \\ &- \epsilon^2 \mathcal{G}_4(\xi) \mathcal{G}_1(\xi) U'(\xi) + i\mathcal{G}_3(\xi) U(\xi), for\ Im\ \xi\ >\ 0; \end{aligned} \tag{2.74}$$

$\mathcal{G}_1(\xi) = G_1(F, F_-)[\xi]$, as given by (2.48), $\mathcal{G}_3(\xi) = G_3(F, F_-)[\xi]$, as given by (2.57) and $\mathcal{G}_4(\xi) = G_4(F, F_-, F_+)[\xi]$, where operator G_4 is defined by:

$$\begin{aligned} &G_4(F, F_-, F_+) \\ &= \frac{(F_+'' + H')\{(F' + H)^2 + (F_+' + H)^2 + (F' + H)(F_+' + H)\}}{(F_-' + \bar{H})^{1/2}(F' + H)^{3/2}(F_+' + H)^{3/2}\{(F' + H)^{3/2} + (F_+' + H)^{3/2}\}} \\ &+ \frac{(F_+'' + \bar{H}')}{(F_-' + \bar{H})^{3/2}(F' + H)^{1/2}(F_+' + H)^{1/2}\{(F' + H)^{1/2} + (F_+' + H)^{1/2}\}}. \end{aligned} \tag{2.75}$$

Proof. Since F satisfies (2.71) in $\mathcal{R}^-$, applying operator $\mathcal{V}$, it is clear that F satisfies equation (2.37) in $\mathcal{R}^-$. Symmetry condition and Schwarz reflection principle that relates F and its derivatives in $\mathcal{R}^+$ to their values in $\mathcal{R}^-$ guarantees that (2.37) is satisfied in $\mathcal{R}$. But equation (2.37) can be rewritten as:

$$F(\xi) = \epsilon^2 I_2(\xi) + \frac{\epsilon^2}{i} G(F, F_-)(\xi), \tag{2.76}$$

Then, on deforming the contour for I_2 from r_2 to r_1,

$$F(\xi) = \epsilon^2 I_3(\xi) + i\epsilon^2 \left[G(F_+, F_-)(\xi) - G(F, F_-)(\xi)\right], \text{ for } \xi \text{ above } r_1 \tag{2.77}$$

where

$$I_3(\xi) = -\frac{1}{2\pi}\int_{r_1} \frac{G(F_+, F_-)(t)}{t - \xi} dt, \text{for } \xi \text{ above } r_1 \ ; \tag{2.78}$$

It is clear that $I_3(\xi)$ is analytic above r_1; indeed from contour deformation of (2.31) and (2.78)and analyticity and decay properties of $G(F_+, F_-)$ on $\mathcal{R}$ itself, it is clear that $I_3 \in \mathbf{A}_0$ and analytic in $\mathcal{R} \cup \{Im\ \xi > 0\}$. Substituting for F from (2.77) into (2.31), it follows that for $Im\ \xi > 0$,

$$\begin{aligned} F_+(\xi) &= \epsilon^2 I_3(\xi) + \frac{\epsilon^2}{2\pi}\int_{r_1} \frac{[G(F_+, F_-)[t] - G(F, F_-)[t]]}{t - \xi} dt \\ &= -\frac{\epsilon^2}{2\pi}\int_{r_1} \frac{G(F, F_-)(t)}{t - \xi} dt = -\frac{\epsilon^2}{2\pi}\int_{-\infty}^{\infty} \frac{G(F, F_-)(t)}{t - \xi} dt \quad \text{for } Im\ \xi > 0; \end{aligned} \tag{2.79}$$

Subtracting (2.79) from (2.77), with integration contour deformed to the real axis, we obtain for $Im\ \xi > 0$:

$$\begin{aligned} U(\xi) \ &= \ i\epsilon^2 \left[G(F_+, F_-)(\xi) - G(F, F_-)(\xi)\right] \\ &\quad - \frac{\epsilon^2}{2\pi}\int_{-\infty}^{\infty} \frac{G(F_+, F_-)(t) - G(F, F_-)(t)}{t - \xi} dt \end{aligned} \tag{2.80}$$

It follows that

$$U(\xi) \ = \ -\frac{\epsilon^2}{2\pi}\int_{-\infty}^{\infty} \frac{G(F_+, F_-)(t) - G(F, F_-)(t)}{t - \xi} dt \quad \text{for } Im\ \xi \ < \ 0 \tag{2.81}$$

which is analytic everywhere for $Im\ \xi\ <\ 0$. However, since each of F and F_+ are analytic in $\mathcal{R}$, it follows that $U = F - F_+$ is also analytic in $\mathcal{R}$; hence statement (1) follows. Using (2.3), and the definition of G_1 and G_4 in (2.48) and (2.75),

$$G(F, F_-) - G(F_+, F_-) = G_1^{-1}(F, F_-)U'' + G_4(F, F_-, F_+)U'; \qquad (2.82)$$

From (2.82), (2.80) $U(\xi)$ satisfies the following homogeneous equation for $Im\ \xi\ >\ 0$:

$$\epsilon^2 \frac{d^2U}{d\xi^2} + L(\xi)U(\xi) = M(U, U')(\xi) \qquad (2.83)$$

Using $F_+ = F - U$ and (2.37), (2.83), $F_+(\xi)$ satisfies following equation for $Im\ \xi\ >\ 0$:

$$\mathcal{V}F_+ \ = \ \bar{N}_2 - \bar{M}(U, U') \qquad (2.84)$$

Using analyticity of F_+ at $\xi = i$, it follows from above equation that for $\xi \in \mathcal{R}^-$,

$$\begin{aligned} F_+(\xi) &= -\frac{1}{\epsilon^2} g_1(\xi) \int_i^\xi \frac{\mathcal{V}F_+(t)}{W(t)} g_2(t)dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^\xi \frac{\mathcal{V}F_+}{W(t)} g_1(t)dt \\ &= \beta_+ g_1(\xi) - \frac{1}{\epsilon^2} g_1(\xi) \int_{i\alpha}^\xi \frac{\mathcal{V}F_+(t)}{W(t)} g_2(t)dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^\xi \frac{\mathcal{V}F_+}{W(t)} g_1(t)dt \\ &= \beta_+ g_1(\xi) + \mathcal{U}(\bar{N}_2 - \bar{M})(\xi); \end{aligned} \qquad (2.85)$$

Subtracting (2.85) from (2.71), we can see that $U(\xi)$ satisfies equation (2.73) in $\mathcal{R}$ and in $\mathcal{R}^-$ in particular.

Definition 2.22. We define the upper boundary of $\mathcal{R}$ to be r_u, and the upper boundary of $\mathcal{R}^-$ to be r_u^-, *i.e.*

$$r_u = r_{u_1} \cup r_{u_2} \cup r_{u_3} \cup r_{u_4} \cup r_{u_5} \cup r_{u_6}$$

$$r_u^- = r_{u_1} \cup r_{u_2} \cup r_{u_3}$$

Remark 2.45. We will show that $U = 0$. Since F, F^+ and hence U is known to be analytic in $\mathcal{R}$ and continuous upto its boundary, it is enough to show that $U = 0$ on r_u^-. We will do so by showing that equation (2.73) combined with its first derivative forms a contraction map in the space of continuous functions (U, U') on r_u^-, with norm

$$\|(U, U')\|_{r_u^-} = \sup_{\xi \in r_u^-} |\xi - 2i|^\tau |U(\xi)| + \sup_{\xi \in r_u^-} |\xi - 2i|^{\tau+1} |U'(\xi)|$$

It is to be noted that the integration in the operator $\mathcal{U}$ is performed on the path r_u^-. Since on this path on any $\mathbf{C}^1$ segment, $\frac{d}{ds}\ Re\ P(t(s))\ >\ 0$ for arc length s increasing in the direction of $-\infty$, Lemmas 2.37 and 2.38 are valid and hence the operator $\mathcal{U}$ is bounded when restricted to functions on r_u^-. Further note that since U, U' are analytic in $\mathcal{R} \cup \{Im\ \xi\ <\ 0\}$ and satisfies symmetry condition, it follows that U, U' on r_u^- completely determines U and its derivatives for ξ real. This is crucial in controlling $\mathcal{M}$ on r_u^-, as necessary.

Lemma 2.46. *Let $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$, then for $\xi \in \mathcal{R}$*

$$|1/G_1(F,F_-)(\xi)| \le C|\xi - 2i|^2; \tag{2.86}$$

where C is independent of ϵ and λ.

Proof. By (2.7),(2.25) and lemma 2.33,we have

$$\begin{aligned}|G_1^{-1}(F,F_-)(\xi)| &\le \left(-\|F'\|_1|\xi-2i|^{-1-\tau} + H_m|\xi-2i|^{-1}\right)^{-3/2} \\ &\quad\times\left(-\|F'_-\|_1|\xi-2i|^{-1-\tau} + H_m|\xi-2i|^{-1}\right)^{-1/2} \\ &\le C\left(K_1 - \|F'\|_1\right)^{-3/2}\left(K_3 - \|F\|_0|\right)^{-1/2}|\xi-2i|^2 \le C|\xi-2i|^2;\end{aligned}$$

Lemma 2.47. *Let $F \in \mathbf{A}_{0,\delta}$,$F' \in \mathbf{A}_{1,\delta_1}$, then for $\xi \in \bar{\mathcal{R}} \cap \Omega_1 \cap \Omega_0$ (which includes r_u^- as well as the real ξ-axis), G_4 as defined in (2.72) satisfies:*

$$|G_4(F,F_-,F_+)(\xi)| \le C|\xi-2i|; \tag{2.87}$$

where C is independent of ϵ and λ.

Proof. From Lemma 2.23:

$$\begin{aligned}|F'_+ + H| &\ge \left(-|\xi-2i|^{-1-\tau}\|F'_+\|_1 + H_m|\xi-2i|^{-1}\right) \\ &\ge \left(-K_2\delta + K_1\right)|\xi-2i|^{-1};\end{aligned} \tag{2.88}$$

Using (2.50), (2.51), Lemmas 2.15 and 2.23,

$$\begin{aligned}|F''_+ + H'| &\le \left(|\xi-2i|^{-2-\tau}\|F''_+\|_2 + C_1|\xi-2i|^{-2}\right) \\ &\le \left(C_2\delta + C_1\right)|\xi-2i|^{-2}\end{aligned} \tag{2.89}$$

$$|F''_- + \bar{H}'| \le \left(|\xi-2i|^{-2-\tau}\|F''_-\|_2 + C_1|\xi-2i|^{-2}\right) \le \left(C_2\delta + C_1\right)|\xi-2i|^{-2}$$

the lemma follows from (2.75) (2.7),(2.25) and the above inequalities.

Lemma 2.48. *let $U(\xi)$ be as in Lemma 2.44,then*

$$\sup_{\xi\in\mathcal{D}}|\xi-2i|^{k+\tau}|U^{(k)}|(\xi)| \le C\|U\|_{0,r_u^-}\ ; \quad k=0,1,2. \tag{2.90}$$

when $\mathcal{D} = \mathcal{R} \cap \{Im\ \xi\ \le\ 0\}$.

Proof. Since U is analytic in $\{Im\,\xi\ <\ 0\} \cup \mathcal{R}$ (lemma 2.44) and continuous upto its boundary, and belongs to $\mathbf{A}_0$, by Cauchy integral formula formula:

$$U^{(k)}(\xi) = -\frac{k!}{2\pi i}\int_{r_u}\frac{U(t)}{(t-\xi)^{k+1}}dt; \text{for } \xi \text{ in } \mathcal{D}$$

Applying Lemma 2.12, with Γ chosen to be r_u, we complete the proof, after realizing from symmetry that $\|U\|_{0,r_u} = \|U\|_{0,r_u^-}$.

Lemma 2.49. *Let* $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$. *Let* $\bar{\mathcal{M}}(U,U')$ *be as defined in (2.74),then*

$$\bar{\mathcal{M}}(U,U') \in \mathbf{A}_{2,r_u^-}, \|\bar{\mathcal{M}}(U,U')\|_{2,r_u^-} \leq C\delta_2(\|U\|_{0,r_u^-} + \epsilon\|U'\|_{1,r_u^-}); \quad (2.91)$$

C is some constant independent of ϵ and

$$\delta_2 = \min\left\{\epsilon, (K_7\|F'\|_1 + 1)^{3/2}(K_6\|F\|_0 + 1)^{1/2} - 1\right\} \quad (2.92)$$

where K_6, K_7 are as in Lemma 2.32.

Proof. Using Lemma 2.46,Lemma 2.47 and Lemma 2.48:

$$|G_1^{-1}U''(t) + G_4U'(t)| \leq C|t-2i|^{-\tau}\|U\|_0, \text{ for } t \in (-\infty,\infty); \quad (2.93)$$

Applying Lemmas 2.12, 2.35 and Lemmas 2.46-2.48 to (2.74)

$$\begin{aligned}&\sup_{r_u^-}|\xi - 2i|^{2+\tau}|M(U,U')| \leq C_1\epsilon^2[\|U\|_{0,r_u^-} + \|U'\|_{1,r_u^-}]\\ &+ C_3\|U\|_{0,r_u^-}\left\{(K_7\|F'\|_1+1)^{3/2}(K_6\|F\|_0+1)^{1/2} - 1\right\} \quad (2.94)\\ &\leq C\delta_2\left(\|U\|_{0,r_u^-} + \epsilon\|U'\|_{1,r_u^-}\right);\end{aligned}$$

Lemma 2.50. *If $F \in \mathbf{A}_{0,\delta}^-$ and $F' \in \mathbf{A}_{1,\delta_1}^-$, and F satisfies the symmetry condition: $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$ then for small enough ϵ,*

$$\|(\hat{\beta} - \beta_+)g_1(\xi)\|_0^- \leq \epsilon^2K_4\|U\|_{0,r_u^-} \quad ; \quad \|(\hat{\beta} - \beta_+)h_1g_1(\xi)\|_1^- \leq \epsilon K_4\|U\|_{0,r_u^-}$$

for some K_4 independent of ϵ and λ.

Proof. From (2.68) and (2.72),

$$(\hat{\beta} - \beta_+)g_1(i\alpha) = \frac{h_2(i\alpha)[\hat{F}(i\alpha) - F_+(i\alpha)] - [\hat{F}'(i\alpha) - F'_+(i\alpha)]}{h_2(i\alpha) - h_1(i\alpha)}$$

Using (2.34) and (2.77), it follows that for any integer l,

$$\hat{F}^{(l)}(i\alpha) - F_+^{(l)}(i\alpha) = -\frac{\epsilon^2 l!}{2\pi}\int_{r_1}\frac{G(F_+,F_-)[t] - G(F,F_-)[t]}{(t-i\alpha)^{l+1}}dt \quad (2.95)$$

But

$$G(F_+,F_-) - G(F,F_-) = G_1^{-1}U'' + G_4U'$$

and from Lemmas 2.46-2.48,

$$\|G_1^{-1}U'' + G_4U'\|_{0,\mathbf{R}} \leq C\|U\|_{0,r_u^-}$$

where $\mathbf{R}$ is the real axis. Using the fact that $h_2(i\alpha)/(h_2(i\alpha) - h_1(i\alpha)) = O(1)$ and $1/(h_2(i\alpha) - h_1(i\alpha)) = O(\epsilon)$, it follows from applying 2.12 to (2.95) that

$$|(\hat{\beta} - \beta_+)g_1(i\alpha)| \leq \epsilon^2C\|U\|_{0,r_u^-}.$$

Noting that each of $(\xi - 2i)^\tau g_1(\xi)/g_1(i\alpha)$ and $\epsilon(\xi - 2i)^{\tau+1}h_1(\xi)g_1(\xi)/g_1(i\alpha)$ can be bounded by constants independent of ϵ and λ for small ϵ in the domain $\mathcal{R}^-$, the lemma follows.

Proof. of Theorem 2.43: Taking derivative in (2.73):

$$U'(\xi) = \mathcal{U}_1 M(U,U') + (\hat{\beta} - \beta_+) h_1 g_1; \tag{2.96}$$

Define the norm of (U,U') on r_{u^-} by:

$$\|(U,U')\|^- = \|U\|_{0,r_u^-} + \epsilon \|U'\|_{1,r_u^-} \tag{2.97}$$

From (2.73), (2.96), Lemmas 2.37, 2.38, 2.39, 2.49, 2.50, we obtain

$$\|(U,U')\|^- \leq C\|M(U,U')\|_{2,r_u^-} + \epsilon C\|(U,U')\|^- \ \leq \ C(\delta_2 + \epsilon)\|(U,U')\|^-; \tag{2.98}$$

where C is some constant independent of ϵ and δ_2. From (2.92), when $\epsilon, \|F\|_0, \|F'\|_1$ are small enough, $C(\delta_2 + \epsilon) < 1$ in (2.98). This implies $U(\xi) \equiv 0$ on r_u^- and hence everywhere by analytic continuation. Hence $F(\xi) = F_+(\xi) = \hat{F}(\xi)$ and $F(\xi)$ is analytic in the upper half plane. Thus for $\xi \in \{Im\ \xi < 0\} \cap \mathcal{R}^-$, $I_2(\xi) = I_1(\xi)$ and $N_2(\xi) = N_1(\xi)$ and equation (2.37) reduces to (2.28) in that region.

Theorem 2.51. *Problem 2 is equivalent to Problem 1 and hence to the original Finger problem.*

Proof. This follows from Lemma 2.42 and Theorem 2.43.

2.3. Formulation of the Weak problem

If $F \in \mathbf{A}_0^-$ and satisfies symmetry condition $Im\ F = 0$ on $\{Re\ \xi\ = 0\} \cap \mathcal{R}$, then Schwartz reflection principle applies and

$$F(\xi) = [F(-\xi^*)]^*, \text{for } \xi \in \mathcal{R}; \tag{2.99}$$

defines F in $\mathcal{R}^+$; consequently $F \in \mathbf{A}_0$ with $\|F\|_0 = \|F\|_0^-$ The reflection principle also implies

$$F'(\xi) = -[F'(-\xi^*)]^*, \text{for } \xi \in \mathcal{R}; \tag{2.100}$$

$$F''(\xi) = [F''(-\xi^*)]^*, \text{for } \xi \in \mathcal{R}; \tag{2.101}$$

For $F \in \mathbf{A}^-$, if we relax the symmetry condition $Im\ F = 0$ on $\{Re\ \xi\ = 0\} \cap \mathcal{R}$, then it is still possible to define F and its derivatives in $\mathcal{R}^+$, based on F in $\mathcal{R}^-$ using (2.99)-(2.101). However, this F in $\mathcal{R}^+$ is not the analytic continuation of F in $\mathcal{R}^-$ since violation of symmetry condition implies that extension of F in $\mathcal{R}^+$ is discontinuous across $\{Re\ \xi = 0\} \cup \mathcal{R}$. Nonetheless, this still allows us to define analytic functions F_- in Ω_0 through (2.16), and F_+ in Ω_1 through (2.32), $\hat{F}$ in $Im\ \xi\ >\ 0$ through (2.34), each of which have vanishing imaginary parts on the $Im\ \xi$ axis segment that are part of their domains of analyticity. Thus, I_2 is still defined as in (2.35) as an analytic function everywhere in $\mathcal{R}$. Also, the norms of these functions F_-, F_+, $\hat{F}$ and I_2 in their respective domains are completely controlled by $\|F\|_0^-$.

Definition 2.23.

$$\beta_w = -i \frac{Im \ \left(\mathcal{U}\bar{N}_2(i\alpha)\right)}{g_1(i\alpha)}; \tag{2.102}$$

Consider solution to:

$$F(\xi) = \hat{\beta} g_1(\xi) + \mathcal{U}\bar{N}_2(\xi) + \beta_w g_1(\xi); \tag{2.103}$$

Weak Problem: *Find function* $F \in \mathbf{A}_\delta^-$, $F' \in \mathbf{A}_{1,\delta_1}^-$ *that are analytic in* $\mathcal{R}^-$ *and continuous in its closure, and satisfies equation (2.103) in* $\mathcal{R}^-$.

Remark 2.52. If F is a solution of Weak Problem, then $Im \ F(\alpha i) = 0$. This follows from the fact that from Lemma 2.40 and equation (2.102), both $Im \ \left(\hat{\beta} g_1(i\alpha)\right) = 0$ and $Im \ \left(\mathcal{U}\bar{N}_2 + \beta_w g_1(i\alpha)\right) = 0$.

Theorem 2.53. *If* $F \in \mathbf{A}_\delta^-, F' \in \mathbf{A}_{\delta_1}^-$ *is a solution of Weak Problem, and* F *satisfies*

$$Im \ F = 0, \ \ on \ \{Re \ \xi = 0\} \cap \mathcal{R}; \tag{2.104}$$

then F *is a solution to* **Problem 2** *and therefore the original Finger problem. Conversely, any solution* F *to* **Problem 2** *(and therefore the original Finger problem) is also a solution to the weak problem.*

Proof. We claim first that when symmetry condition is invoked, $\beta_w = 0$. First, any solution to the weak problem clearly satisfies

$$\mathcal{V}F(\xi) = \bar{N}_2(\xi)$$

Then

$$\mathcal{U}\bar{N}_2(i\alpha) = \mathcal{U}\left(\mathcal{V}F\right)(i\alpha)$$

On integration by parts and using $\mathcal{V}g_1 = 0$ and the identities $g_1 g_2 = \frac{2i}{\epsilon(h_2-h_1)}$ and $W(t) = \frac{2i}{\epsilon}$, we obtain

$$\mathcal{U}\bar{N}_2(i\alpha) = \frac{F'(i\alpha) - h_1(i\alpha)F(i\alpha)}{h_2(i\alpha) - h_1(i\alpha)}$$

which is real since from symmetry condition, $F(i\alpha)$ is real and each of $F'(i\alpha)$ $h_1(i\alpha)$ and $h_2(i\alpha)$ purely imaginary. Therefore, the claim $\beta_w = 0$ is established when symmetry condition holds. Hence (2.71) and (2.103) become identical. Hence a solution to the Weak Problem that satisfies symmetry condition solves **Problem 2**.

Conversely, if F solves **Problem 2**, then symmetry condition implies $Im \ \left[\mathcal{U}\bar{N}_2(i\alpha)\right] = 0$. Hence equations (2.71 and and (2.102) become identical since $\beta_w = 0$. Hence, any solution to **Problem 2** solves the Weak problem as defined above.

3. Solution to Weak problem in $\mathcal{R}^-$

In this section, we are going to prove the existence and uniqueness of solution of Weak Problem by Contraction Theorem in suitable Banach space. We first need to prove some additional lemmas.

Lemma 3.1. *Let $\mathcal{D}, f_k, g_k$ be as in Lemma 2.5.$G_5(f,g)$ is defined by:*

$$G_5(f,g)(t) = (f' + H)^{-3/2}(g' + \bar{H})^{-3/2} \tag{3.1}$$

If $\|f_k'\|_{1,\mathcal{D}}, \|g_k'\|_{1,\mathcal{D}} < K_1/2$,with K_1 as in Lemma 2.5, then

$$|G_5(f_k, g_k)(t)| \leq C|t - 2i|^3; t \in \mathcal{D} \tag{3.2}$$

and

$$|G_5(f_1, g_1) - G_5(f_2, g_2)(t)| \leq C|t - 2i|^3 \left(\|f_1' - f_2'\|_{1,\mathcal{D}} + \|g_1' - g_2'\|_{1,\mathcal{D}}\right); \tag{3.3}$$

where C is independent of ϵ and λ.

Proof. (3.2) follows immediatly from (2.7) in Lemma 2.5. By straight algebra:

$$\begin{aligned} &G_5(f_1, g_1) - G_5(f_2, g_2)(t) = G_5(f_1, g_1)G_5(f_2, g_2) \\ &\times \left\{ \frac{(f_2' - f_1')(g_2' + \bar{H})^{3/2}[(f_2' + H)^2 + (f_2' + H)(f_1' + H) + (f_1' + H)^2]}{(f_2' + H)^{3/2} + (f_1' + H)^{3/2}} \right. \\ &\quad + \frac{(g_2' - g_1')(f_1' + H)^{3/2}[(g_2' + \bar{H})^2 + (g_2' + H)(g_1' + H) + (g_1' + H)^2]}{(g_2' + H)^{3/2} + (g_1' + H)^{3/2}} \end{aligned}$$

(3.3) follows from the above equation and (2.8) applied to f and g in domain $\mathcal{D}$ and using $\|f'\|_{1,\mathcal{D}}, \|g'\|_{1,\mathcal{D}} < K_1/2$.

Lemma 3.2. *Let $\mathcal{D}, f_k, g_k$ be as in Lemma 2.5. Let*

$$G_6(f,g) := f''(g' + \bar{H}) - g''(f' + H) + H'g' - f'\bar{H}; \tag{3.4}$$

If $\|f_k'\|_{1,\mathcal{D}}, \|g_k'\|_{1,\mathcal{D}}$ and $\|f_k''\|_{2,\mathcal{D}}, \|g_k''\|_{2,\mathcal{D}}$ are bounded by C which is independent of ϵ and λ, then for $t \in \mathcal{D}$

$$|G_6(f,g)(t)| \leq C|t - 2i|^{-3-\tau} \left(\|f''\|_{2,\mathcal{D}} + \|g''\|_{2,\mathcal{D}} + \|f'\|_{1,\mathcal{D}} + \|g'\|_{1,\mathcal{D}}\right); \tag{3.5}$$

$$\begin{aligned} |G_6(f_1, g_1) - G_6(f_2, g_2)| \leq C|t - 2i|^{-3-\tau} \Big\{ &\|f_1'' - f_2''\|_{2,\mathcal{D}} + \|g_1'' - g_2''\|_{2,\mathcal{D}} \\ &+ \|f_1' - f_2'\|_{1,\mathcal{D}} + \|g_1' - g_2'\|_{1,\mathcal{D}} \Big\}; t \in \mathcal{D} \end{aligned} \tag{3.6}$$

where C is independent of ϵ and λ.

Proof.

$$|G_6(f,g)(t)| \leq \|f''\|_2|t-2i|^{-2-\tau}\left(\|g'\|_1|t-2i|^{-1-\tau}+C|t-2i|^{-1}\right)$$
$$+\|g''\|_2|t-2i|^{-2-\tau}\left(\|f'\|_1|t-2i|^{-1-\tau}+C|t-2i|^{-1}\right)$$
$$+C\|g'\|_1|t-2i|^{-3-\tau}+C\|f'\|_1|t-2i|^{-3-\tau}$$
$$\leq C|t-2i|^{-3-\tau}\left(\|f''\|_{2,\mathcal{D}}+\|g''\|_{2,\mathcal{D}}+\|f'\|_{1,\mathcal{D}}+\|g'\|_{1,\mathcal{D}}\right);$$

(3.6) follows from the following inequality:

$$|G_6(f_1,g_1)(t)-G_6(f_2,g_2)(t)| \leq \Big\{|(f_1''-f_2'')g_1'|+|(g_1'-g_2')f_2''|+|(f_1''-f_2'')\bar{H}'|$$
$$+|(g_1'-g_2')H'|+|(f_1'-f_2')g_1''|+|(g_1''-g_2'')f_2'|$$
$$+|(g_1''-g_2'')H|+|(f_1'-f_2')\bar{H}'|\Big\}, t\in\mathcal{D};$$

Lemma 3.3. *Let $G(f,g)$ be defined by (2.3). If $\|f_k'\|_{1,\mathcal{D}}, \|g_k'\|_{1,\mathcal{D}} < K_1/2$,K_1 as in Lemma 2.5 and $\|f_k''\|_{2,\mathcal{D}}, \|g_k''\|_{2,\mathcal{D}}$ bounded by constant independent of ϵ and λ, then*

$$\|G(f_1,g_1)-G(f_2,g_2)\|_{0,\mathcal{D}}$$
$$\leq C\left(\|f_1'-f_2'\|_{1,\mathcal{D}}+\|g_1'-g_2'\|_{1,\mathcal{D}}+\|f_1''-f_2''\|_{2,\mathcal{D}}+\|g_1''-g_2''\|_{2,\mathcal{D}}\right); \quad (3.7)$$

where C is independent of ϵ and λ.

Proof. From (2.6),

$$G(f_1,g_1)-G(f_2,g_2)=(H'\bar{H}-H\bar{H}')\left(G_5(f_1,g_1)-G_5(f_2,g_2)\right)$$
$$+G_5(f_1,g_1)\left(G_6(f_1,g_1)-G_6(f_2,g_2)\right)+\left(G_5(f_1,g_1)-G_5(f_2,g_2)\right)G_6(f_2,g_2); \quad (3.8)$$

Using above identity and Lemma 3.1-Lemma 3.2,we have (3.7).

Lemma 3.4. *Let $F_k \in \mathbf{A}^-_{0,\delta}, F_k' \in \mathbf{A}^-_{1,\delta_1}, k=1,2$,then*

$$\|F^{(l)}_{+,1}-F^{(l)}_{+,2}\|_{l,\Omega_1} \leq C\|F_1-F_2\|^-_0; \quad (3.9)$$

where C is independent of ϵ and λ.

Proof. From (2.32), it follows that for ξ above r_1 :

$$F^{(l)}_{+,1}(\xi)-F^{(l)}_{+,2}(\xi)=\frac{l!}{2\pi i}\Big\{\int_{r_1^-}\frac{(F_1(t)-F_2(t))dt}{(t-\xi)^{l+1}}$$
$$+\int_{r_1^+}\frac{([F_1(-t^*)]^*-[F_2(-t^*)]^*dt)}{(t-\xi)^{l+1}}\Big\}$$

Applying lemma 2.12 with $\Gamma=r_1$ and $\mathcal{D}=\Omega_1$, we obtain for $l=0,1,2$,

$$\|F^{(l)}_{+,1}-F^{(l)}_{+,2}\|_{l,\Omega_1} \leq C\|F_1-F_2\|^-_0$$

Lemma 3.5. *Let* $F_k \in \mathbf{A}^-_{0,\delta}, F'_k \in \mathbf{A}^-_{1,\delta_1}, k = 1, 2$. *Then*

$$\|I_2(F_1) - I_2(F_2)\|_0 \le C \|F_1 - F_2\|^-_0; \tag{3.10}$$

where C *is independent of* ϵ *and* λ.

Proof. From (2.35),

$$I_2(F_1)[\xi] - I_2(F_2)[\xi] = -\frac{1}{2\pi} \int_{r_2} \frac{G(F_{+,1}, F_{-,1})(t) - G(F_{2,+}, F_{2,-})(t)}{t - \xi} dt;$$

Using Lemma 2.12 with $\Gamma = r_2$ and $\mathcal{D} = \mathcal{R}$, Lemmas 3.3,3.4, 2.15 and 2.23, we get

$$\begin{aligned} &\|I_2(F_1) - I_2(F_2)\|_0 \le C \|G(F_{+,1}, F_{-,1}) - G(F_{+,2}, F_{-,2})\|_{0,r_2} \\ &\le C \left(\|F''_{+,1} - F''_{+,2}\|_{2,r_2} + \|F''_{-,1} - F''_{-,2}\|_{2,r_2} \right) \\ &+ C \left(\|F'_{+,1} - F'_{+,2}\|_{1,r_2} + \|F'_{-,1} - F'_{-,2}\|_{1,r_2} \right) \\ &\le C \|F_1 - F_2\|^-_0; \end{aligned}$$

Lemma 3.6. *Let* $F_k \in \mathbf{A}^-_{0,\delta}, F'_k \in \mathbf{A}^-_{1,\delta_1}, k = 1, 2$. *Let* $G_1(F, F_-)$ *be defined as in (2.48) in lemma 2.33. Then for* $\xi \in \mathcal{R}^-$

$$|G_1(F_1, F_{-,1})(\xi) - G_1(F_2, F_{-,2})(\xi)| \le C |\xi - 2i|^{-2} \left(\|F_1 - F_2\|^-_0 + \|F'_1 - F'_2\|^-_1 \right); \tag{3.11}$$

where C *is independent of* ϵ *and* λ.

Proof. From straight forward algebra:

$$G_1(F_1, F_{-,1}) - G_1(F_2, F_{-,2}) = \frac{(F'_2 + H)^{3/2} (F'_{-,1} - F'_{-,2})}{(F'_{-,1} + \bar{H})^{1/2} + (F'_{-,2} + \bar{H})^{1/2}}$$

$$+ \frac{(F'_1 - F'_2)(F'_{-,1} + \bar{H})^{1/2} [(F'_1 + H)^2 + (F'_1 + H)(F'_2 + H) + (F'_2 + H)^2]}{(F'_1 + H)^{3/2} + (F'_2 + H)^{3/2}}$$

Applying (2.50), (2.51), (2.56) and Lemma 2.32 to above equation, we complete the proof.

Lemma 3.7. *Let* $F_k \in \mathbf{A}^-_{0,\delta}, F'_k \in \mathbf{A}^-_{1,\delta_1}, k = 1, 2$, $G_2(F, F_-)$ *be defined as in (2.52) in lemma 2.34. Then*

$$\|G_2(F_1, F_{-,1}) - G_2(F_2, F_{-,2})\|^-_2 \le C \left(\|F_1 - F_2\|^-_0 + \|F'_1 - F'_2\|^-_1 \right); \tag{3.12}$$

Proof. From Lemma 2.15 and (2.54):

$$|(F''_{-,1} - F''_{-,2})| \le C |\xi - 2i|^{-2-\tau} \|F_1 - F_2\|^-_0;$$

$$|(F''_- + \bar{H}')| \le \frac{C}{|\xi - 2i|^2} (1 + \|F\|^-_0);$$

By straight forward algebra:

$$G_2(F_1, F_{-,1}) - G_2(F_2, F_{-,2}) = (F''_{-,1} - F''_{-,2})\frac{F'_{-,1} + H}{F'_{-,1} + \bar{H}}$$
$$+ \frac{(F''_{-,2} + \bar{H}')}{(F'_{-,1} + \bar{H})}(F'_1 - F'_2) + \frac{(F''_{-,2} + \bar{H}')(F'_2 + H)}{(F'_{-,1} + \bar{H})(F'_{-,2} + \bar{H})}(F'_{-,2} - F'_{-,1});$$

Applying (2.50),(2.51),(2.56) and Lemma 2.32 to above equation, we obtain the lemma.

Lemma 3.8. *Let* $F_k \in \mathbf{A}^-_\delta, F'_k \in \mathbf{A}^-_{\delta_1}, k = 1, 2,$ *then*

$$\begin{aligned} &\|\bar{N}(F_1, I_2(F_1), F_{-,1}) - \bar{N}(F_2, I_2(F_2), F_{-,2}\|_2^- \\ &\quad \le C\Big\{ \left(\epsilon^2 + \|F_1\|_0^- + \|F_2\|_0^- + \|F'_1\|_1^-\right) \|F_1 - F_2\|_0^- \\ &\qquad + (\epsilon^2 + \|F_2\|_0^-)\|F'_1 - F'_2\|_1^- \Big\} \end{aligned} \tag{3.13}$$

Proof. From Definition 2.15 and equation (2.45):

$$\begin{aligned} &\bar{N}(F_1, I_2(F_1), F_{-,1}) - \bar{N}(F_2, I_2(F_2), F_{-,2}) \\ &= -i\epsilon^2 I_2(F_1)\left(G_1(F_1, F_{-,1}) - G_1(F_2, F_{-,2})\right) \\ &\ - i\epsilon^2 G_1(F_2, F_{-,2})\left(I_2(F_1) - I_2(F_2)\right) \\ &\ + i(F_1 - F_2)G_3(F_1, F_{-,1}) + iF_2\left(G_1(F_1, F_{-,1}) - G_1(F_2, F_{-,2})\right) \\ &+ \epsilon^2 L_1(\xi)(F_1 - F_2) + \epsilon^2\left(G_2(F_1, F_{-,1}) - G_2(F_2, F_{-,2})\right); \end{aligned} \tag{3.14}$$

Using the above identity, (2.42), Remark 2.30, Lemmas 2.33, 2.35, 3.5-3.7 and 2.28, we complete the proof.

Definition 3.1.

$$T(\xi) \equiv \bar{N}(0, I_2(0), 0)(\xi) \equiv \epsilon^2\left(\frac{\bar{H}'H}{\bar{H}} - H'\right)(\xi) - i\epsilon^2 H^{3/2}\bar{H}^{1/2} I_2(0)(\xi); \tag{3.15}$$

Lemma 3.9. $T \in \mathbf{A}^-_2$, *with* $\|T\|_2^- \le C\,\epsilon^2$, *where* C *is independent of* ϵ *and* λ.

Proof. The proof follows from (2.60) and Lemma 2.28 applied to (3.15).

Lemma 3.10. *Let* $F \in \mathbf{A}^-_{0,\delta}, F' \in \mathbf{A}^-_{1,\delta_1}$*, then*

$$\begin{aligned} &\|\bar{N}(F, I_2(F), F_-) - T\|_2^- \\ &\quad \le C\left((\epsilon^2 + \|F\|_0^- + \|F'\|_1^-)\|F\|_0^- + (\epsilon^2 + \|F\|_0^-)\|F'\|_1^-\right); \end{aligned} \tag{3.16}$$

Proof. In Lemma 3.8, Let $F_1 = F, F_2 = 0$,we get the lemma.

Lemma 3.11. *Let $F_k(\xi) \in \mathbf{A}^-, k = 1,2$, with functional $\hat{\beta}(F_k)$ defined as in (2.68), then*

$$\|\Big(\hat{\beta}(F_1) - \hat{\beta}(F_2)\Big) g_1\|_0^- \leq C\epsilon^2 \|F_1 - F_2\|_0^- \ ,$$
$$\|\Big(\hat{\beta}(F_1) - \hat{\beta}(F_2)\Big) h_1 g_1\|^- \leq C\epsilon \|F_1 - F_2\|_0^- .$$

Proof. From (2.68),

$$(\hat{\beta}(F_1) - \hat{\beta}(F_2)) g_1(i\alpha) = \frac{h_2(i\alpha)[\hat{F}_1(i\alpha) - \hat{F}_2(i\alpha)]}{h_2(i\alpha) - h_1(i\alpha)} + \frac{\hat{F}_1'(i\alpha) - \hat{F}_2'(i\alpha)}{h_2(i\alpha) - h_1(i\alpha)}$$

It follows that

$$|(\hat{\beta}(F_1) - \hat{\beta}(F_2)) g_1(i\alpha)| \leq C \Big[|\hat{F}_1(i\alpha) - \hat{F}_2(i\alpha)| + \epsilon |\hat{F}_1'(i\alpha) - \hat{F}_2'(i\alpha)|\Big] \tag{3.17}$$

where C is independent of ϵ. However, we know that for integer $l \geq 0$,

$$\hat{F}_1^{(l)}(i\alpha) - \hat{F}_2^{(l)}(i\alpha) = -\frac{\epsilon^2 l!}{2\pi} \int_{-\infty}^{\infty} \frac{G(F_{+,1}, F_{-,1}) - G(F_{+,2}, F_{-,2})}{(t - i\alpha)^{l+1}}$$

But, using Lemmas 3.3, 2.15 and 2.23, on the real axis $\mathbf{R}$,

$$\|G(F_{+,1}, F_{-,1})[t] - G(F_{+,2}, F_{-,2})[t]\|_{0,\mathbf{R}} \leq C \|F_1 - F_2\|_0^-$$

Therefore, using Lemma 2.12, we get

$$|(\hat{\beta}(F_1) - \hat{\beta}(F_2)) g_1(i\alpha)| \leq \epsilon^2 C \|F_1 - F_2\|_0^-$$

Since $(\xi - 2i)^\tau g_1(\xi)/g_1(i\alpha)$ and $\epsilon(\xi - 2i)^{1+\tau} h_1(\xi) g_1(\xi)/g_1(i\alpha)$ have upper bounds in $\mathcal{R}^-$ independent of ϵ for small enough ϵ, the lemma follows.

Lemma 3.12. *If $F_k \in \mathbf{A}^-, F_k' \in \mathbf{A}_{1,\delta_1}^-$ for $k = 1,2$, then*

$$\begin{aligned} \|\beta_w(F_1) g_1 - \beta_w(F_2) g_1\|_0^- \leq C\Big\{ & (\epsilon^2 + \|F_2\|_0^-) \|F_1' - F_2'\|_1^- \\ & + \left(\epsilon^2 + \|F_1\|_0^- + \|F_2\|_0^- + \|F_1'\|_1^-\right) \|F_1 - F_2\|_0^- \Big\} \end{aligned} \tag{3.18}$$

$$\begin{aligned} \|\beta_w(F_1) h_1 g_1 - \beta_w(F_2) h_1 g_1\|_1^- \leq \frac{C}{\epsilon}\Big\{ & (\epsilon^2 + \|F_2\|_0^-) \|F_1' - F_2'\|_1^- \\ & + \left(\epsilon^2 + \|F_1\|_0^- + \|F_2\|_0^- + \|F_1'\|_1^-\right) \|F_1 - F_2\|_0^- \Big\} \end{aligned} \tag{3.19}$$

where C is independent of ϵ and λ.

Proof. Using (2.102),

$$|\beta_w(F_1)g_1 - \beta_w(F_2)g_1| \le |\mathcal{U}\left(\bar{N}_2(F_1) - \bar{N}_2(F_2)\right)(\alpha i)| \quad |\frac{g_1(\xi)}{g_1(i\alpha)}|;$$

The Lemma follows from Lemma 2.36, Lemma 3.8 and Lemma 3.11, after we note that $|\frac{g_1(\xi)}{g_1(i\alpha)}|$ is bounded independent of ϵ and asymptotes to $|\xi|^{1/2-C_1/\epsilon}$ for large $|\xi|$, while $|h_1(\xi)| \le \frac{C}{\epsilon|\xi-2i|}$. Thus, for small enough ϵ, $|\frac{g_1(\xi)}{g_1(i\alpha)}|$ and $|\frac{\epsilon h_1 g_1(\xi)}{g_1(i\alpha)}|$ are $O(|\xi-2i|^{-\tau}$ and $O(|\xi-2i|^{-1-\tau}$ for all ξ. The proof immediately follows from definition of $\|.\|_0^-$ and $\|.\|_1^-$.

Lemma 3.13. *If $F \in \mathbf{A}_{0,\delta}, F' \in \mathbf{A}_{1,\delta_1}$ is a solution to the Weak Problem for small enough δ and δ_1 (though each a priori independent of ϵ), then $\|F\|_0^- \le C\epsilon^2, \|F'\|_1^- \le C\epsilon$, with C independent of ϵ.*

Proof. Since

$$F = (\hat{\beta}+\beta_w)g_1+\mathcal{U}\bar{N}(F, I_2(F), F_-), \;\; F' = (\hat{\beta}+\beta_w)h_1 g_1+\mathcal{U}_1\bar{N}(F, I_2(F), F_-); \tag{3.20}$$

Using Lemmas 2.37-2.38, Lemmas 3.11 and (3.9):

$$\begin{aligned}\|F\|_0^- &\le \|(\hat{\beta}+\beta_w)g_1\|_0^- + \|\mathcal{U}T\|_0^- + \|\mathcal{U}(N_2 - T)\|_0^- \\ &\le \|(\hat{\beta}+\beta_w)g_1\|_0^- + C\epsilon^2 + C(\delta+\delta_1)\|F\|_0^-\end{aligned}$$

and

$$\begin{aligned}\|F'\|_1 &\le \|(\hat{\beta}+\beta_w)h_1 g_1\|_1^- + \|\mathcal{U}_1 T\|_1^- + \|\mathcal{U}_1(N_2 - T)\|_1^- \\ &\le \|(\hat{\beta}+\beta_w)h_1 g_1\|_1^- + C\epsilon + \frac{C}{\epsilon}(\delta+\delta_1)\|F\|_0\end{aligned}$$

But from Lemma 2.40, $\|\hat{\beta}g_1\|_0^- \le C\epsilon^2$ and $\|\hat{\beta}h_1 g_1\|_1^- \le C\epsilon$. Again, from (2.102) and Lemmas 2.37-2.38 and Lemma 3.11,

$$\begin{aligned}\|\beta_w g_1\|_0^- &\le C\left(\|\mathcal{U}(\bar{N}_2 - T)\|_0^- + \|T\|_0^-\right)\|g_1^{-1}(\alpha i)g_1(\xi)\|_0^- \\ &\le C\epsilon^2 + C(\delta+\delta_1)\|F\|_0^-\end{aligned}$$

$$\begin{aligned}\|\beta_w h_1 g_1\|_1^- &\le C\left(\|\mathcal{U}(\bar{N}_2 - T)\|_0^- + \|T\|_0^-\right)\|g_1^{-1}(\alpha i)h_1(\xi)g_1(\xi)\|_1^- \\ &\le C\epsilon + \frac{C}{\epsilon}(\delta+\delta_1)\|F\|_0^-\end{aligned}$$

Thus,

$$\|F\|_0^- \le C\epsilon^2 + C(\delta+\delta_1)\|F\|_0^-; \tag{3.21}$$

So, for $C(\delta+\delta_1) \le \frac{1}{2}$, we obtain $\|F\|_0^- \le C\epsilon^2$. Using this in the inequality for F', we get $\|F\|_1^- \le C\epsilon$.

We define spaces:

Definition 3.2.

$$\mathbf{E} := \mathbf{A_0}^- \oplus \mathbf{A}_1^-$$

For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}$,

$$\|\mathbf{e}\|_{\mathbf{E}} := \|u(\xi)\|_0^- + \epsilon \|v(\xi)\|_1^-$$

It is easy to see that $\mathbf{E}$ is Banach space.

Definition 3.3.

$$\mathbf{B} := \{\mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E} : \|u(\xi)\|_0^- \le K\epsilon^2, \|v(\xi)\|_1^- \le K\epsilon\}. \tag{3.22}$$

Remark 3.14. For $(F, F') \in \mathbf{B}$, we have $F \in \mathbf{A}_{0,\delta}^-, F' \in \mathbf{A}_{1,\delta_1}^-$ with $\delta = O(\epsilon^2), \delta_1 = O(\epsilon)$; hence for sufficiently small ϵ, $\delta \le K_3/2, \delta_1 \le K_1/2$.

In (2.103), we replace $F(\xi)$ by $u(\xi)$,$F'(\xi)$ by $v(\xi)$ in the expression of $\bar{N}_2$. We show explicit dependence of $\bar{N}_2$ on u and v by using the notation $\bar{N}_2(u, v)(\xi)$ which is a departure from previous notation. Similarly, the dependence of functionals $\hat{\beta}$ and β_w on u, v is shown explicitly as $\hat{\beta}(u), \beta_w(u)$.

Definition 3.4. Let

$$\mathbf{O} : \mathbf{E} \longmapsto \mathbf{E}$$

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathbf{e}) = (\mathbf{O}_1(\mathbf{e}), \mathbf{O}_2(\mathbf{e}))$$

where

$$\mathbf{O}_1(\mathbf{e}) = \left(\hat{\beta}(u) + \beta_w(u)\right) g_1 + \mathcal{U}\bar{N}_2(u, v); \tag{3.23}$$

$$\mathbf{O}_2(\mathbf{e}) = \left(\hat{\beta}(u)) + \beta_w(u)\right) h_1 g_1 + \mathcal{U}_1\bar{N}_2(u, v); \tag{3.24}$$

Lemma 3.15. *For K suitably chosen but independent of ϵ, $(\mathcal{U}T(\xi), \mathcal{U}_1 T(\xi)) \in \mathbf{B}$.*

Proof. The lemma follows from (3.15), and the bounds for $|h_1|$ and $|h_2|$ in addition to Lemmas 2.38, 2.37 and 3.9, applied to equation (3.15) for K chosen large enough (but independent of ϵ).

Lemma 3.16. *Let $(u, v) \in \mathbf{B}$, then $(\hat{\beta}(u) g_1, \hat{\beta}(u) h_1 g_1) \in \mathbf{B}$. for small ϵ.*

Proof. The Lemma follows from Lemma 2.40.

Lemma 3.17. *If $(u, v) \in \mathbf{B}$, then $(\hat{\beta}(u) g_1, \hat{\beta}(u) h_1 g_1) \in \mathbf{B}$.*

Proof. The lemma follows from Lemma 3.11 with $F_1 = u$ and $F_2 = 0$ and from bounds on h_1.

Theorem 3.18. *For sufficiently small ϵ, the operator $\mathbf{O}$ is a contraction mapping from $\mathbf{B}$ to $\mathbf{B}$. Therefore, there exists unique solution (u, v) to the weak problem.*

Proof. For $(u, v) \in \mathbf{B}$, from lemmas 3.10 and 3.12, $\|\bar{N}_2(u, v) - T\|_2 \leq C\epsilon^3$. From Lemmas 2.37, 2.38, 3.15, 3.16, 3.9 and 3.17,

$$\begin{aligned} \|\mathbf{O}_1(\mathbf{e})\|_0 &\leq \|\hat{\beta} g_1\|_0 + \|\mathcal{U}T\|_0 + \|\mathcal{U}(\bar{N}_2 - T)\|_0 + \|\beta_w(u) g_1\|_0 \\ &\leq 3C\epsilon^2 + C\epsilon^3 \end{aligned} \tag{3.25}$$

From Lemmas 2.37, 2.38, 3.15, 3.16 and 2.39, in a similar manner,

$$\epsilon \|\mathbf{O}_2(\mathbf{e})\|_1 \leq 3C\epsilon^2 + C\epsilon^3 \tag{3.26}$$

Let $\mathbf{e}_k = (u_k, v_k) \in \mathbf{B}, k = 1, 2$. From Lemma 3.8 and equation (3.21), we have

$$\|\bar{N}_2(u_1, v_1) - \bar{N}_2(u_2, v_2)\|_2 \leq C\epsilon \|\mathbf{e}_1 - \mathbf{e}_2\|;$$

So, from Lemmas 2.37-2.38, 3.11 and 3.12, we get

$$\begin{aligned} \|\mathbf{O}_1(e_1) - \mathbf{O}_1(e_2)\|_0^- &\leq \|(\hat{\beta}(u_1) - \hat{\beta}(u_2)) g_1\|_0^- + \|\mathcal{U}\bar{N}_2(u_1, v_1) - \bar{N}_2(u_2, v_2)\|_0^- \\ &\quad + \|(\beta_w(u_1) - \beta_w(u_2)) g_1\|_0^- \\ &\leq C\epsilon^2 \|u_1 - u_2\|_0^- + C\|\bar{N}_2(u_1, v_1) - \bar{N}_2(u_2, v_2)\|_2^- \\ &\leq C\epsilon \|\mathbf{e}_1 - \mathbf{e}_2\|; \end{aligned} \tag{3.27}$$

$$\begin{aligned} \epsilon \|\mathbf{O}_2(e_1) - \mathbf{O}_2(e_2)\|_1 &\leq \|\epsilon(\hat{\beta}(u_1) - \hat{\beta}(u_2)) h_1 g_1\|_1^- \\ &\quad + \|\epsilon \mathcal{U}_1 \left(\bar{N}_2(u_1, v_1) - \bar{N}_2(u_2, v_2)\right)\|_1^- \\ &\quad + \|\epsilon(\beta_w(u_1) - \beta_w(u_2)) h_1 g_1\|_1^- \\ &\leq C \left(\epsilon^2 \|u_1 - u_2\|_0^- + \|\bar{N}_2(u_1, v_1) - \bar{N}_2(u_2, v_2)\|_2^-\right) \\ &\leq C\epsilon \|\mathbf{e}_1 - \mathbf{e}_2\|; \end{aligned} \tag{3.28}$$

Therefore, adding the two of the above equations, it follows that $\mathbf{O}$ is a contraction for small enough ϵ.

Therefore, the main result of this section is that for arbitrary λ in a compact subset of (0, 1), the Weak Problem has a solution $F \in \mathbf{A}_{0,\delta}^-$ and $F' \in \mathbf{A}_{1,\delta_1}^-$ for small enough δ and δ_1 and for all sufficiently small ϵ. From Theorem (2.53), this solution is a solution to the Finger problem if and only if the symmetry condition $Im\ F = 0$ is satisfied on $\{Re\ \xi = 0\} \cap \mathcal{R}$. In the following section, we investigate whether or not this symmetry condition is satisfied. We limit this part of the investigation to $\lambda \in [\frac{1}{2}, \lambda_m)$, where $\lambda_m - \frac{1}{2}$ is appropriately small, though independent of ϵ.

4. Selection of Finger Width: Analysis near $\xi = -i$

4.1. Derivation of Equation Near $\xi = -i$

In order to investigate whether or not the symmetry condition $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$ is satisfied, it is necessary to investigate a neighborhood of a turning point ($\xi = -i\gamma$ in our formulation), as first suggested from formal calculations of Combescot *et al* (1986). To that effect, we rewrite (2.37) from §2:

$$F(\xi) = \epsilon^2 I_2(\xi) + \frac{\epsilon^2}{i(F'(\xi)+H)^{1/2}(\hat{F}'(\xi)+\bar{H})^{1/2}}\left[\frac{F''(\xi)+H'}{F'(\xi)+H} - \frac{F''_{-}(\xi)+\bar{H}'}{F'_{+}(\xi)+\bar{H}}\right]; \quad (4.1)$$

We introduce

$$a = \frac{\gamma - 1}{\epsilon^{4/3}}; \quad (4.2)$$

$$\xi = -i + i2^{1/3}\epsilon^{4/3}y^2, \ \ G(y) = -i2^{-1/3}\left(i2^{1/3}y^2F'(\xi) - \frac{1}{2}(2^{1/3}y^2 + a)\right)^{-1/2}; \quad (4.3)$$

then (4.1) becomes:

$$y\frac{d}{dy}(y\frac{dG}{dy}) + G^{-2} = y^2 + \bar{\delta}_1 + \epsilon^{2/3}E_2(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1}) \quad (4.4)$$

where

$$\bar{\delta}_1 = (\frac{\bar{\delta}}{2})^{2/3} = \frac{a}{2^{1/3}}; \quad (4.5)$$

where $E_2(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1})$ is analytic function of $\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1}$.

Note that the leading order equation obtained from dropping ϵ terms in (4.4) is the same as equation (37) in Combescot et al (1988). In order to get the equation close to the normal form discussed of Costin (1998), it is convenient to introduce additional change in variables:

$$\eta = \frac{2\sqrt{2}}{3}y^{3/2}\ ,\ \psi(\eta) = 1 - yG(y); \quad (4.6)$$

then (4.4) becomes:

$$\frac{d^2\psi}{d\eta^2} - \frac{1}{3\eta}\frac{d\psi}{d\eta} - \psi = -\frac{4}{9\eta^2} - \frac{4}{9\eta^2}\psi + \frac{2^{2/3}a}{3^{4/3}\eta^{4/3}} - \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n+1)\psi^n + \epsilon^{2/3}E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3}), \quad (4.7)$$

where $E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3}$ with a series representation convergent for small values of each argument. It is to be noted

$$E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3}) = \sum_{m,n=0}^{\infty} E_{m,n}(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \eta^{-2/3})\psi^m \psi'^n$$

and since each of the arguments for $E_{m,n}$ can be safely be assumed to be in a compact set, it follows that there exits numbers A, ρ_2 are each independent of any parameter so that

$$|E_{m,n}| \ < \ A\rho_2^{m+n} \tag{4.8}$$

Theorem 4.1. *Let $F(\xi)$,$F'(\xi)$ be the solution of the Weak problem as in Theorem 3.18. After change of variables:*

$$\xi = -i + i\epsilon^{4/3}\frac{3^{4/3}}{2^{5/3}}\eta^{4/3}; \tag{4.9}$$

$$\psi(\eta, \epsilon, a) = \left(1 + \frac{2^{5/3}a}{3^{4/3}\eta^{4/3}} - 2iF'(\xi(\eta))\right)^{-1/2} - 1; \tag{4.10}$$

$\psi(\eta, \epsilon, a)$ satisfies equation (4.7) for $k_0\epsilon^{-1} \le |\eta| \le k_1\epsilon^{-1}, 0 \le \arg\eta < \frac{5\pi}{8}$ (where k_0 and k_1 are some constants independent of ϵ) and the asymptotic condition

$$\psi(\eta, \epsilon, a) \to 0, 0 \le \arg\eta < \frac{5\pi}{8}, \ \textit{as } \epsilon \to 0; \tag{4.11}$$

in that domain.

Proof. Since $\eta = O(\epsilon^{-1})$ in the given domain, using (4.9), we have $|\xi + i| = O(1)$ as $\epsilon \ \to \ 0$. Applying Theorem 3.18 and transformations (4.9) and (4.10), (2.37) implies (4.7). For $\xi \in \mathcal{R}^-, \frac{\pi}{2} < \arg(\xi + i) < \frac{4\pi}{3}$, which on using transformation (4.9) implies $0 < \arg\eta < \frac{5}{8}\pi$. Continuity implies that (4.7) is satisfied for $\arg\eta = 0$ as well. Since $F'(\xi) \sim O(\epsilon)$, and $\eta^{-4/3} \sim O(\epsilon^{4/3})$, using (4.10), we obtain (4.11).

4.2. Leading Inner problem analysis

Setting $\epsilon = 0$ in equation (4.7), we get the leading order equation:

$$\frac{d^2\psi}{d\eta^2} - \frac{1}{3\eta}\frac{d\psi}{d\eta} - \psi = -\frac{4}{9\eta^2} - \frac{4}{9\eta^2}\psi + \frac{2a}{3^{4/3}\eta^{4/3}} - \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n+1)\psi^n \tag{4.12}$$

with far-field matching condition:

$$\psi(\eta, a) \to 0, \text{as } |\eta| \to \infty, 0 \le \arg\eta < \frac{5\pi}{8}; \tag{4.13}$$

We shall prove the following theorem:

Theorem 4.2. *There exists large enough $\rho_0 > 0$ such that (4.12), (4.13) have a unique analytic solution $\psi_0(\eta, a)$ in the region $|\eta| \geq \rho_0, \arg\eta \in (-\frac{\pi}{8}, \frac{5\pi}{8})$.*

The proof of this Theorem will be given after some definitions and lemmas.

Definition 4.1. the region

$$\mathcal{R}_1 = \{\eta : |\eta| > \rho_0, \arg\eta \in (-\frac{\pi}{8}, \frac{5\pi}{8})\}$$

for some large ρ_0 independent of ϵ.

Definition 4.2.

$$\psi_1(\eta) = \eta^{1/6}e^{-\eta}, \psi_2(\eta) = \eta^{1/6}e^{\eta}; \tag{4.14}$$

$\psi_1(\eta), \psi_2(\eta)$ satisfy the following equation exactly:

$$\mathcal{L}\psi \equiv \frac{d^2\psi}{d\eta^2} - \frac{1}{3\eta}\frac{d\psi}{d\eta} - \left(1 - \frac{7}{36\eta^2}\right)\psi = 0; \tag{4.15}$$

Equation (4.12) can be rewritten as

$$\mathcal{L}\psi = \mathcal{N}_1(\eta, \psi) \equiv -\frac{4}{9\eta^2} - \frac{1}{4\eta^2}\psi + \frac{2^{2/3}a}{3^{4/3}\eta^{4/3}} - \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n+1)\psi^n; \tag{4.16}$$

We consider solution ψ of the following integral equation:

$$\psi = \mathcal{L}_1\psi \equiv \Bigg\{ -\psi_1(\eta)\int_{\infty e^{5\pi i/8}}^{\eta} \frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}_1(t, \psi(t))dt + \psi_2(\eta)\int_{\infty}^{\eta} \frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}_1(t, \psi(t))dt\Bigg\}; \tag{4.17}$$

Definition 4.3.

$$\mathbf{B}_1 = \Bigg\{\psi(\eta) : \psi(\eta)\text{analytic in } \mathcal{R}_1 \text{ and continuous on } \overline{\mathcal{R}_1}, \sup_{\mathcal{R}_1}|\eta^{4/3}\psi(\eta)| < \infty\Bigg\} \tag{4.18}$$

$\mathbf{B}_1$ is Banach space with norm

$$\|\psi\| = \sup_{\mathcal{R}_1}|\eta^{4/3}\psi(\eta)|; \tag{4.19}$$

Lemma 4.3. *Let $\mathcal{N} \in \mathbf{B}_1$,then*

$$\phi_1(\eta) := \psi_1(\eta)\int_{\infty e^{i5\pi/8}}^{\eta} \frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}dt \in \mathbf{B}_1, \phi_2(\eta) := \psi_2(\eta)\int_{\infty}^{\eta} \frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}dt \in \mathbf{B}_1;$$

and $\|\phi_1\| \leq K\|\mathcal{N}\|, \|\phi_2\| \leq K\|\mathcal{N}\|$; where K is independent of ρ_0.

Proof. For $\eta \in \mathcal{R}_1$, we use straight lines in the t-plane to connect η to $+\infty$ (or to $\infty e^{i5\pi/8}$) so $Re\ t$ is increasing monotonically from η to $+\infty$ (or from $\infty e^{i5\pi/8}$ to η) and on that path, characterized by arc-length s, $\frac{d}{ds}\ Re\ t(s) > C > 0$. Further, $C_1|\eta| \le |t|$ for nonzero C_1. Then,

$$\begin{aligned}|\phi_2(\eta)| &= \left|\eta^{1/6}\int_\infty^\eta \frac{e^{\eta-t}}{2t^{1/3}}\mathcal{N}dt\right| \\ &\le C|\eta|^{1/6}\int_\infty^\eta |t|^{-4/3-1/6}|e^{\eta-t}||t^{4/3}\mathcal{N}||dt| \\ &\le \frac{C\|\mathcal{N}\|}{|\eta|^{4/3}}\int_\infty^\eta |e^{\eta-t}||dt| \le \frac{C\|\mathcal{N}\|}{|\eta|^{4/3}}\end{aligned}$$

Very similar steps can be used to prove the bound for $\phi_1(\eta)$.

Definition 4.4. Define $T_1(\eta)$ so that $T_1(\eta) := \mathcal{L}_1 0$.

Remark 4.4. Since $|\eta^{4/3}\mathcal{N}_1(\eta, 0)|$ is bounded, Lemma 4.3 implies $T_1 \in \mathbf{B}_1$.

Definition 4.5. $\sigma_1 = \|T_1\|$; $\quad \mathbf{B}_{\sigma_1} := \{\psi \in \mathbf{B}_1 : \|\psi\| \le 2\sigma_1\}$

Lemma 4.5. *If $\psi \in \mathbf{B}_{\sigma_1}, \phi \in \mathbf{B}_{\sigma_1}$, then $\mathcal{N}_1(\eta,\psi) \in \mathbf{B}_1$ and*

$$\begin{aligned}\|\mathcal{N}_1(\eta,\psi)\| &\le 2K_1\sigma_1\left[\rho_0^{-4/3}\sigma_1 + \rho_0^{-2}\right] + \left[\frac{4}{9\rho_0^2} + \frac{3^{2/3}a}{3^{4/3}}\right] \\ \|\mathcal{N}_1(\eta,\psi) - \mathcal{N}_1(\eta,\phi)\| &\le K_1\left[\rho_0^{-4/3}\sigma_1 + \rho_0^{-2}\right](\|\phi-\psi\|)\end{aligned}$$

for some numerical constant K_1 and for $8\sigma_1\rho_0^{-4/3} < 1$.

Proof. It is clear from (4.16) that

$$|\mathcal{N}_1(\eta,\psi) - \mathcal{N}_1(\eta,\phi)| \le \frac{|\psi(\eta)-\phi(\eta)|}{4\|\eta\|^2} + \frac{1}{2}\sum_{k=2}^{\infty}(k+1)|\psi^k - \phi^k| \qquad (4.20)$$

Noting, $|\psi| \le 2\sigma_1|\eta|^{-4/3}, |\phi| \le 2\sigma_1|\eta|^{-4/3}$ and from simple induction,

$$|\psi^k - \phi^k| \le k(|\psi|+|\phi|)^{k-1}|\psi-\phi|, \text{ for } k \ge 1,$$

we obtain for $\sigma_1\rho_0^{-4/3} < \frac{1}{8}$,

$$\|\mathcal{N}_1(\eta,\psi) - \mathcal{N}_1(\eta,\phi)\| < \left[\frac{1}{4\rho_0^2} + \frac{K_2\sigma_1}{\rho_0^{4/3}}\right]\|\psi-\phi\| \qquad (4.21)$$

for some numerical constant K_2. On the other hand,

$$\|\mathcal{N}_1(\eta,0)\| \le \frac{4}{9\rho_0^{2/3}} + \frac{3^{2/3}a}{3^{4/3}}$$

So, it is clear from adding the two results above (with $\phi = 0$), it follows that that for $\psi \in \mathbf{B}_{\sigma_1}$,

$$\|\mathcal{N}_1(\eta,\psi)\| < \left[\frac{4}{9\rho_0^{2/3}} + \frac{3^{2/3}a}{3^{4/3}}\right] + \sigma_1\left[\frac{1}{4\rho_0^2} + \frac{K_2\sigma_1}{\rho_0^{4/3}}\right]$$

Lemma 4.6. *For sufficiently large ρ_0, the operator $\mathcal{L}_1$ as defined in (4.17) is a contraction from $\mathbf{B}_{\sigma_1}$ to $\mathbf{B}_{\sigma_1}$; hence there is a unique solution ψ in this function space.*

Proof. From Lemma 4.3 and Lemma 4.5:

$$\begin{aligned}\|\mathcal{L}_1\psi - T_1\| \le \|\mathcal{L}_1\psi - \mathcal{L}_1 0\| &\le 2K\|\mathcal{N}_1(\eta,\psi) - \mathcal{N}_1(\eta,0)\| \\ &\le 2KK_1\left[\rho_0^{-4/3}\sigma_1 + \rho^{-2}\right]\|\psi\|\end{aligned}$$

$$\|\mathcal{L}_1\psi - \mathcal{L}_1\phi\| \le 2K\|\mathcal{N}_1(\eta,\psi) - \mathcal{N}_1(\eta,\phi)\| \le 2KK_1[\rho_0^{-4/3}\sigma_1 + \rho^{-2}]\|\psi - \phi\|;$$

So,

$$\|\mathcal{L}_1\psi\| \le \|\mathcal{L}_1\psi - \mathcal{L}_1 0\| + \|T_1\| \le 2\sigma_1$$

for sufficiently large ρ_0.

Remark 4.7. It is easy to see that the previous lemma holds when we change the restriction on *arg* η in the definition of $\mathcal{R}_1$ to $(0,\ \frac{5\pi}{8})$. This comment is relevant to the following lemma.

Lemma 4.8. *Any solution ψ to (4.12) satisfying condition (4.13) in the domain $\mathcal{R}_1$ must be in $\mathbf{B}_{\sigma_1}$ and satisfy integral equation: $\psi = \mathcal{L}_1\psi$ for sufficiently large ρ_0*

Proof. First, we note that if we use variation of parameter, the most general solution to (4.11) satisfies the integral equation

$$\psi = \mathcal{L}_1\psi + C_1\psi_1 + C_2\psi_2$$

Now, if we assume $\|\psi\|_\infty$ to be small, as implied by condition (4.13), when ρ_0 is chosen large, it follows from inspection of the the right hand side of (4.12) that $\|\mathcal{N}_1(\eta,\psi)\|_\infty$ is also small. Since Lemma 4.3 is easily seen to hold when the norm is replaced by $\|.\|_\infty$, it follows that $\mathcal{L}_1\psi$ is also small. However, $C_1\psi_1(\eta) + C_2\psi_2(\eta)$ is unbounded in $\mathcal{R}_1$ unless $(C_1, C_2) = (0,\ 0)$. Therefore, any solution to (4.12) satisfying condition (4.13) must satisfy integral equation $\psi = \mathcal{L}_1\psi$. If we were to use the norm $\|.\|_\infty$ instead of the the weighted norm $\|.\|$ in the definition of the Banach Space $\mathbf{B}_1$, it is easily seen that each of Lemmas 4.3-4.6 would remain valid for small enough σ_1, as appropriate when condition (4.13) holds and ρ_0 is large. Thus, it can be concluded that the solution to $\psi = \mathcal{L}_1\psi$ is unique in the bigger space of functions for which ψ satisfies (4.13) and ρ_0 is chosen large enough. However, from previous Lemma 4.6, it follows that this unique solution must be in the function space $\mathbf{B}_1$ and therefore satisfies $\psi = O(\eta^{-4/3})$ for large η.

Proof of Theorem 4.2 follows immediately from Lemmas 4.6 and 4.8

Theorem 4.9. *If $\psi_0(\eta, a)$ is the solution in Theorem 4.2, then*
Im $\psi_0(\eta, a) = S(a)\eta^{1/6}e^{-\eta}\,(1 + o(1))$ on the real η axis and $\eta \to \infty$.

Proof. Plugging $\psi_0 = Re\ \psi_0 + iIm\ \psi_0$ in equation (4.12), then taking imaginary part,then Im $\psi_0(\eta)$ satisfies the following linear homogeneous equation on real positive η axis:

$$\frac{d^2\ Im\ \psi_0}{d\eta^2} - \frac{1}{3\eta}\frac{d\ Im\ \psi_0}{d\eta} - (1 - \frac{4}{9\eta^2} + E(\eta))Im\ \psi_0 = 0; \tag{4.22}$$

where $E(\eta)$ is obtained from an homogeneous expression of $Re\ \psi_0$ and $Im\ \psi_0$. Since *apriori* both $Re\ \psi_0 \sim O(\eta^{-4/3})$ and $Im\ \psi_0 \sim O(\eta^{-4/3})$ as $\eta \to \infty$, we obtain

$$E(\eta) \sim O(\eta^{-4/3}),\ \text{as}\ \eta \to \infty; \tag{4.23}$$

From Theorem 6.2.1 in Olver (1974),there are two independent solutions of linear equation (4.22) $\tilde{\phi}_1, \tilde{\phi}_2$ so that $\tilde{\phi}_1(\eta) = \eta^{1/6}e^{\eta}(1 + o(1)), \tilde{\phi}_2(\eta) = \eta^{1/6}e^{-\eta}(1 + o(1))$, hence there are constants $S_1(a)$ and $S(a)$ so that $Im\ \psi_0(\eta) = S_1(a)\tilde{\phi}_1(\eta) + S(a)\tilde{\phi}_2(\eta)$, the condition $\psi_0 = O(\eta^{-4/3})$ for large η implies $S_1 = 0$; so $Im\ \psi_0(\eta) = S(a)\tilde{\phi}_2(\eta)$.

Remark 4.10. From Theorem 4.9, $Im\ \psi_0(\eta, a) = 0$ iff $S(a) = 0$. Previous numerical results and formal asymptotic results (Combescot et al 1986,1988, Tanveer 1987, Dorsey & Martin 1987 and Chapman 1999) suggest that $S(a) = 0$ if and only if a takes on a discrete set of values. In order of their increasing values, the first few were determined to be $\{1.0278, 3.7168, 7.0934, ...\}$ to within the decimal precision quoted.

4.3. Full Inner Problem Analysis

Now we go back to the full inner equation (4.7). From Theorem 4.1, (4.7) with matching condition (4.11) has unique solution in the domain $k_1\epsilon^{-1} \geq |\eta| \geq k_0\epsilon^{-1}, \arg\eta \in (0, \frac{5\pi}{8})$. We shall first prove that this solution can be extended to the region:
$\mathcal{R}_2 = \{\eta : \rho_0 < Im\ \eta + Re\ \eta < \tilde{k}_0\epsilon^{-1}, \arg\eta \in [0, \frac{5\pi}{8}); -Im\ \eta + \rho_0 < Re\ \eta < Im\ \eta + \tilde{k}_0\epsilon^{-1}, \arg\eta \in (-\frac{\pi}{8}, 0]\}$, where $k_0 < \tilde{k}_0 < k_1$.

Definition 4.6. Let $\psi = \tilde{\psi}(\eta)$ be the unique analytic solution in Theorem 4.1 for $|\eta| \geq k_0\epsilon^{-1}, \arg\eta \in (0, \frac{5\pi}{8})$, restricted to the line segment $\{\eta : Im\ \eta + Re\ \eta = \tilde{k}_0\epsilon^{-1}, \arg\eta \in [0, \frac{5\pi}{8})\}$.

Definition 4.7.

$$\eta_0 = \tilde{k}_0\epsilon^{-1},\quad \eta_1 = \tilde{k}_0\epsilon^{-1}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}}e^{\frac{5i\pi}{8}},\quad \eta_2 = i\tilde{k}_0\epsilon^{-1}; \tag{4.24}$$

Lemma 4.11. *Each of $\tilde{\psi}(\eta_0)$ and $\tilde{\psi}(\eta_1)$ are $O(\epsilon)$, while $\tilde{\psi}'(\eta_0)$ and $\tilde{\psi}'(\eta_1)$ are each $\epsilon^{4/3}$*

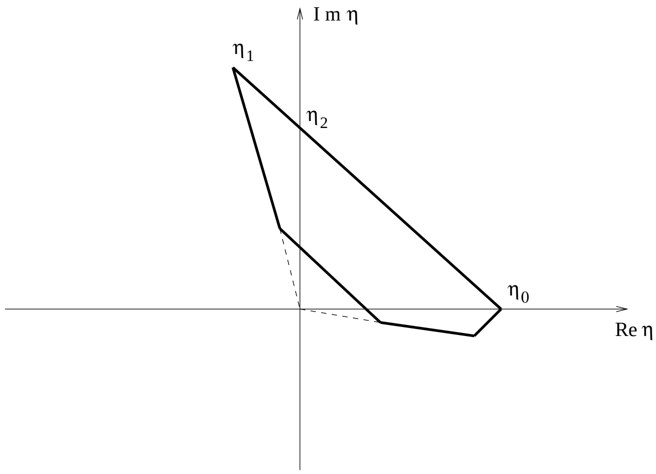

Fig. 4.1. Region $\mathcal{R}_2$.

Proof. Note from Theorem 3.18, that for $|\xi + i\gamma| = O(1)$ in domain $\mathcal{R}^-$, $F' = O(\epsilon)$. Noting transformation (4.9) and (4.10), solution $\psi = O(\eta^{-4/3}, \epsilon)$. So, $\tilde{\psi}(\eta^0)$ and $\tilde{\psi}(\eta_1)$ are each $O(\epsilon)$. Also, from (4.1), it is clear that when $F = O(\epsilon^2)$, $F'(\xi) = O(\epsilon)$, then $F''(\xi) = O(1)$. Also, from transformation (4.9) and (4.10), it is clear that $\psi'(\eta) = O(\epsilon^{4/3})$ in such cases; so $\tilde{\psi}'(\eta_0)$ and $\tilde{\psi}'(\eta_1)$ are each $O(\epsilon^{4/3})$.

Equation (4.7) can be rewritten as

$$\begin{aligned}\mathcal{L}\psi &= \mathcal{N}_2(\eta, \epsilon, \psi, \psi') \\ &\equiv -\frac{4}{9\eta^2} - \frac{1}{4\eta^2}\psi + \frac{2^{2/3}a}{3^{4/3}\eta^{4/3}} \\ &- \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n+1)\psi^n + \epsilon^{2/3}E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3});\end{aligned} \tag{4.25}$$

We consider solution ψ of the following integral equation:

$$\begin{aligned}\psi = \mathcal{L}_2\psi \equiv -\psi_1(\eta)\int_{\eta_1}^{\eta}\frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}_2(t,\psi)dt + \psi_2(\eta)\int_{\eta_0}^{\eta}\frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}_2(t,\psi)dt \\ + \ a_1\psi_1(\eta) + a_2\psi_2(\eta)\end{aligned} \tag{4.26}$$

where

$$a_1 = -\frac{\left(\psi_2(\eta_1)\tilde{\psi}'(\eta_1) - \psi_2'(\eta_1)\tilde{\psi}(\eta_1)\right)}{2\eta_1^{1/3}} \tag{4.27}$$

$$a_2 = \frac{\left(\psi_1(\eta_0)\tilde{\psi}'(\eta_0) - \psi_1'(\eta_0)\tilde{\psi}(\eta_0)\right)}{2\eta_0^{1/3}}; \tag{4.28}$$

On taking the derivative of $\mathcal{L}_2$ we obtain

$$\psi' = \mathcal{L}_2'\psi \equiv -\psi_1'(\eta)\int_{\eta_1}^{\eta} \frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}_2(t,\psi)dt + \psi_2'(\eta)\int_{\eta_0}^{\eta} \frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}_2(t,\psi)dt + a_1\psi_1'(\eta) + a_2\psi_2'(\eta) \tag{4.29}$$

Lemma 4.12. *For a_1, a_2 as defined, and $\eta \in \mathcal{R}_2$,*

$$|a_1\psi_1(\eta) + a_2\psi_2(\eta)| \; < \; K\epsilon$$
$$|a_1\psi_1'(\eta) + a_2\psi_2'(\eta)| \; < \; K\epsilon$$

Proof. First, it is clear from the expressions for ψ_1 and ψ_2 that each of $\psi_1(\eta)\psi_2(\eta_1)\eta_1^{-1/3}$, $\psi_1(\eta)\psi_2'(\eta_1)\eta_1^{-1/3}$, $\psi_2(\eta)\psi_1(\eta_0)\eta_0^{-1/3}$ and $\psi_2(\eta)\psi_1(\eta_0)\eta_0^{-1/3}$ and their first derivatives with respect to η are bounded for $\eta \in \mathcal{R}_2$ by a numerical constant independent of any parameter. Hence using $\tilde{\psi}(\eta_0), \tilde{\psi}(\eta_1) = O(\epsilon)$ and $\tilde{\psi}'(\eta_0), \tilde{\psi}'(\eta_1) = O(\epsilon^{4/3})$ from Lemma 4.11, we obtain the proof from the expressions for a_1 and a_2.

Definition 4.8.

$$\mathbf{B}_2 = \{\psi(\eta) : \psi(\eta) \text{ analytic in } \mathcal{R}_2, \text{ and continuous on } \overline{\mathcal{R}}_2.\} \tag{4.30}$$

$\mathbf{B}_2$ is Banach space with norm

$$\|\psi\| = \sup_{\overline{\mathcal{R}_2}} \rho_1|\psi(\eta)|, \text{ where } \rho_1 = \rho_0^{4/3}; \tag{4.31}$$

Lemma 4.13. *Let $\mathcal{N} \in \mathbf{B}_2$. Then*

$$\phi_1(\eta) := \psi_1(\eta)\int_{\eta_1}^{\eta} \frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}(t)dt \in \mathbf{B}_2, \;\; \phi_2(\eta) := \psi_2(\eta)\int_{\eta_0}^{\eta} \frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}(t)dt \in \mathbf{B}_2;$$

$$\phi_3(\eta) := \psi_1'(\eta)\int_{\eta_1}^{\eta} \frac{\psi_2(t)}{2t^{1/3}}\mathcal{N}(t)dt \in \mathbf{B}_2, \;\; \phi_4(\eta) := \psi_2'(\eta)\int_{\eta_0}^{\eta} \frac{\psi_1(t)}{2t^{1/3}}\mathcal{N}(t)dt \in \mathbf{B}_2;$$

and each of $\|\phi_1\|$, $\|\phi_2\|$, $\|\phi_3\|$ and $\|\phi_4\| \le K\|\mathcal{N}\|$, where K is a numerical constant independent of any parameters.

Proof. For $\eta \in \mathcal{R}_2$, we use straight lines to connect η to η_0(or η_1) so that Re t is increasing from η to η_0(from η_1 to η) and $C_1|\eta| \le |t| \le C_2|\eta|$.

$$\begin{aligned} |\phi_2(\eta)| &= \left|\eta^{1/6}\int_{\eta_0}^{\eta} \frac{e^{\eta-t}}{2t^{1/6}}\mathcal{N}dt\right| \\ &\le \frac{K}{\rho_0^{4/3}}\int_{\eta_0}^{\eta} |e^{\eta-t}||\rho_0^{4/3}\mathcal{N}||dt| \\ &\le \frac{K\|\mathcal{N}\|}{\rho_0^{4/3}}\int_{\eta_0}^{\eta} |e^{\eta-t}||dt| \le \frac{K\|\mathcal{N}\|}{\rho_0^{4/3}} \end{aligned}$$

The same steps work for $\phi_1(\eta)$. Since $\frac{\psi_1'}{\psi_1}$ and $\frac{\psi_2'}{\psi_2}$ are bounded in the domain $\mathcal{R}_2$ by numerical constant, independent of any parameters, it is easily seen that the same conclusions must be valid for $\phi_3(\eta)$ and $\phi_4(\eta)$

Definition 4.9. $T_2(\eta) := \mathcal{L}_2 0, \quad \sigma_2 \equiv Max\{\|T_2\|, \|T_2'\|\}$

Remark 4.14. Since $\mathcal{N}_2(\eta, \epsilon, 0, 0) = O(\eta^{-4/3}, \epsilon^{2/3})$ in $\mathcal{R}_2$, it follows from Lemma 4.13 and 4.12 that $\sigma_2 = O(1)$, as $\epsilon \to 0^+$.

We define space

$$\mathbf{B}_{\sigma_2} = \{\psi \in \mathbf{B}_2 : \|\psi\| \leq 2\sigma_2\}$$

Lemma 4.15. *If $\psi \in \mathbf{B}_{\sigma_2}$, $\phi \in \mathbf{B}_{\sigma_2}$, $\psi' \in \mathbf{B}_{\sigma_2}$, $\phi' \in \mathbf{B}_{\sigma_2}$, then for $\rho_1 > Max\{4\sigma_2\rho_2, 8\sigma_2\}$, ρ_2 as defined in equation (4.8),*

$$\|\mathcal{N}_2(\eta, \epsilon, \psi, \psi') - \mathcal{N}_2(\eta, \epsilon, \phi, \phi')\| \leq \left[\frac{1}{4\rho_0^2} + \frac{K_2\sigma_1}{\rho_0^{4/3}} + 8A\sigma_2\epsilon^{2/3}\right] \times (\|\phi - \psi\| + \|\phi' - \psi'\|),$$

for sufficiently small ϵ, where K_2 is a numerical constant.

Proof. Using (4.25), $|\psi| \leq 2\sigma_2\rho_1^{-1}, |\phi| \leq 2\sigma_2\rho_1^{-1}$ and inequality

$$|\psi^k - \phi^k| \leq k(|\psi| + |\phi|)^{k-1}|\psi - \phi|, \text{ for } k \geq 2$$

we have from (4.8)

$$\epsilon^{2/3}|E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \psi', \eta^{-2/3}) - E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \phi, \phi', \eta^{-2/3})|$$
$$\leq \frac{8A\rho_2\epsilon^{2/3}}{\rho_1}\left[\|\phi' - \psi'\| + \|\phi - \psi\|\right]$$

Combining with (4.21), the lemma follows.

Definition 4.10.

$$\mathbf{E}_{\sigma_2} := \mathbf{B}_{\sigma_2} \oplus \mathbf{B}_{\sigma_2}, \|(\psi, \psi')\| = \|\psi\| + \|\psi'\|; \tag{4.32}$$

Theorem 4.16. *There exists a unique solution $\psi \in \mathbf{B}_{\sigma_2}$ of equation (4.2) for all sufficiently large ρ_0 and small ϵ.*

Proof. In (4.25) and (4.26),replacing ψ with u,ψ' with v, we define

$$\mathcal{O}_1(u, v) := \mathcal{L}_2(u.v), \mathcal{O}_2(u, v) := \mathcal{L}_2'(u.v); \tag{4.33}$$

$$\mathcal{O}(u, v) = (\mathcal{O}_1(u, v), \mathcal{O}_2(u, v)) \tag{4.34}$$

Using (4.26), Lemma 4.13 and Lemma 4.15, it is easily seen that

$$\|\mathcal{L}_2(u, v)\| \leq \|\mathcal{L}_2(u, v) - \mathcal{L}_2(0, 0)\| + \|T\|_2$$
$$\leq \sigma_2 + 4\sigma_2 K\left[\frac{1}{4\rho^2} + \frac{\sigma_1 K_2}{\rho_0^{4/3}} + 8A\sigma_2\epsilon^{2/3}\right] < 2\sigma_2$$

for sufficiently large ρ_0 and small ϵ. Also, the same holds for $\|\mathcal{L}_2'\|$. On the other hand,

$$\|\mathcal{L}_2(u_1, v_1) - \mathcal{L}_2(u_2, v_2)\| \leq K\left[\frac{1}{4\rho_0^2} + \frac{\sigma_1 K_2}{\rho_0^{4/3}} + 8A\sigma_2\epsilon^{2/3}\right]\|(u_1, v_1) - (u_2, v_2)\|$$

and similar results hold for $\mathcal{L}_2'$. Hence the proof follows from contraction mapping theorem on a Banach space.

Lemma 4.17. *Let $\psi(\eta)$ be the solution of (4.26) as in Theorem 4.16 then $\psi(\eta)$ is a solution of (4.25) with $\psi(t) \equiv \tilde{\psi}(t)$ for $t \in \{\eta : \; Re\ \eta + \; Im\ \eta = \tilde{k}_0 \epsilon^{-1}, \arg \eta \in [0, \frac{5\pi}{8}]\}$.*

Proof. Since $\tilde{\psi}(t)$ is a solution of (4.25) for $t \in \{\eta : \; Re\ \eta + \; Im\ \eta = \tilde{k}_0 \epsilon^{-1}, \arg \eta \in [0, \frac{5\pi}{8}]\}$, by variation of parameters:

$$\begin{aligned} \tilde{\psi}(t) = &-\psi_1(t) \int_{\eta_2}^{t} \frac{\psi_2(s)}{2s^{1/3}} \mathcal{N}_2(s) ds + \psi_2(t) \int_{\eta_2}^{t} \frac{\psi_1(s)}{2s^{1/3}} \mathcal{N}_2(s) ds \\ &+ A_1 \psi_1(t) + A_2 \psi_2(t); \end{aligned} \tag{4.35}$$

Pluging $t = \eta_2$ in the above equation,we have

$$A_1 = \frac{\tilde{\psi}(\eta_2)\psi_2'(\eta_2) - \tilde{\psi}'(\eta_2)\psi_2(\eta_2)}{2\eta_2^{1/3}}; \tag{4.36}$$

$$A_2 = \frac{\tilde{\psi}(\eta_2)\psi_1'(\eta_2) - \tilde{\psi}'(\eta_2)\psi_1(\eta_2)}{2\eta_2^{1/3}}; \tag{4.37}$$

Rewrite equation (4.35):

$$\begin{aligned} \tilde{\psi}(t) = &\mathcal{L}_2 \tilde{\psi} + A_1 \psi_1(\eta) + A_2 \psi_2(\eta) \\ &- \psi_1(t) \int_{\eta_2}^{\eta_1} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{N}(\tilde{\psi})(t) dt + \psi_2(t) \int_{\eta_2}^{\eta_0} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{N}(\tilde{\psi})(t) dt \\ &+ \psi_1(t) \frac{\left(\psi_2(\eta_1)\tilde{\psi}'(\eta_1) - \psi_2'(\eta_1)\tilde{\psi}(\eta_1)\right)}{2\eta_1^{1/3}} \\ &- \psi_2(\eta) \frac{\left(\psi_1(\eta_0)\tilde{\psi}'(\eta_0) - \psi_1'(\eta_0)\tilde{\psi}(\eta_0)\right)}{2\eta_0^{1/3}}; \end{aligned} \tag{4.38}$$

Using integration by parts twice:

$$\begin{aligned} &- \psi_1(t) \int_{\eta_2}^{\eta_1} \frac{\psi_2(t)}{2t^{1/3}} \mathcal{L}_1 \tilde{\psi}(t) dt + \psi_2(t) \int_{\eta_2}^{\eta_0} \frac{\psi_1(t)}{2t^{1/3}} \mathcal{L}_1 \tilde{\psi}(t) dt \\ &= -\psi_1(t) \frac{\left(\psi_2(\eta_1)\tilde{\psi}'(\eta_1) - \psi_2'(\eta_1)\tilde{\psi}(\eta_1)\right)}{2\eta_1^{1/3}} \\ &+ \psi_2(t) \frac{\left(\psi_1(\eta_0)\tilde{\psi}'(\eta_0) - \psi_1'(\eta_0)\tilde{\psi}(\eta_0)\right)}{2\eta_0^{1/3}} \\ &- A_1 \psi_1(t) - A_2 \psi_2(t); \end{aligned} \tag{4.39}$$

So equation (4.38) is reduced to $\tilde{\psi}(t) = \mathcal{L}_2 \tilde{\psi}$. By equation (4.26),we have

$$\begin{aligned} \psi(t) - \tilde{\psi}(t) = &-\psi_1(t) \int_{\eta_1}^{t} \frac{\psi_2(t)}{2t^{1/3}} \left(\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\right) dt \\ &+ \psi_2(t) \int_{\eta_0}^{t} \frac{\psi_1(t)}{2t^{1/3}} \left(\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\right) dt \end{aligned} \tag{4.40}$$

Taking derivative in above equation:

$$\begin{aligned}\psi'(t) - \tilde{\psi}'(t) = -\psi_1'(t)\int_{\eta_1}^{t} \frac{\psi_2(t)}{2t^{1/3}}\left(\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\right) dt \\ + \psi_2'(t)\int_{\eta_0}^{t} \frac{\psi_1(t)}{2t^{1/3}}\left(\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\right) dt\end{aligned} \tag{4.41}$$

Using Lemma 4.13 and Lemma 4.15 ,we have

$$\|(\psi,\psi') - (\tilde{\psi},\tilde{\psi}')\| \le K\left[\frac{1}{4\rho^2} + \frac{\sigma_1 K_2}{\rho_0^{4/3}} + 8A\sigma_2\epsilon^{2/3}\right]\|(\psi,\psi') - (\tilde{\psi},\tilde{\psi}')\|$$

for sufficiently large ρ_0 and small ϵ_0. So $\psi(t) \equiv \tilde{\psi}(t)$.

Theorem 4.18. *(1)For large enough ρ_0, there exists a unique solution $\psi(\eta,\epsilon,a)$ of (4.11), (4.11) in region $R_1 \ge |\eta| \ge \rho_0$ for $\arg\eta \in [0, 5\pi/8]$,where R_1 is some constant chosen to be independent of ϵ.*
(2) The solution $\psi(\rho_0,\epsilon,a)$ of (1) is analytic in $\epsilon^{2/3}$,a,as $\epsilon \to 0$.
(3) Furthermore, $\lim_{\epsilon\to 0^+}\psi(\eta,\epsilon,a) = \psi_0(\eta,a)$ for $R_1 \ge |\eta| \ge \rho_0$.

Proof. Part (1) follows from Theorem 4.1 and Theorem 4.16 and Theorem 4.17 . Note that as $\epsilon \to 0$, $\epsilon^{2/3}E_3(\epsilon,\eta,\psi) \to 0$ uniformly in the region given. Part (2) follows from the theorem of dependence of solution on parameters (see, for instance, Theorem 3.8.5 in Hille 1976)

Lemma 4.19. *Let $F(\xi)$ be the solution of the weak problem in Theorem 3.18, we define $q(\xi)$ so that $q(\xi) = \frac{F(\xi)-[F(-\xi^*)]^*}{2i}$ (Note this is the same as $Im\ F$ on $\{\ Re\ \xi = 0\} \cap \mathcal{R}$). Then q satisfies the following homogeneous equation on imaginary ξ axis: $\{\xi = i\nu\}$:*

$$\epsilon^2\frac{d^2q}{d\nu^2} - (L(i\nu) + \tilde{L}(\nu))q = 0; \tag{4.42}$$

where $\tilde{L}(\nu)$ is some real function and $\tilde{L}(\nu) \sim O(\epsilon^2)$ as $\epsilon \to 0$.

Proof. On imaginary axis $\{\xi = i\nu\}$: Using (2.11), $L(i\nu) = \frac{\sqrt{\gamma^2-\nu^2}(\nu+\gamma)}{(1-\nu^2)^2}$ is real. Using (2.60),$\left(\frac{\bar{H}'H}{\bar{H}} - H'\right)(i\nu) = \frac{2\gamma}{(1-\nu^2)(\gamma-\nu)}$ is real. By taking imaginary part in equation (2.44), we have the lemma.

Remark 4.20. F is analytic in $[-i + i\epsilon^{4/3}\rho_0, ib)$.

Lemma 4.21. *If $q(i\nu_1) = 0$ with $-1+\epsilon^{4/3}\rho_0 \le \nu_1 < -b$, then $q(i\nu) \equiv 0$ for all $\nu \in [-i+\epsilon^{4/3}\rho_0, \alpha]$. Conversely, if $q(\nu_1) \ne 0$ for $\nu_1 \in [-1+\epsilon^{4/3}\rho_0, -b)$, then F cannot satisfy symmetry condition: $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$.*

Proof. By equation (4.42) and Theorem 6.2.1 of Olver(1974), there exist two linearly independent solutions $q_1(\nu)$ and $q_2(\nu)$ so that:

$$q_1(\nu) = L^{-1/4}(i\nu)\exp\{\frac{1}{\epsilon}\int_{\nu_1}^{\nu} L^{1/2}(it)dt\}\{1+o(1)\}$$

$$q_2(\nu) = L^{-1/4}(i\nu)\exp\{-\frac{1}{\epsilon}\int_{\nu_1}^{\nu} L^{1/2}(it)dt\}\{1+o(1)\}$$

Now $q(i\nu) = c_1 q_1(\nu) + c_2 q_2(\nu)$; $q(i\alpha) = 0$ (see remark 2.52); hence if $q(i\nu_1) = 0$, then $c_1 = 0$, $c_2 = 0$ since q_1 and q_2 involve non-oscillatory exponentials. Conversely, if $q(i\nu_1) \neq 0$, then c_1, c_2 cannot be identically zero. Hence F cannot satisfy symmetry condition $Im\ F = 0$ on $\{Re\ \xi = 0\} \cap \mathcal{R}$.

Theorem 4.22. *Assume $S(a_n) = 0$, but $S'(a_n) \neq 0$, then for small enough ϵ and large enough ρ_0,there is analytic function $\beta(\epsilon^{2/3})$ such that $\lim_{\epsilon\to 0}\beta(\epsilon^{2/3}) = a_n$, and if λ satisfies (1.30), then $Im\ F(\xi) = 0$ on $\{\xi = i\nu\} \cap \mathcal{R}$.*

Proof. For fixed large enough $\rho_0 > 0$, $S'(a_n) \neq 0$ implies

$$\frac{\partial \text{Im}\ \psi_0}{\partial a}(\rho_0, a_n) \neq 0, \tag{4.43}$$

Using Theorem 4.18, (4.43) and implicit function theorem,there exists analytic function $\beta(\epsilon^{2/3})$ so that Im $\psi(\rho_0, \epsilon, \beta(\epsilon^{2/3})) = 0$. This implies that $q(i\nu)$ is zero at some point in $[-i + i\rho_0^{4/3}, -bi)$. Then using Lemma 4.21, we complete the proof .

Proof of Theorem 1.11: If λ satisfies restriction (1.31), from Theorem 4.22 and Theorem 2.53, $F(\xi)$ is solution of Finger Problem. Part (3) follows from Lemma 3.13, Theorem 4.1,Theorem 4.18 and Theorem 4.9.

Appendix A. Properties of Function $P(\xi)$

In this section, we disscuss the properties of the following function:

$$\begin{aligned} P(\xi) &= \int_{-i\gamma}^{\xi} iL^{1/2}(t)dt \\ &= i\int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4}(\gamma + it)^{1/4}}{(1+t^2)}dt \end{aligned} \tag{A.1}$$

we choose branch cut $\{\xi : \xi = \rho i, \rho > \gamma\}, -\pi \leq \arg(\gamma + i\xi) \leq \pi$ for the function $(\gamma + i\xi)^{1/4}$ and branch cut $\{\xi : \xi = -\rho i, \rho > \gamma\}, -\pi \leq \arg(\gamma - i\xi) \leq \pi$ for the function $(\gamma - i\xi)^{3/4}$.

Definition A.1. The curves in complex ξ-plane on which Im $P(\xi) = constant$ will be called Stokes lines, where as curves in complex ξ-plane on which Re $P(\xi) = constant$ will be called anti-Stokes lines.

Lemma A.1. *Re $P(\xi)$ decreases with increasing ξ on the negative Re ξ-axis $(-\infty, 0)$. Further, for any $\nu \geq \nu_0 > 0$,*

$$-\frac{d}{d\xi} \; Re \; P \; > \; \frac{C_1}{|\xi - 2i|} \; > \; 0 \quad for \; \xi \in (-\infty, \; -\nu]$$

where constant C_1 only depends on ν_0 for λ in a compact subset of (0, 1).

Proof. From (A.1), it follows that

$$-\frac{d}{d\xi} P(\xi) = -i \frac{(\gamma^2 + \xi^2)^{1/4} (\gamma - i\xi)^{1/2}}{\xi^2 + 1}$$

Therefore, for λ in a compact subset of (0, 1), *i.e.* γ in a compact subset of $(0, \infty)$, for $\xi \in (-\infty, -\nu]$,

$$-|\xi - 2i| \frac{d}{d\xi} \; Re \; P(\xi) \; = \; \frac{(\xi^2 + \gamma^2)^{1/2} |\xi - 2i|}{\xi^2 + 1} \; \sin \; \left[\frac{1}{2} arctan \left(-\frac{\xi}{\gamma} \right) \right] > C_1$$

where $C_1 > 0$ is only dependent on the lower bound of $\sin \left[arctan \left(\frac{\nu}{\gamma} \right) \right]$, which depends on ν_0. Hence, $Re \; P(\xi)$ is monotonically decreasing with increasing ξ for $\xi \in (-\infty, \; \nu]$.

Lemma A.2. *For λ in a compact subset of (0, 1), on the imaginary ξ axis segment $(-ib, \; i\alpha)$, where $0 < \alpha < b < Min(\gamma, 1)$ and each of b, α independent of ϵ,*

$$-\frac{d}{d\rho} P(i\rho) \; > \; C \; > \; 0$$

where constant C is independent of ϵ and γ (and hence of λ).

Proof. Note from (A.1), that on the $Im \; \xi$ axis-segment,

$$P(\rho i) = P(0) - \int_0^\rho \frac{(\gamma + s)^{3/4} (\gamma - s)^{1/4}}{(1 - s^2)} ds,$$

It follows that $P(i\rho) - P(0)$ is real and decreasing from $\rho = -bi$ to $\rho = i\alpha$. Further,

$$-\frac{d}{d\rho} P(i\rho) \; = \; \frac{(\gamma + \rho)^{3/4} (\gamma - \rho)^{1/4}}{1 - \rho^2} \; > \; C \; > \; 0$$

where constant C is only dependent on b and α and independent of ϵ, λ (or γ), when λ is in a compact subset of (0, 1).

Remark A.3. Property 1 follows from combining Lemmas A.1 and A.2.

Lemma A.4. *(Property 2) For $\xi \in \mathcal{R}^-$, as $|\xi| \to \infty$, $P'(\xi) \sim e^{-i\pi/4}/\xi$ and $P(\xi) \sim e^{-i\pi/4} \ln \xi$. In particular for λ in a compact subset of (0, 1), there exists constant R sufficiently large but independent of λ and ϵ so that for $|t| > R$ on the ray $r := \{t : t = \xi - se^{i\phi}, 0 \le s < \infty\}$, with ϕ restricted by the requirement*

$$arg \left[-\xi e^{-i\phi}\right] \in \left[-\frac{\pi}{2} + \hat{\delta}, \frac{\pi}{4} - \hat{\delta}\right]$$

for some $\hat{\delta} > 0$ independent of any parameter,

$$\frac{d}{ds} Re\ P(t(s)) > \frac{C}{|t(s) - 2i|} > 0$$

where C is independent of ϵ and λ.

Proof. From (A.1),

$$P'(\xi) = i\frac{(\gamma - i\xi)^{3/4}(\gamma + i\xi)^{1/4}}{1 + \xi^2}$$

In $\mathcal{R}^-$, as $|\xi| \to \infty$, $arg\ (\xi + i\gamma) \sim arg(\xi)$, but $arg\ (\xi - i\gamma) \sim -2\pi + arg(\xi)$. So,

$$P'(\xi) \sim \frac{e^{-i\pi/4}}{\xi}$$

$$P(\xi) \sim e^{-i\pi/4} \ln \xi$$

Now, for λ in a compact subset of (0, 1), and $\gamma - \alpha$ and $(1 - \alpha^2)$ chosen independent of ϵ, there exists R independent of λ so that for $|t| > R$, on $t = \xi - se^{i\phi}$, $0 \le s < \infty$,

$$\frac{d}{ds} Re\ P(t(s)) > C\ Re \left[\frac{e^{i\phi}e^{-i\pi/4}}{se^{i\phi} - \xi}\right]$$

$$> \frac{C}{|s - \xi e^{-i\phi}|} \cos \left[\frac{\pi}{4} + arg(s - \xi e^{-i\phi})\right] > \frac{C}{|t(s) - 2i|}$$

for $\xi \in \mathcal{R}^-$ with $Arg\ (-\xi e^{-i\phi})$ appropriately constrained.

Corollary A.5. *On line segment r_{u_1}, $Re\ P$ increases towards ∞, with $\frac{d}{ds} Re\ P(t(s)) > \frac{C}{|t(s)-2i|} > 0$ for constant C independent of ϵ and λ when the latter is restricted to a compact subset of (0, 1).*

Proof. This follows very simply from the previous lemma.

Lemma A.6. *(Property 3) For λ in a compact subset of (0, 1), there exists small ν, independent of ϵ and λ, so that*

$$\frac{d}{ds}\left[Re\ P(t(s))\right] \ge C > 0$$

on the line $t(s) = i\alpha + se^{-i3\pi/4}$ for $0 \le s \le \sqrt{2}\nu$, where C is independent of ϵ and λ.

Proof. From differentiating (A.1), and doing a Taylor expansion of P' at $i\alpha$, it follows that for $0 \le s \le \sqrt{2\nu}$

$$t'(s)\ P'(t(s))\ =\ e^{-i\pi/4}\frac{(\gamma+\alpha)^{3/4}(\gamma-\alpha)^{1/4}}{1-\alpha^2}\ +\ O(\nu)$$

for small enough ν. Since γ is in a compact subset of $(0,\infty)$, and $\gamma+\alpha$, $\gamma-\alpha$, $(1-\alpha^2)$ are independent of ϵ, it follows that there exists ν small enough and independent of ϵ and λ so that

$$\frac{d}{ds}\ Re\ P(t(s))\ =\ Re\ \left(t'(s)P'(t(s))\right)\ > C\ >\ 0$$

where C is independent of ϵ and λ.

Lemma A.7. *For λ in a compact subset of (0, 1) and for given ν_0 and R independent of ϵ, with $0 < \nu_0 < R$, consider the line segment*

$$t = i\nu_1 - \nu - se^{i\phi}\quad ,\quad 0 \le s \le R$$

with $\nu \ge \nu_0$. Then, there exists real ν_1, ϕ sufficiently small in absolute value and depending only on ν_0, R so that on this line segment

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ C\ >\ 0$$

where C is only dependent on ν_0

Proof. Note for $\nu_1 = 0$ and $\phi = 0$, result holds from Lemma A.1, with $C = C_1$ only depending on the lower bound for ν. Since $\frac{d}{ds}\ Re\ P(t(s))$ is clearly a continuous function of ϕ and ν_1, and uniformly continous for for s restricted in a compact set, it follows that there exists ϕ and ν_1 small enough that

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C_1}{2} = C\ >\ 0$$

where C is only dependent on ν_0

Corollary A.8. *For small enough ϕ_0 and $\alpha-\nu > 0$, on line segment r_{u_2}, parametrized by arclength s increasing towards ∞,*

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ C\ >\ 0$$

where constant C is independent of ϵ and λ, and only depends on ν.

Proof. For r_{u_2}, we use previous Lemma A.7 with $\nu_1 = \alpha - \nu$, $\phi = 0$ to obtain desired result.

Lemma A.9. *For λ in any compact subset of (0, 1), there exists $\phi_0 > 0$, $\mu > 0$ and $b > 0$ each small enough so that $Re\ P(t(s))$ monotonically increases on $t = -ib - se^{i(\mu+\phi_0)}$ with s as s goes from 0 to ∞ and*

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C}{|t(s)-2i|}$$

where C is independent of ϵ, λ.

Proof. First we note that on a straight line through the origin $t = -se^{i\phi}$, straight forward algebra gives

$$\frac{\partial^2 P(t(s,\phi))}{\partial\phi\partial s}|_{\phi=0} = \frac{(\gamma+is)^{1/2}[2\gamma^2+2s^2(2-\gamma^2)+i\gamma s(1+s^2)]}{2(1+s^2)^2(\gamma^2+s^2)^{3/4}}$$

If we define $2\theta_1 =$ arctan (s/γ) and $\theta_2 =$ arctan $\frac{\gamma s(1+s^2)}{2\gamma^2+2s^2(2-\gamma^2)}$ then it is clear that

$$\frac{\partial^2 \ Re \ P(t(s,\phi))}{\partial\phi\partial s}|_{\phi=0} > 0$$

when $\theta_1+\theta_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ For γ in a compact subset of $(0,\infty)$, it is possible to choose a number $s_m > 0$ independent of any parameter so that $0 \le \theta_2 \le \frac{\pi}{6}$ when $0 \le s \le s_m$ and in that case for $\theta_1+\theta_2 \in [0, \frac{5\pi}{12}]$ since θ_1 is clearly in $[0, \frac{\pi}{4})$ interval for all s. Thus

$$\frac{\partial^2 \ Re \ P(t(s,\phi))}{\partial\phi\partial s}|_{\phi=0} > C > 0$$

Since Lemma A.1 shows that for $\phi = 0$, $\frac{d}{ds} Re \ P(t(s)) \ge 0$ for $s \in [0, s_m]$, it follows that for small enough $\phi > 0$,

$$\frac{d}{ds} Re \ P(t(s,\phi)) > C > 0$$

for $s \in [0, s_m]$, where C only depends on ϕ. On the other hand, we know that for ϕ and b each restricted to be in some fixed convenient interval, there exists R independent of ϕ and b so that for $s > R$, from Lemma A.4,

$$\frac{d}{ds} \ Re \ P(t(s)) > \frac{C}{|t(s)-2i|} > 0$$

on $t = -ib - se^{i\phi}$, with C independent of ϵ and λ. From continuity, there exists small enough b, ϕ_0 and μ each positive, so that on $t = -ib - se^{i(\phi_0+\mu)}$ for $s \in [s_m, R]$,

$$\frac{d}{ds} Re \ P(t) > \frac{C}{2} > 0$$

where C only depends on $\phi_0+\mu$. Combining all the above results, we complete the proof of the lemma.

Remark A.10. The above holds for any λ in a compact subset of $(0,1)$. In the special case when $\lambda \in [\frac{1}{2}, \lambda_m]$, $\lambda_m - \frac{1}{2}$ small enough (i.e. γ somewhat bigger than 1), we demonstrate more directly the above lemma for $\mu+\phi_0 = \frac{\pi}{3}$ where b may be chosen close to 1 with the only requirement $10|\gamma-1| \le b-1$.

Lemma A.11. *If* $10|\gamma-1| \le |b-1|$,*then* $Re \ P(t)$ *is increasing along ray* $r_l = \{\xi = -bi + se^{i4\pi/3}, 0 \le s < \infty\}$.

Proof. We want to show that

$$\frac{d}{ds} Re P(\xi(s)) = Re\{P'(\xi) e^{i4\pi/3}\} > 0 \text{ on } r_l. \tag{A.2}$$

Note:

$$P'(\xi) = i \frac{(\xi - i\gamma)}{(\xi + i)} \left(\frac{(\gamma + i\xi)}{(\gamma - i\xi)} \right)^{1/4} \frac{1}{(1 + i\xi)}; \tag{A.3}$$

Let $B(s)$ be the positive angle between $\xi(s) + \gamma i$ and $\xi(s) + i$,then by geometry:

$$\arg P'(\xi) \in (-B + \frac{\pi}{4}, \frac{2\pi}{3}); \tag{A.4}$$

we can see that $B < \frac{\pi}{12}$ implies (A.2).
Let $d_1 = |b-1| + |\gamma - 1|$, by geometry:

$$\cos B = \frac{(s^2 - 2s|b-1|\sin\frac{\pi}{3} + |b-1|^2) + (s^2 - 2sd_1 \sin\frac{\pi}{3} + d_1^2) - |\gamma - 1|^2}{2\sqrt{(s^2 - 2s|b-1|\sin\frac{\pi}{3} + |b-1|^2)}\sqrt{(s^2 - 2sd_1\sin\frac{\pi}{3} + d_1^2)}} \tag{A.5}$$

Let $t = \frac{|\gamma-1|^2 s}{|b-1|\sin\frac{\pi}{3}}, d = \frac{|\gamma-1|}{|b-1|}$,

$$\cos B = \frac{(t - 1 - \frac{d}{2})^2 + (1+d)\cot^2\frac{\pi}{3} - \frac{1}{4}d^2}{\sqrt{(t-1)^2 + \cot^2\frac{\pi}{3}}\sqrt{(t-1-d)^2 + (1+d)^2\cot^2\frac{\pi}{3}}} \tag{A.6}$$

The min of the above function over $0 < t < \infty, d \leq 0.1$ is .9688749307,but $\cos\frac{\pi}{12} = 0.9330127$,so $B < \frac{\pi}{12}$.

Lemma A.12. *(Property 4) For any $\xi \in \mathcal{R}^-$, there is a connected path $\mathcal{P}(i\alpha, \xi)$ which is at least a piecewise $\mathbf{C}^1$ curve in the t-plane from $t = i\alpha$ to $t = \xi$ so that on each $\mathbf{C}^1$ segment $t = t(s)$, parametrized by arclength increasing towards $t = \xi$,*

$$\frac{d}{ds}\ Re\ P\ (t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

where C is a constant independent of ϵ. Further $Re\ P(t)$ attains a minimum in $\mathcal{R}^-$ at $i\ \alpha$.

Proof. First, let $\xi \in \partial\mathcal{R}^-$.
Case a: $\xi \in r_{u_1} \cup r_{u_2} \cup r_{u_3}$. Combining previous lemmas A.4, A.7 and corollary A.8, it is clear that $Re\ P$ is increasing monotonically from $\xi = -i\alpha$ to $\xi = -\infty$ on this boundary of $\mathcal{R}^-$ and that on this piecewise $\mathbf{C}^1$ curve on each segment

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

where C is a constant independent of ϵ and s is arclength on each segment increasing towards $t = -\infty$.

Case b: $\xi \in (-ib, i\alpha)$. From Lemma A.2, it follows that with parametrization $s = \alpha - \rho$,

$$\frac{d}{ds} \, Re \, P(t(s)) \; > \; \frac{C}{|t(s) - 2i|} > \; 0$$

for C independent of ϵ and λ since λ is in a fixed compact subset of (0,1).
Case c: $\xi \in r_l^-$. In this case we choose path $\mathcal{P}(i\alpha, \xi)$ to be the union of imaginary line segment $(-ib, \; i\alpha)$ and part of r_l^- upto $t = \xi$. For the imaginary ξ axis segment, we know from Lemma A.2 that

$$\frac{d}{ds} \, Re \, P(t(s)) \; > \; \frac{C}{|t(s) - 2i|} > \; 0$$

where arclength s is taken to be increasing from $i\alpha$ to $-ib$. Previous two lemmas A.9 and A.11 imply that on the straight line segment r_l^- connecting $t = -ib$ to $t = \xi$, we have

$$\frac{d}{ds} \, Re \, P(t(s)) \; > \; \frac{C}{|t(s) - 2i|} \; > \; 0$$

Now consider $\xi \in \mathcal{R}^-$ (in the interior). Consider the Stokes line in the t plane passing through $t = \xi$: $Im \; P(t(s)) = \; Im \; P(\xi)$ for which $Re \; P$ is decreasing. On such a steepest descent path, it is clear from the asymptotics of P' for large $|\xi|$ and the fact that there are no critical points of $P(t)$ in $\mathcal{R}^-$ that

$$\frac{d}{ds} Re \, P(t(s)) \; > \; \frac{C}{|t(s) - 2i|} \; > \; 0$$

where C is independent of ϵ. This Stokes line must intersect the boundary of $\mathcal{R}^-$ at a finite point ξ_0 since asymptotic behavior $Im \; P \; (t) \; \sim \; - sin \, (\frac{\pi}{4}) \; \ln \; |t|$ for large $|t|$ precludes the Stokes line from going to ∞. Denote the Stokes line traversed from $t = \xi_0$ to $t = \xi$ by $\mathcal{P}(\xi_0, \xi)$. Now, from what has been already argued, there exists a piecewise $\mathbf{C}^1$ path $\mathcal{P}(i\alpha, \xi_0)$ connecting $t = i\alpha$ to $t = \xi_0$, where on each $\mathbf{C}^1$ segment,

$$\frac{d}{ds} Re \, P(t(s)) \; > \; \frac{C}{|t(s) - 2i|} \; > \; 0$$

Hence the desired path $\mathcal{P}(i\alpha, \xi)$ consists of the union of $\mathcal{P}(i\alpha, \xi_0)$ and $\mathcal{P}(\xi_0, \xi)$ and this satisfies all the desired conditions.

Lemma A.13. *(Property 5) For any $\xi \in \mathcal{R}^-$, there is a connected path $\mathcal{P}(\xi, -\infty)$ which is at least a piecewise $\mathbf{C}^1$ curve in the t-plane from $t = \xi$ to $t = -\infty$ so that on any $\mathbf{C}^1$ segment parametrized by arclength s (chosen to increase towards $t = -\infty$),*

$$\frac{d}{ds} \, Re \; P \; (t(s)) \; > \; \frac{C}{|t(s) - 2i|} \; > \; 0$$

where C is a constant independent of ϵ.

Proof. First, let $\xi \in \partial \mathcal{R}^-$.
Case a: $\xi \in r_{u_1} \cup r_{u_2} \cup r_{u_3}$. From previous lemmas, A.4, A.7 and corollary A.8, it is clear that $Re\ P$ is increasing monotonically from $t = i\alpha$ to $t = -\infty$ on this boundary of $\mathcal{R}^-$. Thus, there is a piecewise $\mathbf{C}^1$ curve connecting $t = \xi$ to $t = -\infty$ consisting of appropriate parts of $r_{u_1} \cup r_{u_2} \cup r_{u_3}$. On each $\mathbf{C}^1$ segment,

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

where C is a constant independent of ϵ and s is arclength on each segment increasing towards $t = -\infty$.
Case b: $\xi \in r_l^-$. In this case we choose path $\mathcal{P}(\xi, -\infty)$ along r_l^-. We know that on this straight line, with arclength chosen to increase towards $t = -\infty$, from Lemmas A.9 and A.11,

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

Case b: $\xi \in (-ib, i\alpha)$. In this case, we choose $\mathcal{P}(\xi, -\infty) = \mathcal{P}(\xi, -ib) \cup \mathcal{P}(-ib, -\infty)$ where the first path segment goes down along the imaginary ξ axis segment and the second coincides with r_l^-. Clearly, on each segment, we may write

$$\frac{d}{ds}\ Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

for constant C independent of ϵ and λ.

Now consider $\xi \in \mathcal{R}^-$ in the interior. Consider the Stokes line in the t plane passing through $t = \xi$: $Im\ P(t(s)) = Im\ P(\xi)$ for which $Re\ P$ is increasing. On such a steepest ascent path, it is clear from the asymptotics of P' for large $|\xi|$ and the fact that there are no critical points of $P(t)$ in $\mathcal{R}^-$ that

$$\frac{d}{ds} Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

where C is independent of ϵ. This Stokes line must intersect the boundary of $\mathcal{R}^-$ at a finite point ξ_0 since asymptotic behavior $Im\ P\ (t) \sim -\sin\left(\frac{\pi}{4}\right) \ln\ |t|$ for large $|t|$ precludes the Stokes line from going to ∞. Call the Stokes line traversed from $t = \xi_0$ to $t = \xi$ as $\mathcal{P}(\xi, \xi_0)$. From what has been already argued, there exists a path $\mathcal{P}(\xi_0, -\infty)$ connecting $t = \xi_0$ to $t = -\infty$ where

$$\frac{d}{ds} Re\ P(t(s))\ >\ \frac{C}{|t(s) - 2i|}\ >\ 0$$

Hence the desired path $\mathcal{P}(\xi, -\infty)$ consists of the union of $\mathcal{P}(\xi, \xi_0)$ (along Stokes line) and $\mathcal{P}(\xi_0, -\infty)$ and this satisfies all the desired conditions.

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References

1. C.J.Amick,J.B.Mcleod,A singular perturbation Problem in Needle Crystals, *Arch.Rational Mech.Anal.*109(1990),139-171.
2. Bensimon,D.,Kadanoff,L.P.,Liang,S.,Shraiman,B.I.,Tang,C. Viscous flow in two dimensions.*Rev. mod. phys.58* (1986),977.
3. G.F.Carrier,M.Krook & C.E.Pearson,*Functions of a complex variable,* McGraw-Hill,1966.
4. S.J.Chapman,On the role of Stokes lines in the selection of Saffman-Taylor fingers with small surface tension,Eur.J.Appl Math. 10(1999),513-534.
5. R.Combescot,V.Hakim,T.Dombre.Y.Pomeau,A.Pumir,shape selection for Saffman-Taylor fingers,*Physical Review Letter*vol.56,number 19(1986),pp 2036.
6. R.Combescot,V.Hakim,T.Dombre.Y.Pomeau,A.Pumir,Analytic theory of Saffman-Taylor fingers,*Physical Review A*vol.37,number 4(1988),pp1270-1283.
7. R.Combescot,T.Dombre,Selection in the Saffman-Taylor bubble and assymmetrical finger problem,*Physical Review A* 38(5)(1988),pp2573.
8. O.Costin,Exponential Asymptotics,Transseries,and Generlized Borel Summation for Analytic,Nonlinear,Rank-one Systems of Ordinary Diffenretial Equations, IMRN,No.8 (1995), pp377-417.
9. O.Costin,On Borel Summation and Stokes Phenomena for Rank-one Systems of Ordinary Diffenretial Equations,Duke Math.J. 93(2)(1998),pp289-343.
10. DeGregoria,A.J.& Schwartz,L.W, Saffman-Taylor finger width at low interfacial tension, *Phy.Rev. Lett.*,**58**,(1987) 1742
11. Dorsey,A.T.& Martin O. Saffman-Taylor fingers with anisotropic surface tension. Phys.Rev.A 35 (1987),pp3989.
12. Duchon,J.& Robert,R. Evolution d'une interface par capillarite' t difusion de volume, Ann l'Inst.H.Poincare' 1 (1984),361.
13. E.Hille,Ordinary Differential Equations in the Complex Domain, Wiley-Interscience,1976.
14. Homsy,G.M Viscous fingering in porous media.*Ann.Rev Fluid Mech.19*(1987),271.
15. D.C.Hong, J.S.Langer, Analytic theory for the selection of Saffman-Taylor finger, Phys.Rev.Lett.56(1986),pp2032.
16. D.Kessler,J.Koplik ,H.Levine,Patterned Selection in Fingered Growth Phenomena.*Adv.Phys.37*(1988),255.
17. D.Kessler,H.Levine,The theory of Saffman-Taylor finger,Phys.Rev.A 32(1985),1930.
18. D.Kessler, H.Levine, Stability of finger patterns in Hele-Shaw Cells.*Phy. Rev. A***33**, (1986) 2632.
19. M.Kruskal, H.Segur, Aero.Res.Ass.Tech.Memo.85-25 (1985).
20. M.Kruskal,H.Segur,Asymptotics beyond all ordersin a Model of Crystal Growth, Studies in applied math.85(1991),pp129-181.
21. J.Mclean,Ph.D Thesis,Caltec,1980.
22. J.W.Mclean,P.G.Saffman,The effect of surface tension on the shape of fingers in a Hele Shaw cell,*J.Fluid Mech.*vol.102(1981),pp 455-469.
23. N.I.Muskhelishvili,Singular Integral Equations,Noordhoff International Publishing,1977.
24. F.W.J.Olver,Asymtotics and Special Functions,Academic Press,New York,1974.
25. Park,C.W.& Homsey,G.M. Two-phase displacement in Hele-Shaw cells :theory. J.Fluid Mech. 139(1985),pp 291.
26. Pelce,P.Dynamics of Curved Fronts,Academic,1988.
27. Reinelt,D.A. The effect of thin film variations and transverse curvature on the shape of fingers in a Hele-Shaw cell. Phys.Fluids 30(1987),pp2617.

28. L.A.Romero,Ph.D thesis,Caltech,1982.
29. P.G.Saffman,Viscous Fingering in a Hele-shaw cell.*J.Fluid Mech.173*(1986),73.
30. P.G.Saffman,G.I.Taylor,The penetration of a fluid into a porous medium of Hele-Shaw cell containing a more viscous fluid,*Proc.R.Soc.London Ser.A.* 145(1958),312.
31. P.G.Saffman,Exact solutions for the growth of fingers from a flat interface between two fluids in a porous medium or Hele-Shaw cell, *Q.J.Mech.appl.math,*12(1959),pp146-150.
32. B.I.Shraiman, On velocity selection and the Saffman-Taylor problem, Phys. Rev. Lett.56(1986),2028.
33. J.Su,On the Existence of Finger Solution in Hele-Shaw cell, Nonlinerity 14(2001),153-166.
34. S.Tanveer,The effect of surface tension on the shape of a Hele-Shaw cell bubble,.*Phys.Fluids 29*(1986),pp3537.
35. S.Tanveer,Analytic theory for the selection of a symmetric Saffman-Taylor finger in a Hele Shaw cell.*Phys.Fluids 30(6)*(1987a),pp1589-1605
36. S.Tanveer, Analytic theory for the linear stability of Saffman-Taylor finger. *Phys.Fluids 30*(1987b), pp2318
37. S.Tanveer,New solutions for steady bubbles in a Hele-Shaw cell. *Phys.Fluids 30*(1987c),pp651.
38. S.Tanveer,Analytic Theory for the Determination of Velosity and Stability of Bubbles in a Hele-Shaw Cell,Part I:Velocity Selection,Theoretical and Computational Fluid Dynamics,1(1989),135.
39. S.Tanveer, Analytic theory for the selection of Saffman-Taylor finger in the presence of thin-film effects.Proc.R.Soc.Lond.A 428(1990),511.
40. S.Tanveer,Viscous Displacement in a Hele-Shaw cell,*Asymptotics Beyond all orders*,Eds:H.Segur,S.Tanveer,H.Levine,Plenum,New York,1991.
41. S.Tanveer,Surprises in Viscous fingering,J.Fluid Mech.,vol.409(2000),pp.273-308.
42. S.Tanveer & X. Xie, Analyticity and Nonexistence of Classical Steady Hele-Shaw Fingers (Submitted), 2002.
43. G.I.Taylor,P.G.Saffman,A note on the motion of bubbles in a Hele-Shaw cell and Porous medium,*Q.J.Mech.Appl.Math.*12(1959),259.
44. J.M.Vanden-Broeck,fingers in a Hele-Shaw cell with surface tension, Phy.Fluids,26(1983),2033.
45. W.Wasow,*Asymtotic Expansion for Odinary Diffential Equations*, Interscience Publishers,New York,1968.
46. Weinstein,S.J.,Dussan V.,E.B.& Ungar,L.H. A theoretical study of two phase flow through a narrow gap with a moving contact line: viscous fingering in a Hele-shaw cell. J.Fluid Mech. 221(1990),53.
47. Xie,X. Rigorous results in steady finger selection in viscous fingering.Ph.D Thesis,The Ohio State University,2000
48. J.J.Xu,Globally unstable oscillatory modes in viscous fingering,*European J.of Appl. Math.* 2(1991),105.

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