Analytic approximation for nonlinear problems

The Case of Blasius Similarity Solution

Saleh Tanveer

(Ohio State University)

Collaborator: O. Costin

Background

Analytic solution for nonlinear problems, including ones arising in vortex dynamics are very desirable.

Unfortunately, this can be only achieved for only a special class of problems

When small asymptotic parameters are available; we can make some headway if the limiting problem is solvable, even rigorously.

$$\mathcal{N}[u; \epsilon] = 0 \ , \ \text{suppose} \, \mathcal{N}[u_0; 0] = 0$$

Define $u = u_0 + E$**, then** $\mathcal{L}[E] = -\delta - \mathcal{N}_1[E]$**, where**

$\delta = \mathcal{N}[u_0; \epsilon]$**,** $\mathcal{L} := \frac{\partial \mathcal{N}}{\partial u}[u_0; \epsilon], \mathcal{N}_1[E] := \mathcal{N}[u_0 + E] - \mathcal{L}[E]$

Inversion of $\mathcal{L}$ **and contraction leads to** $E = O(\epsilon)$ **estimates**

New non-perturbative methods

Recently, in some problems with no perturbation parameter, Costin, Huang, Schlag [2012] and Costin, Huang & T [2012] determined u_0 for which $\mathcal{N}[u_0]$ is small. Proof of error estimates $E = u - u_0$ found by inversion of $\mathcal{L}$ in $\mathcal{L}[E] = -\delta - \mathcal{N}_1[E]$

The idea is simple and natural. Use empirical data to construct orthogonal polynomial approximation in a finite part of domain; for the rest use far-field asymptotics; matching gives a composite u_0. The method can in principle be applied to any nonlinear problem that allows convenient theoretical estimates of $\mathcal{L}^{-1}$

We illustrate this for the classical Blasius Similarity solution:

$$f''' + ff'' = 0\,, \text{ with } f(0) = 0 = f'(0) = 0\,, \quad \lim_{x\to\infty} f'(x) = 1\,,$$

Blasius equation: Background

Blasius (1912) introduced this as similarity solution for Prandtl Boundary layer equation for flow past a flat plate

Much work since then. Topfer (1912) introduced transformation $f(x) = a^{-1/2} F(x a^{-1/2})$ that makes this equivalent to

$$F''' + FF'' = 0 \quad F(0) = 0 = F'(0)\ , F''(0) = 1\ ,$$

provided $\lim_{x \to \infty} F'(x) = a > 0$ exists. Weyl (1942) proved existence and uniqueness. Series representation using hodograph transformations (Calleghari-Friedman, 1968) have been applied, but representation not very convenient for finding f. Liao ('99) obtained an emprically accurate solution through a formal process. No rigorous control on the error available for accurate and efficient approximation.

Definitions

$$\text{Define } P(y) = \sum_{j=0}^{12} \frac{2}{5(j+2)(j+3)(j+4)} p_j y^j \tag{1}$$

where $[p_0, \ldots, p_{12}]$ are given by

$$\left[-\frac{510}{10445149}, -\frac{18523}{5934}, -\frac{42998}{441819}, \frac{113448}{81151}, -\frac{65173}{22093}, \frac{390101}{6016}, -\frac{2326169}{9858}, \frac{4134879}{7249}, -\frac{1928001}{1960}, \frac{20880183}{19117}, -\frac{1572554}{2161}, \frac{1546782}{5833}, -\frac{1315241}{32239}\right] \tag{2}$$

$$\text{Define } t(x) = \frac{a}{2}(x + b/a)^2, \;\; I_0(t) = 1 - \sqrt{\pi t} e^t \operatorname{erfc}(\sqrt{t})$$

$$J_0(t) = 1 - \sqrt{2\pi t} e^{2t} \operatorname{erfc}(\sqrt{2t}),$$

$$q_0(t) = 2c\sqrt{t} e^{-t} I_0 + c^2 e^{-2t} \left(2J_0 - I_0 - I_0^2\right), \tag{3}$$

Main Theorem

Theorem: **Let F_0 be defined by**

$$F_0(x) = \begin{cases} \frac{x^2}{2} + x^4 P\left(\frac{2}{5}x\right) \text{ for } x \in [0, \frac{5}{2}] \\ ax + b + \sqrt{\frac{a}{2t(x)}}\, q_0(t(x)) \text{ for } x > \frac{5}{2} \end{cases} \tag{4}$$

Then, there is a unique triple (a, b, c) in $\mathcal{S}$ close to $(a_0, b_0, c_0) = \left(\frac{3221}{1946}, -\frac{2763}{1765}, \frac{377}{1613}\right)$

$$\mathcal{S} = \left\{(a, b, c) : \sqrt{(a - a_0)^2 + \frac{1}{4}(b - b_0)^2 + \frac{1}{4}(c - c_0)^2} \le 5 \times 10^{-5}\right\} \tag{5}$$

with the property that $F \approx F_0$ in the sense

$$F(x) = F_0(x) + E(x)\ , \tag{6}$$

Rigorous Error bounds for approximation

$$\|E''\|_\infty \leq 3.5 \times 10^{-6} , \|E'\|_\infty \leq 4.5 \times 10^{-6} , \|E\|_\infty \leq 4 \times 10^{-6} \text{ on } [0, \frac{5}{2}] \tag{7}$$

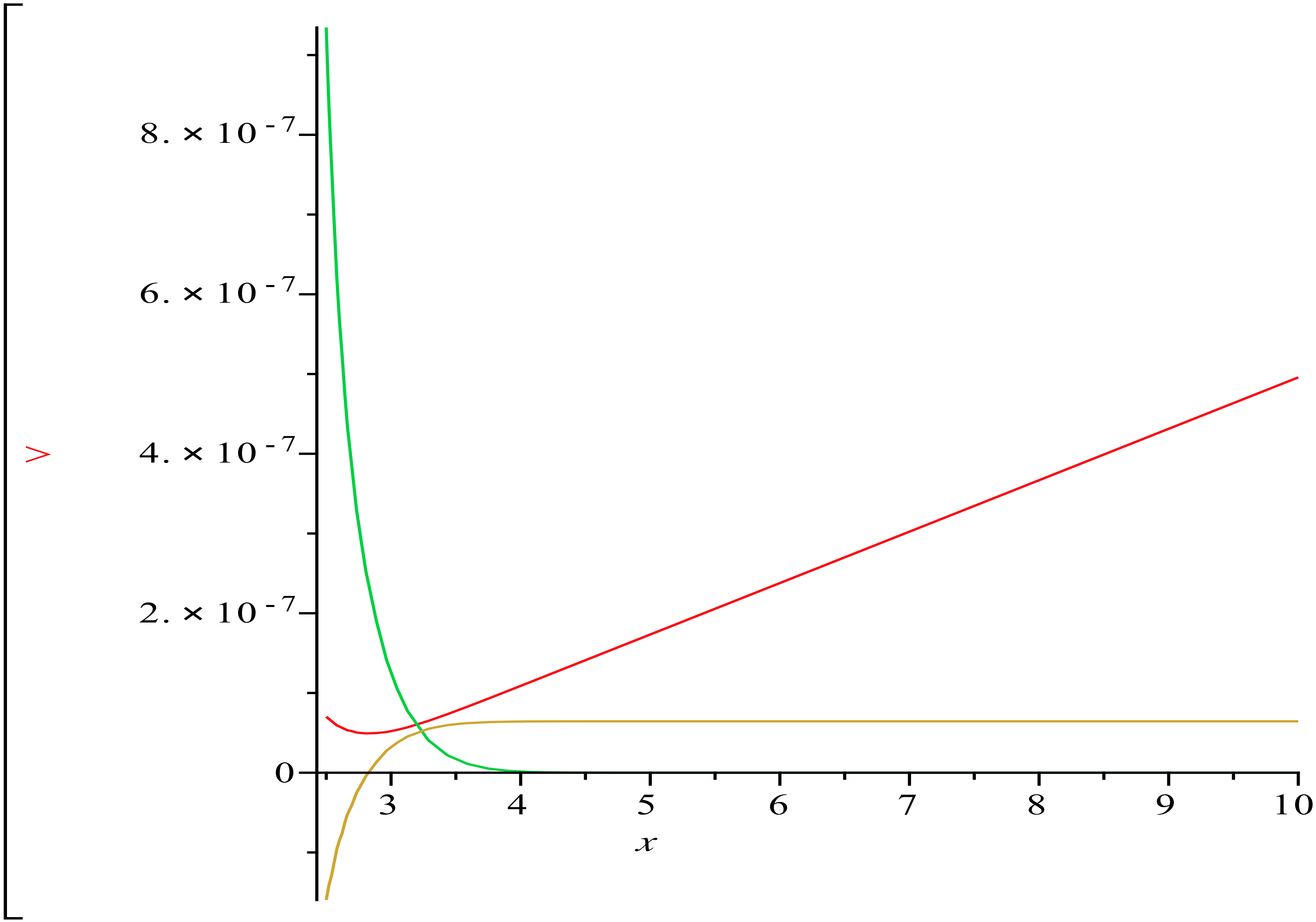
and for $x > \frac{5}{2}$,

$$\left| E \right| \leq 1.69 \times 10^{-5} t^{-2} e^{-3t} , \left| \frac{d}{dx} E \right| \leq 9.20 \times 10^{-5} t^{-3/2} e^{-3t}$$

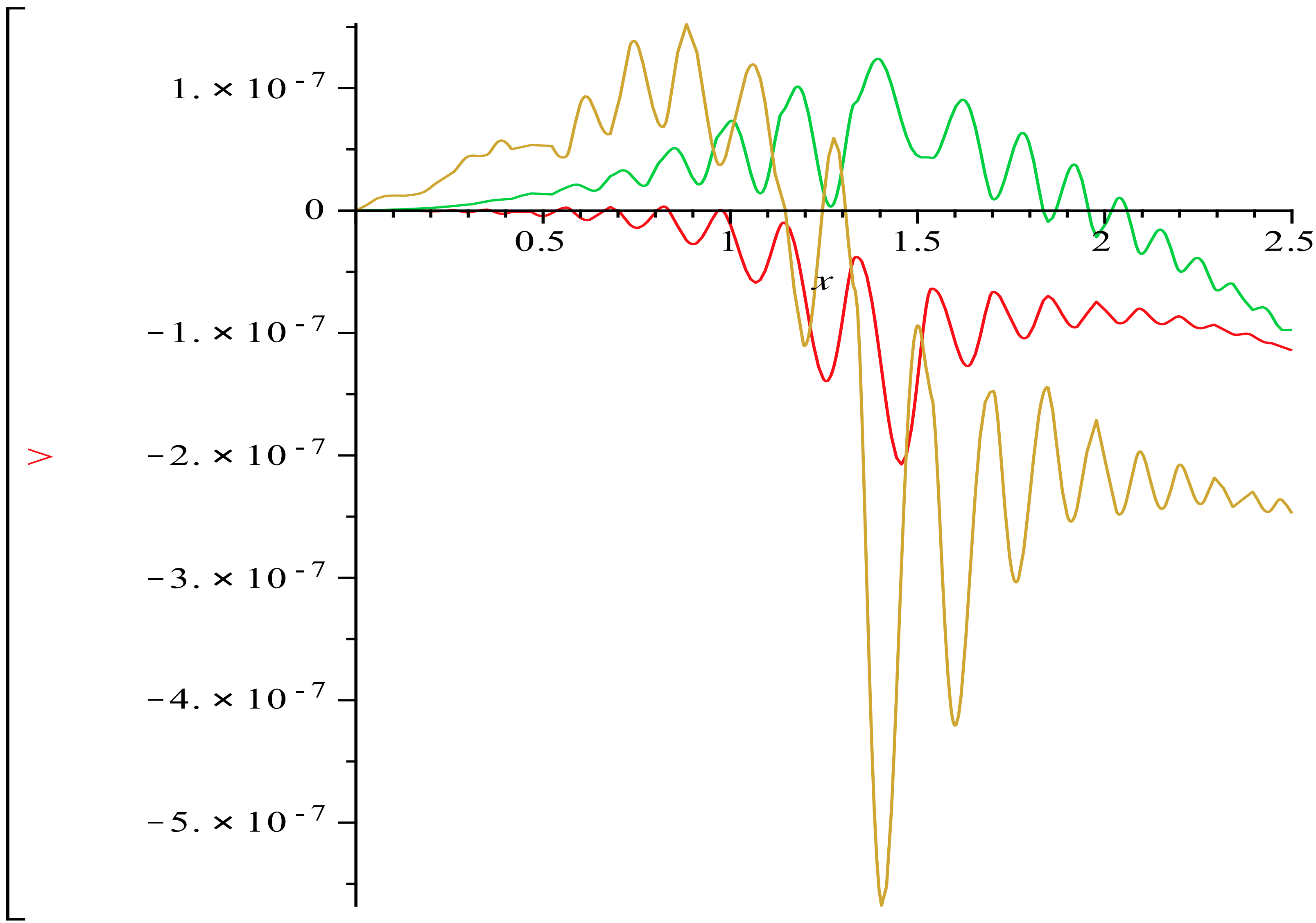
$$\left| \frac{d^2}{dx^2} E \right| \leq 5.02 \times 10^{-4} t^{-1} e^{-3t} \tag{8}$$

Remark: These rigorous bounds are likely overestimates since comparison with numerical solutions give at least ten times smaller errors.

$F_N - F_0$, $F'_N - F'_0$, $F''_N - F''_0$ for $x > \frac{5}{2}$



$F_N - F_0$, $F'_N - F'_0$, $F''_N - F''_0$ for $x \in [0, \frac{5}{2}]$



Guess F_0

Guess F_0 in $\left[0, \frac{5}{2}\right]$ found by using empirical data for $F''' = -FF''$ on $[0, \frac{5}{2}]$ and projecting it to a low order Chebyshev polynomial.

Guess for F_0 in $\left(\frac{5}{2}, \infty\right)$ found from asymptotics for large x, rigorously controlled, that includes upto $O\left[\exp\left\{-a\left(x + \frac{b}{a}\right)^2\right\}\right]$ contribution exactly, for each $a > 0$, b and c.

Parameters (a, b, c) determined from continuity of solution representation of F at $x = \frac{5}{2}$, casting it as a contraction map in $\mathbb{R}^3$

Once guess F_0 is found so that residual $R = F_0''' + F_0 F_0''$ is uniformly small, rest is a mathematical analysis of E, where $F = F_0 + E$, helped by the smallness of R, which allows a contraction mapping argument.

Conclusion

We have shown how an accurate and efficient analytical approximation of Blasius solution with rigorous error bounds is possible.

The method is general and simply relies on finding a candidate F_0 so that $\mathcal{N}[F_0]$ is small.

Candidate F_0 on finite domain $I = [0, \frac{5}{2}]$ obtained through orthogonal projection of empirical solution using a small number of Chebyshev polynomials. More accuracy requires more modes.

Outside I rigorous exponential asymptotics ideas was used to come up with highly accurate $F \approx F_0$ approximation

The ideas equally applicable to problems with parameters or to class of nonlinear PDEs

Full analysis available in

http://www.math.ohio-state.edu/$\sim$tanveer