Viscous fingering

–a tale of unexpected singular perturbation

Saleh Tanveer

(Ohio State University)

Collaborators: X. Xie, J. Ye, M. Siegel, M. Kunka, M. Foster

Viscous Fingering in Porous media

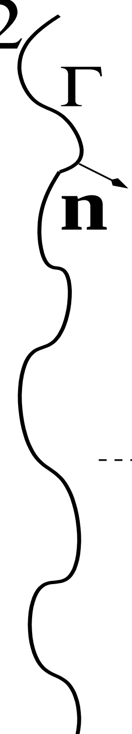
Planar interface between less viscous fluid pushing a more viscous fluid unstable (Hill, '52, Saffman & Taylor, '58)

Darcy's Law: $\mathbf{v} = -\mathbf{k}/\mu$ grad p, p: pressure. **v: velocity**

Incompressibility: $\mathbf{div\ v=0.}$ $\varphi = -\mathbf{k}\ \mathbf{p}/\mu$ μ: viscosity.

On Γ : $-\mu_2\ \varphi_2 + \mu_1 \varphi_1 = \varepsilon\ \kappa$, κ: curvature.

$\mathbf{n}.\mathrm{grad}\,\varphi_2 = \mathbf{n}.\mathrm{grad}\,\varphi_1 = v_n$, normal speed



$\Delta\ \varphi_2 = 0$ in Ω_2

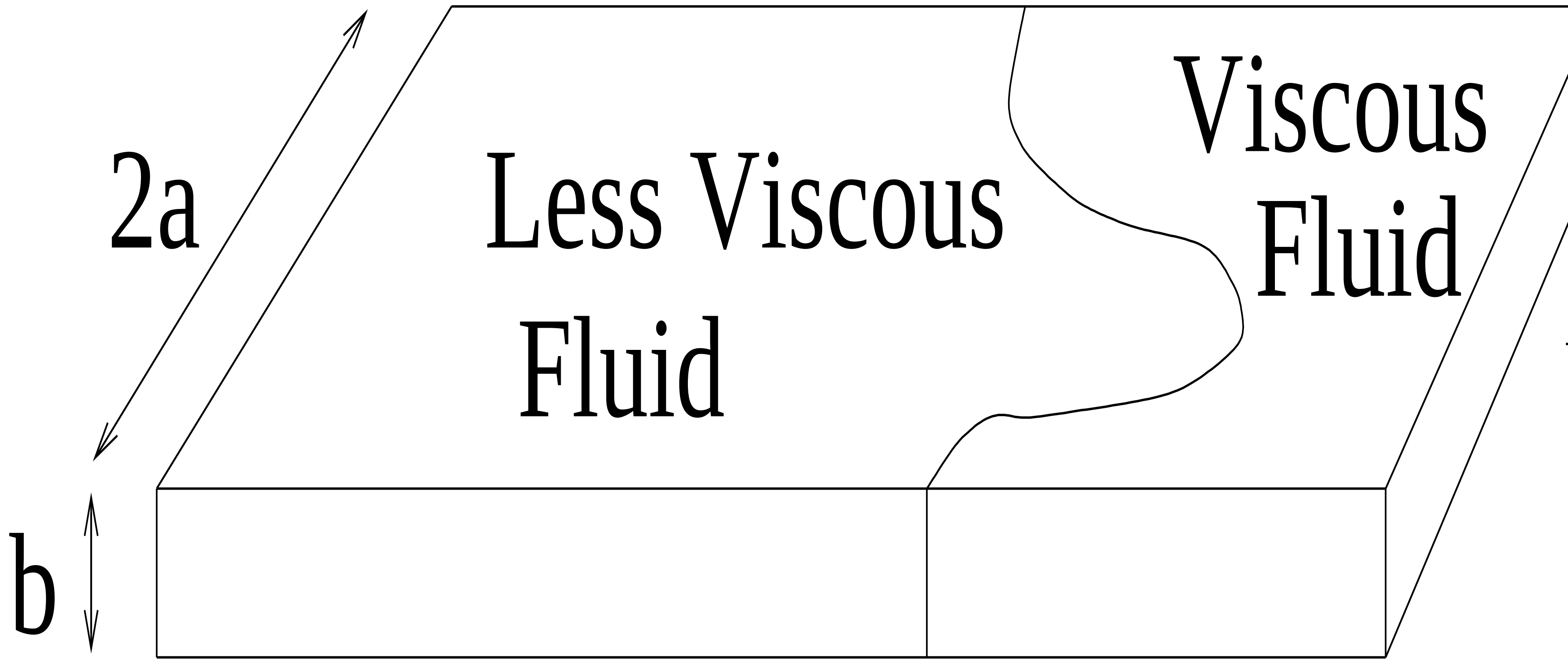
$\Delta\ \varphi_1 = 0$ in Ω_1

Less viscous (viscosity μ_1)

More viscous Fluid (viscosity μ_2)

Fingering in Hele-Shaw cell

Mathematically, the porous media flow related to displacement of more viscous by less viscous fluid in a Hele-Shaw cell, where $b/a << 1$



Saffman-Taylor Experiment in 1958

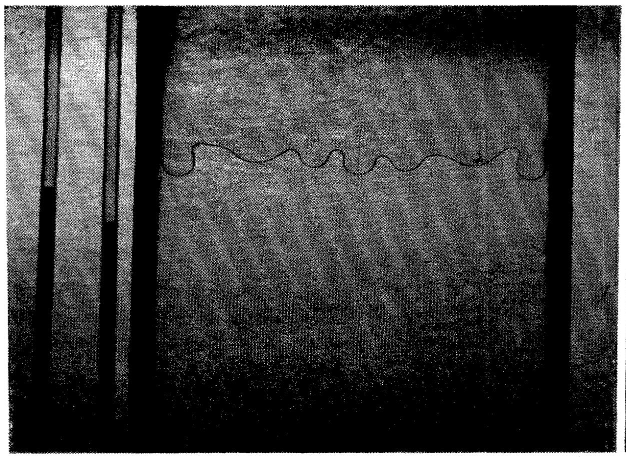


FIGURE 2

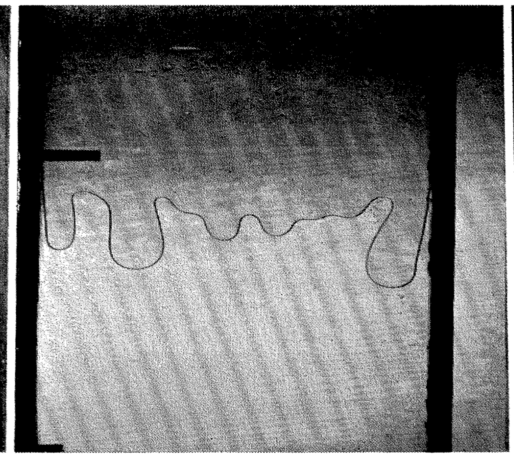


FIGURE 3

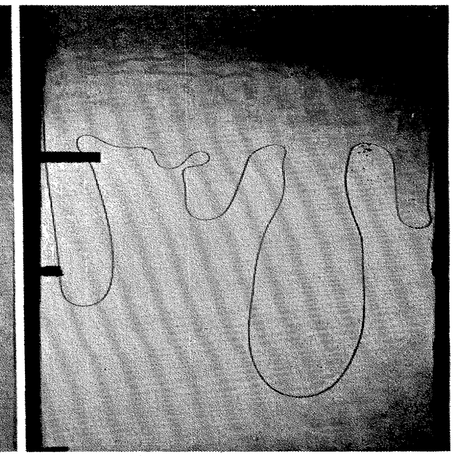


FIGURE 4

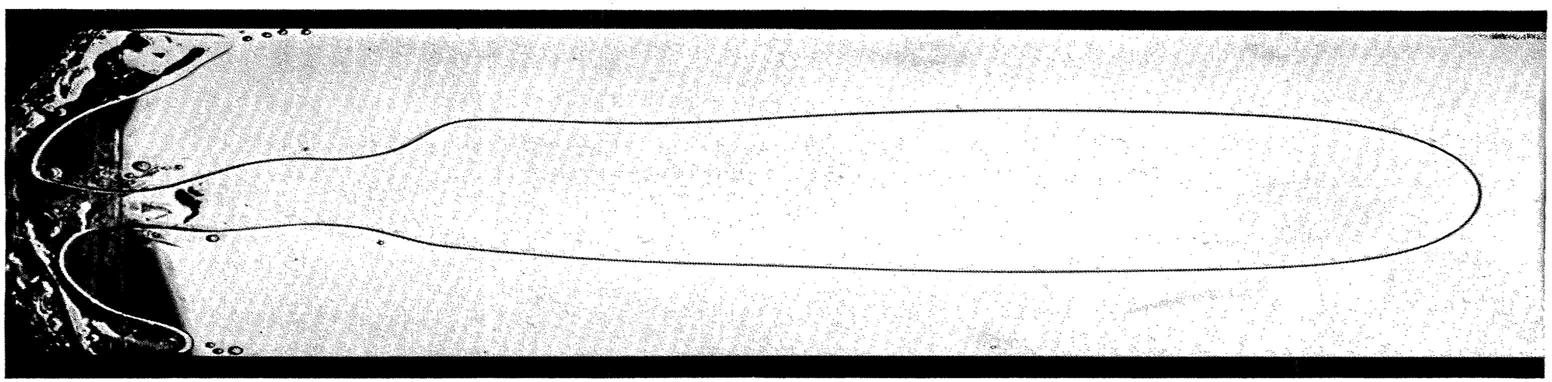
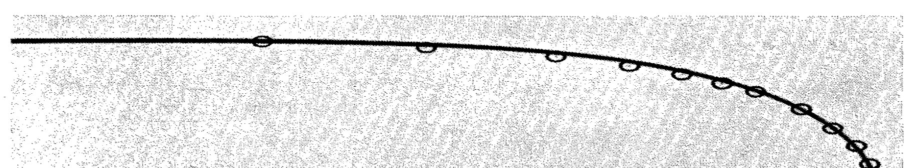
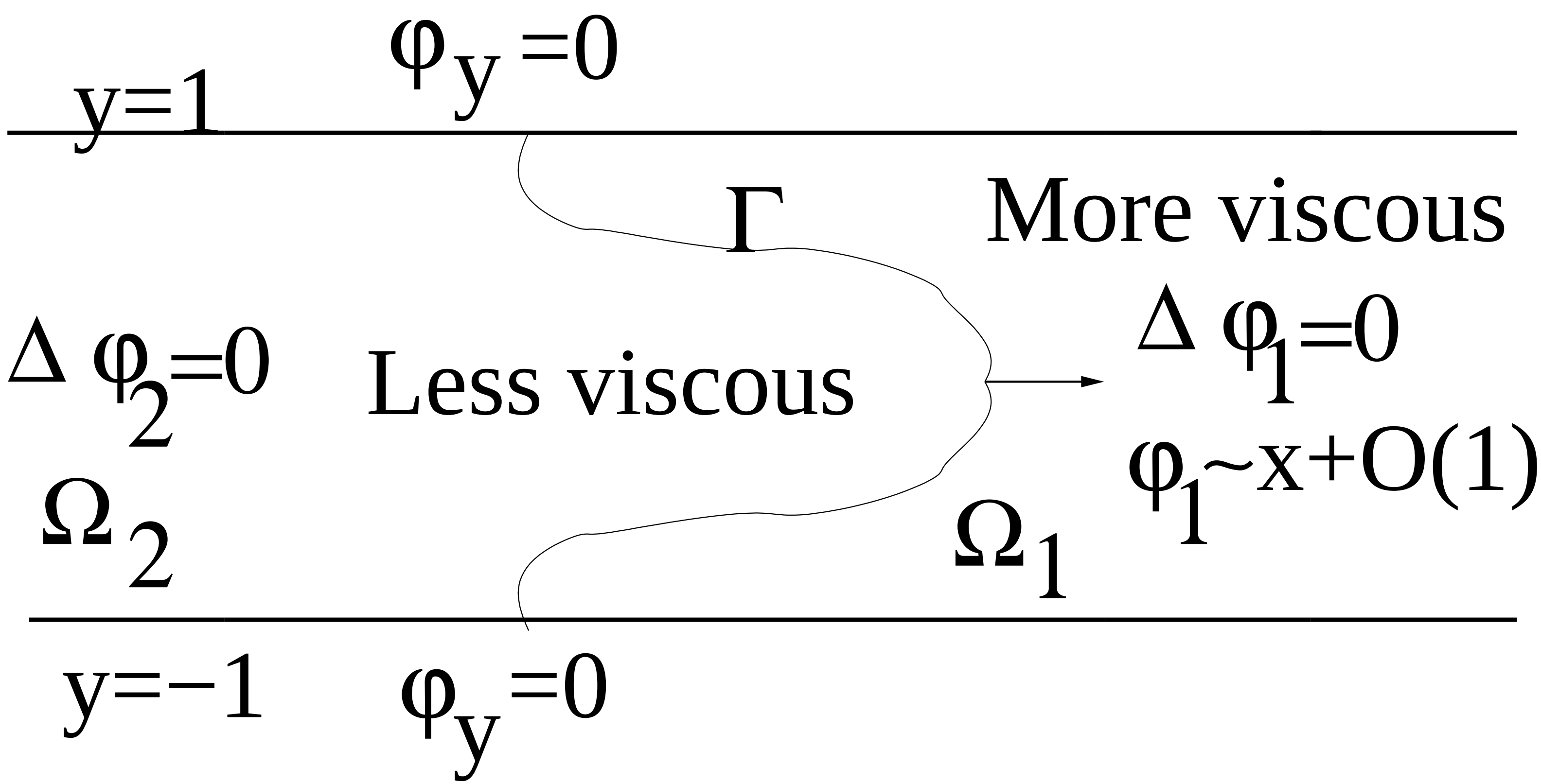


FIGURE 8



Idealized model for Hele-Shaw flow

Gap averaged Stokes flow: $\mathbf{u}_1 = -\frac{b^2}{12\mu_1}\nabla p_1$ **in** Ω_1 **and** $\mathbf{u}_2 = -\frac{b^2}{12\mu_2}\nabla p_2$ **in** Ω_2**. With** $\phi_1 = -\frac{b^2}{12\mu_1}p_1$, $\phi_2 = -\frac{b^2}{12\mu_2}$**, incompressibility gives harmonic** ϕ_1, ϕ_2**. Nondimensionalizing:**



On Γ, $\phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa$, $\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n} = v_n$**, normal interface speed**

Zero viscosity ratio simplification

In this case, we only need consider one domain $\Omega = \Omega_1$, where

$\Delta\phi = 0$

Far-field and wall conditions in non-dimensionalized form:

$$\phi \sim x + O(1) \text{ as } x \to +\infty \ , \ \text{ and } \frac{\partial \phi}{\partial y}(x, \pm 1) = 0$$

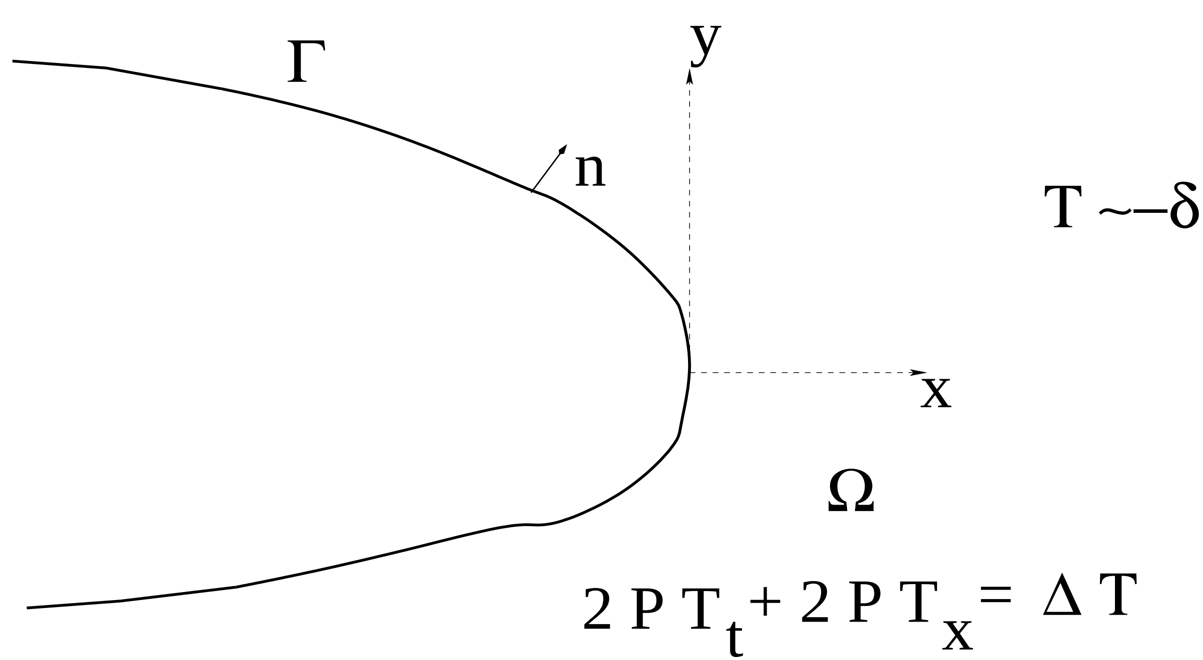
Interfacial conditions:

$$v_n = \frac{\partial \phi}{\partial n}, \ \text{ and } \ \phi = \epsilon\kappa$$

where κ is the curvature and ϵ surface tension coefficient. These interfacial conditions ignores 3-D thin-film effects. It turns out (Taylor and Saffman,'59), for steady flow, the problem with nonzero viscosity ratio is equivalent to a zero-viscosity problem with change of parameters.

Mathematically Related Crystal Growth

$$\text{On} \quad \Gamma: \; T = -d_0 \kappa \, , \quad [n . \operatorname{grad} T \,] = -2\, P v$$



Here d_0, P, δ, v_n and κ are capillary lengths, Peclet number, non-dimensional undercooling, normal interface speed and curvature respectively.

Small undercooling limit of Dendrite equation

For an interface that approaches a parabola in the far-field, P is related to δ. For $\delta << 1$, $P << 1$ and limit for small undercooling for $(x, y) << O(P^{-1/2})$ for a one-sided model gives the following equations for $u = -\frac{T}{2P}$ (Kunka *et al, '97):*

$$\Delta u = 0, \quad \text{for} \quad (x, y) \in \Omega$$

$$\frac{\partial u}{\partial n} = v_n \quad , \quad u = \epsilon \kappa \quad \text{for} \quad (x, y) \in \partial\Omega,$$

$$u \to x + O(1) \quad , \quad \text{as } (x, y) \to \infty, \text{far from the interface}$$

Except for the geometry, this limiting small undercooling 1-sided dendritic crystal growth model, reduces to the Hele-Shaw 1-sided model

Other Mathematically Related Problems

Directional Solidification, where growth is driven by concentration diffusion rather than Temperature (Pelce, '87, Kessler *et al*, **'87)**

Streamers in Electric Discharge (Ebert *et al*, **'11)**

Continuum model for probability density function in Diffusion Limited Aggregation (DLA) (Witten & Sanders, '81)

Tumor Growth models in limiting cases (Bazaliy & Friedman, '03)

Zero surface tension evolution model related to Random Matrix Theory (Wiegmann, '06)

Mathematical Issues

1. Determination of steady translating shapes and critical role of surface tension or other regularization

2. Stability of translating states and role of regularization

3. Initial Value Problem: Global Existence of solutions

Mathematics simplifies considerably when $\epsilon = 0$. In many physical situation, as in Saffman-Taylor experiment, $\epsilon << 1$. So, it is tempting to set $\epsilon = 0$. Unfortunately, the model is structurally unstable and ill-posed for $\epsilon = 0$ in any physically relevant norm and $\epsilon = 0$ model predictions need not be physically relevant.

Also, mathematics simplifies, without being structurally unstable or ill-posed, when $\epsilon \neq 0$, and one looks at small bubbles that translate along the channel or when side-walls are far apart and initial bubbles are nearly circular.

Structural stability and physical relevance

Any mathematical model can be described abstractly by

$$\mathcal{N}\left[\mathrm{u};\epsilon\right] = 0,$$

where operator $\mathcal{N}$ can describe arbitrary differential, integral or algebraic operator and ϵ describes parameters.

The above problem is *structurally stable* if solution set $\{u\}$ depends continuously on ϵ.

Consider initial value problem (IVP): $u_t = \mathcal{N}[u], \;\; u(0) = u_0$. The problem above is *well-posed* if there exists unique solution to the IVP that depends continuously on u_0

Models that are not structurally stable or well-posed are generally irrelevant physically since ϵ and u_0 not known exactly in experiment. Singular ϵ effects cannot be ignored.

Determination of Steady Translating Shapes

$\epsilon = 0$ exact solutions by Zhuravlev, '56, Saffman & Taylor '58, Taylor & Saffman '59, T., '87 reveals degeneracy of solutions, indexed by translation speed U (equivalent to width λ for a finger) and symmetry about channel centerline. For symmetric fingers,

$$x = 2\frac{1-\lambda}{\pi} \log \ \cos\left(\frac{\pi y}{2\lambda}\right) \ , \quad y \in (-\lambda, \lambda) \ , \ \lambda \in (0,1)$$

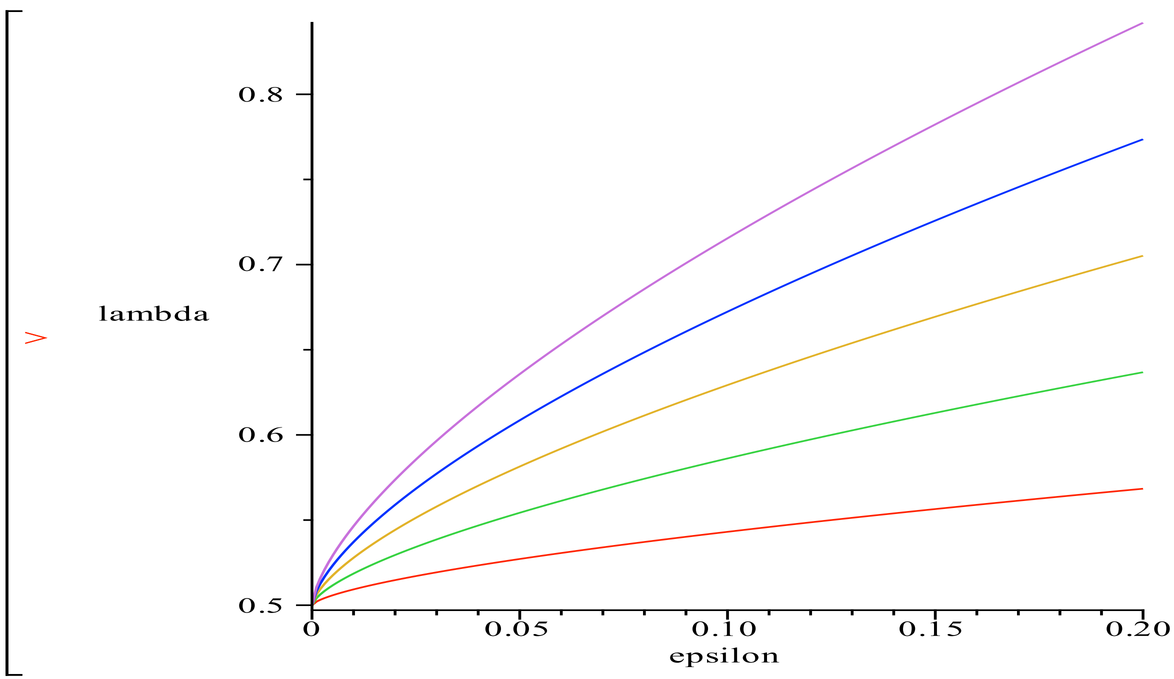
However, experiment resulted in $\lambda \approx \frac{1}{2}$, and $\lambda = \frac{1}{2}$ theoretical shape agreed with it. Thus, there was a "selection problem."

Similar problems for translating bubbles of specified area (Taylor & Saffman, 1959), where bubble speed U was undetermined; also in other pattern formation problems

Selection quandary worse than believed. There exists $\epsilon = 0$ non-symmetric fingers (Taylor-Saffman, '59) and bubbles (T., '87).

Steady State Selection for $\epsilon \neq 0$

Numerical work (McLean-Saffman,'80, VandenBroeck, '82, Kessler & Levine, '86) suggested selection for $\epsilon \neq 0$ of a discrete family of solution branches as below, though formal perturbation in powers of ϵ suggested no selection !



Selection Through Exponential Asymptotics

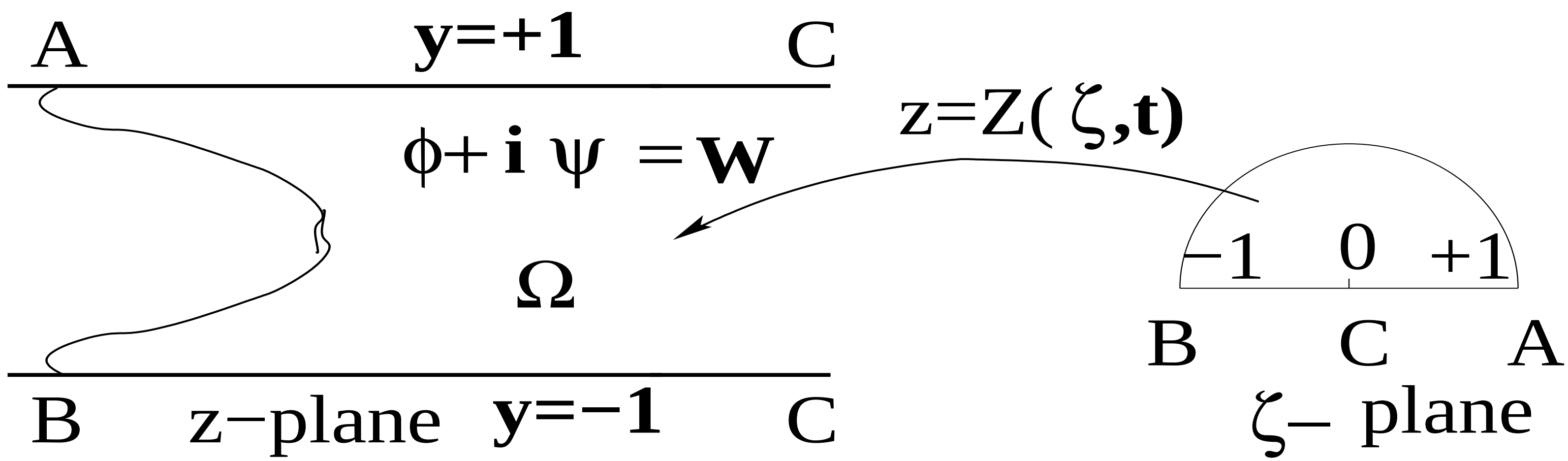
Formal exponential asymptotics (Combescot et al ’86, Hong & Langer ’86, Shraiman, ’86, T. ’86, ’87b, Dorsey-Martin ’87) suggested selection and stability of one branch only (T., 87c, Bensimon-Pelce ’87), though differing conclusions by Xu ’91, Scwartz & DeGregoria ’87

Selection results proved rigorously (Xie & T. ’03, T. & Xie, 2003, Xie, ’08)

Formal exponential asymptotics with 3-D effects available (T., 1990); qualitatively similar results

Other $\lambda \neq \frac{1}{2}$ or non-symmetric finger solutions accessible in experiment with etching of glasses (BenJacob *et al*, 1985), Needle Piercing the interface (Zocchi *et al*, ’87), small bubble in front of a finger (Couder *et al* ’86).

Conformal map formulation for 1-fluid



Conformal map from unit−semi−circle to physical domain Ω in one fluid problem

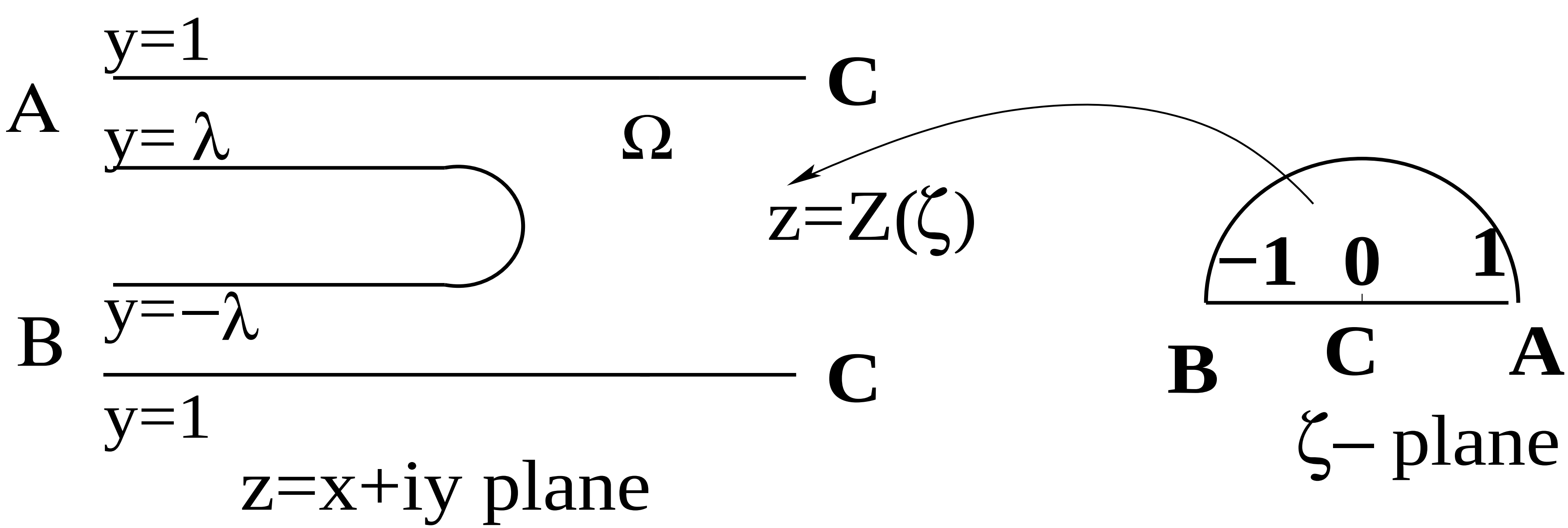
$$\mathbf{Z}(\zeta, \mathbf{t}) = -2/\pi \, \mathbf{log}(\zeta) + \mathbf{f}(\zeta, t)$$

$$\mathbf{W}(\zeta, \mathbf{t}) = -2/\pi \, \mathbf{log}(\zeta) + \omega(\zeta, \mathbf{t})$$

Interface boundary conditions imply on $|\zeta| = 1$:

$$\mathrm{Re}\left[\frac{Z_t}{\zeta Z_\zeta}\right] = \mathrm{Re}\,\frac{\zeta W_\zeta}{|Z_\zeta|^2}\ , \quad \mathrm{Re}\,W = -\frac{\epsilon}{|Z_\zeta|}\,\mathrm{Re}\left[1 + \frac{\zeta Z_{\zeta\zeta}}{Z_\zeta}\right]$$

Steady finger formulation in co-moving frame



With $Z(\zeta) = Z_0(\zeta;\lambda) + f(\zeta)$, where $Z_0(\zeta;\lambda)$ is the ZST solution, obtain $\mathrm{Im}\, f = 0$ on $(-1,1)$ and on $|\zeta| = 1$:

$$\mathrm{Re}\, f = -\frac{\epsilon}{|f' + h|}\, \mathrm{Re}\left[1 + \zeta \frac{f'' + h'}{f' + h}\right], \;\; \text{where}\; h(\zeta) = \frac{1 - (2\lambda - 1)\zeta^2}{\zeta(\zeta^2 - 1)}$$

Formal expansion $f \sim \epsilon f_1 + \epsilon^2 f_2 \; + \; ..$ consistent for $\lambda \in (0,1)$!

Toy problem in exponential asymptotics selection

Consider the solution $\phi(x, y)$ to

$$\Delta\phi = 0 \ \text{ for } y > 0$$

On $y = 0$, require Boundary Condition

$$\epsilon\phi_{xxx}(x, 0) + (1 - x^2 + a)\phi_x(x, 0) - 2x\phi_y(x, 0) = 1,$$

where $a \in (-1, \infty)$ is real. Also require that as $x^2 + y^2 \to \infty$, $(x^2 + y^2)\,|\nabla\phi|$ bounded.

Can show $W(x + iy) = \phi_x(x, y) - i\phi_y(x, y)$ satisfes

$$\epsilon W'' + \left[-(z + i)^2 + a\right] W = 1$$

For $\epsilon = 0$, $W = W_0 \equiv \frac{1}{-(z+i)^2+a}$. *Ansatz* $W = W_0 + \epsilon W_1 + \,..$ **consistent. Suggests no restriction on a. Yet, we will discover this conclusion to be incorrect !**

Toy problem for exponential asymptotics–II

With scaling of dependent and independent variable, obtain:

$$z + i = i 2^{-1/2} \epsilon^{1/2} Z \; ; \; W = 2^{-1} \epsilon^{-1} G(Z) \; ; a = 2\epsilon\alpha$$

$$G'' - (\frac{1}{4} Z^2 + \alpha) G = -1$$

Using parabolic cylinder functions, the above problem has an explicit solution. Requiring $G \to 0$ as $Z \to \infty$ for $\arg Z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is possible if and only for integer $n \geq 0$

$$\alpha = (2n + \frac{3}{2}) \; , \; i.e. \; a = 2\epsilon\alpha = 2\epsilon \left(2n + \frac{3}{2}\right)$$

$\lim_{\epsilon \to 0+}$ solution not equal $\epsilon = 0$ solution, unless a is as above. Discontinuity of solution set at $\epsilon = 0$. So ϵ term cannot be discarded, despite consistency of regular perturbation series.

Surprising sensitivity to other small effects

Suppose $\epsilon_1 << \epsilon$ in the following variation of the toy problem:

$$\epsilon W'' + \left[-(z+i)^2 + a - \frac{\epsilon_1}{(z+i)^2} \right] W = 1 \ \text{ for } \ y = \operatorname{Im} z \geq 0$$

Question: Should we ignore ϵ_1 term ? Appears reasonable since a scales as ϵ without ϵ_1 term and $\frac{\epsilon_1}{(z+i)^2} << a$ for $y = \operatorname{Im} z \geq 0$.

This reasoning is incorrect.

Explanation: what matters is the size of ϵ_1-term in an $\epsilon^{1/2}$ neighborhood of $z = -i$. It is $O(\epsilon)$ when $\epsilon_1 = O(\epsilon^2)$.

This explains the dramatic effect of small perturbation in experiment (BenJacob *et al*, Zocchi *et al*, Couder *et al*)

The toy problem illustrates disparate length (and time) scales can interact close to structural instability.

Time evolution for $\epsilon = 0$, $\epsilon \neq 0$

$\epsilon = 0$ dynamics is rich (Polabarinova-Kochina, '46, Galin, '46, Richardson, Gustaffson, ...

However, $\epsilon = 0$ evolution problem has *no continuity* **with respect to I.C. in a physically reasonable norm (say H^1) Howison ('86), Fokas & T. ('98).**

For $\epsilon \neq 0$, local existence (Duchon & Roberts). Global existence in unforced case (Constantin & Pugh) and for pressure gradient causing a near circular bubble to translate (Ye & T.)

Denote solution as $u(t;\epsilon)$ for $\epsilon \neq 0$. Question: When is $\lim_{\epsilon \to 0^+} u(t;\epsilon) = u(t;0)$?.

Define $T_0 \leq \infty$ to be the singularity time for $u(t;0)$. Evidence shows that in some cases, there exists $T_d < T_0$, independent of ϵ so that $\lim_{\epsilon \to 0} u(t;\epsilon) \neq u(t,\epsilon)$ for $t \in (T_d, T_s)$ (Siegel *et al* **'96)**

Toy problem for time evolution

Consider the following PDE for $\mathrm{Im}\,\xi \geq 0$:

$$G_t + iG_\xi = 1 \; + \; 2i\epsilon \left[G^{-1/2}\right]_{\xi\xi\xi} \qquad \text{with} \quad G(\xi, 0) = 1 - 2i\xi$$

Formal expansion $G \sim G^{(0)} + \epsilon G^{(1)} \; + \;$ .. gives:

$$G^0(\xi, t) \; = \; 2i\,(\xi_0(t) - \xi) \;\; , \quad \text{where} \;\; \xi_0(t) \; = \; -\frac{i}{2}(1 - t)$$

$$G_t^1 + iG_\xi^1 = 30(2i\xi_0(t) - 2i\xi)^{-7/2} \;\; , \;\; \text{where} \;\; G^1(\xi, 0) = 0$$

$$G^1(\xi, t) = -12\,(2i\xi_0(t) - 2i\xi)^{-5/2} + 12\,(2i\xi_d(t) - 2i\xi)^{-5/2},$$

$$\text{where} \quad \xi_d(t) = \xi_0(0) + it \; = \; -\frac{i}{2} + i\,t$$

Note $\xi_d(t)$ moves faster than $\xi_0(t)$ towards real axis

Inner scale and singular effects on real axis

When $\xi - \xi_d(t) \; = \; O(\mathcal{B}^{1/3}), t = O_s(1),$

$$G(\xi, t) \; \sim \; t \, M^{-2} \left\{ \mathcal{B}^{-1/3} [-i(\xi - \xi_d(t))] t^{1/6} \right\},$$

where $M(\eta)$ satisfies

$$-\frac{1}{2} M + \frac{1}{6} \eta M' = \left[-\frac{1}{2} + M''' \right] M^3 \; \text{with matching condition}$$

The inner ODE admits $(\eta - \eta_s)^{2/3}$ singularities; corresponding to $(\xi - \xi_s)^{-4/3}$ singularity for G, clustered near $\xi = \xi_d$

These singularities affect evolution on real ξ axis before $\xi_0(t)$ reaches real axis !

Similar singular effects occur for Hele-Shaw cell for small ϵ. Other regularizations cause similar effect

Conclusion

The classic zero surface tension model is structurally unstable to small regularizing effects such as surface tension.

This structural instability for steady shapes implies that most zero surface tension shapes are physically irrelevant.

Near structurally unstable system are unpredictably sensitive to other small effects. No universality independent of regularization

Structural instability in time evolution problems occur: $\lim_{\epsilon \to 0^+} u(t; \epsilon) \neq u(t; 0)$ even for $t < T_s$, the singular time for $u(t; 0)$. Strong evidence that this is the case, though mathematical proof of this part is an open problem.

Steadily translating bubbles in a channel under the action of a pressure gradient are nonlinearly stable with a shrinking basin of attraction as $\epsilon \to 0$ for large sidewall distances.