THE BAR SPECTRAL SEQUENCE CONVERGING TO $h_*(SO(n))$

Vidhyānāth Rao

Abstract. We study the bar spectral sequence converging to $h_*(SO(2n{+}1))$, where h is an algebra theory over BP. The differentials are determined completely if $h = P(l)$ and $n < 2^l$. These results will be used in a future paper on the Morava K-theories of $SO(2n{+}1)$, with no restriction on n. As another application, we determine $BP_*(Spin(7))$ including much of its algebra structure.

1. Introduction.

Although BP-theory has been very useful in stable homotopy theory, it has not become equally popular in the unstable world. One hurdle has been computing the BP-theories of well known spaces. The Morava K-theories have been proposed as substitutes [RW].

Among Lie groups, the only computation is by Yagita [Ya], who computes the BP-cohomology (at odd primes) of the exceptional Lie groups. The orthogonal groups or the Spin groups (for BP-theory at 2) would be the logical next step. In hindsight, Petrie's paper [Pe] is seen to compute only their K-theory. This paper represents the second step in an assault on this problem. The example of $Spin(7)$ in the last section will give an idea of the complications involved.

Our approach is inspired by [Pe]. The first step, completed in [Ra1], was to get a good grip on $BP_*(\Omega SO(q))$. Next we look at the Bar spectral sequence, which we will refer to as the Bss,

$$\mathrm{Tor}_{**}^{h_*(\Omega SO(q))}(h_*, h_*) \Rightarrow h_*(SO(q)),$$

where h is a BP-algebra theory and q is odd. Our main aim is to determine the differentials of this spectral sequence for the case $h = P(l)$ and $q < 2^{l+1}$ (see the next section for a description of $P(l)$). This is done in Theorem 5.1, and will be used in [Ra2] to deal with the case of h being a Morava K-theory (with no restriction on q). As another application, we compute the Bss converging to $BP_*(Spin(7))$. This is done by mapping it into the Bss for $P(2)_*(SO(7))$. As a consequence we obtain the BP_*-module structure of $BP_*(Spin(7))$, as well as nearly complete information about the algebra structure.

We concentrate on the case of odd q, because, for any BP-algebra theory h, we have the following short exact sequence of bialgebras:

$$h_*(SO(2n+1)) \longrightarrow h_*(SO(2n+2)) \xrightarrow{g} E\,,$$

where E is an exterior algebra on a single primitive element of degree $2n+1$, and g splits; we may think of E as $h_*(\mathbb{S}^{2n+1})$ endowed with the only algebra structure possible. See Section 4 for the details.

There is a mild irritant in all this, namely the fact that $P(l)$ are *non-commutative* ring spectra. The next two sections recall the precise details and explain why this does not affect the bar spectral sequence. [But sadly the extension problems are affected]. A description of $BP_*(\Omega SO(2n+1))$ and of a resolution of BP_* over it, due to Petrie [Pe], will be found in the fourth section. Section 5 contains the main result and its proof. In the last section we calculate $BP_*(Spin(7))$.

We refer the reader to [Wi] for an exposition of the neccessary background information concerning BP and related spectra.

Throughout this paper, BP will refer to the 2-local theory.

2. Products in $P(k)$-theories.

In this section we collect a few facts concerning products in the $P(k)$ theories at the prime 2. These are results of Würgler [Wu]. These also apply to Morava K-theories. The source of difficulty is that these are not commutative ring spectra. However they have only two viable products and the commutator is explicitly known. Furthermore they satisfy the following substitute for full commutativity: We say that a ring spectrum h is quasi-commutative if for any $x \in h_*(X)$, $y \in h_*(Y)$ and $a \in h_*$, $x \wedge (ay) = a(x \wedge y) = (ax) \wedge y$ in $h_*(X \wedge Y)$. We will assume that all ring spectra that we meet with are quasi-commutative. This is true for all 'interesting' BP-algebra theories by the results of Würgler.

Recall that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ where v_i has degree $2(2^i-1)$ [Qu]. For the sake of definiteness, we will take the v_i's to be the Hazewinkel generators [Hz]. Let $k > 0$. Using the Sullivan-Baas technique ([Su], [Bs]), we can kill $2, v_1, \dots, v_{k-1}$. The resulting spectrum is $P(k)$, a BP-module spectrum with $P(k)_* = \mathbb{Z}/2[v_k, v_{k+1}, \dots]$. (This and the next few statements are due to Jack Morava. See [JW], or [Wu], for a source in print.) It is consistent to define $P(0) = BP$ and $P(\infty) = H\mathbb{Z}/2$. For $0 \le i < k$ there are the 'Milnor' operations Q_i, of degree $2^{i+1}-1$, in $P(k)^*P(k)$. Define Q_i to be 0 if $i \ge k$. We have a direct system $P(0) \to P(1) \to \dots$, and the Q_i's are preserved by these maps. The reader should be aware that we will use the Q_i's as *homology* operations.

The source for this paragraph is [Wu]. For $0 < k < \infty$, there are two products on $P(k)$ which make $P(k)$ into a BP-algebra theory. Let μ be one of these two products; then the other product is $\tau\mu$ where $\tau\colon P(k) \wedge P(k) \to P(k) \wedge P(k)$ is the switch map. The commutator of $P(k)$ is $\mu - \tau\mu = 1 + v_k\mu(Q_{k-1} \wedge Q_{k-1})$.

Proposition 1. (1). *Suppose that $P(k)_*(X)$ and $P(k)_*(Y)$ are flat over $P(k)_*$. Then the diagonal of $P(k)_*(X \times Y)$ is given by*

$$\Delta(x \times y) = (1 + v_k\, id_X \otimes Q_{k-1} \otimes Q_{k-1} \otimes id_Y)(\Delta(x) \otimes \Delta(y)).$$

(2). *Suppose that $P(k)_*(X) \to H_*(X, \mathbb{Z}/2)$ is onto. Then $P(i)_*(X)$ is a cocommutative coalgebra if $i > k$. If, in addition, X is an H-space, then $P(i)_*(X)$ is a Hopf algebra.*

Proof. Consider DIAGRAM A, where $P = P(k)$ and

$$\theta(a \otimes b \otimes c \otimes d) = a \otimes b \otimes c \otimes d + v_k a \otimes Q_{k-1} b \otimes Q_{k-1} c \otimes d \;;$$
$$\phi(a \otimes b \otimes c \otimes d) = a \times b \times c \times d + v_k\, a \times Q_{k-1} b \times Q_{k-1} c \times d \;.$$

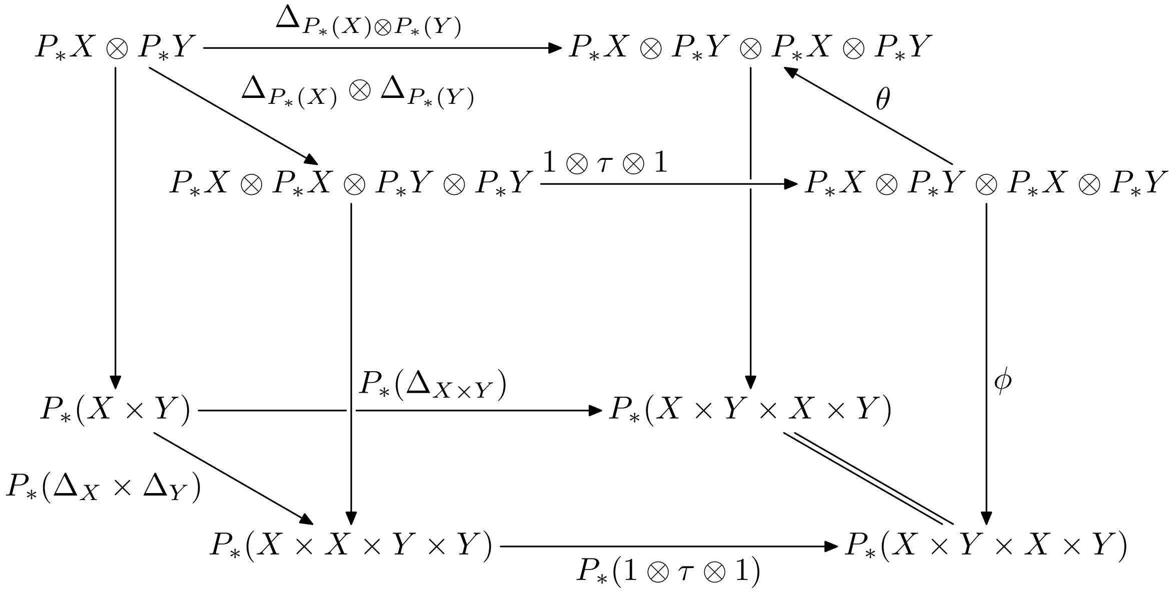

FIGURE A

All faces except the top and the front commute for trivial reasons; the front commutes by the cited formula of Würgler. The vertical arrows are isomorphisms by the flatness assumption. Thus the top face commutes, proving (1).

Under the assumption of (2), $P(k)_*(X)$ maps *onto* $P(i)_*(X)$ and so $Q_j = 0$ if $j \geq k$. Now a standard argument together with the commutator formula and (1) completes the proof. ∎

3. The Bar Spectral Sequence.

In this section we summarize the basic facts about the bar spectral sequence that we need. According to [RW, *pp.* 704-705], where the reader will find a short history and further references, the idea of this spectral sequence goes further back than [RS]. The proofs usually given work only for commutative ring spectra. As we need to deal with $P(k)$, we outline a proof that works for any BP-algebra spectrum. The ideas behind this proof go back to [Mi] and [Se]. See also [Ma, *p.* 80].

Theorem 1. (1). *Let X be a path-connected space whose loop-space has no 2-torsion in homology. Let h be a BP-module spectrum. Then there is a (first and fourth quadrant) spectral sequence*

$$E^2_{**}(X,h) = \mathrm{Tor}^{BP_*(\Omega X)}_{**}(h_*, BP_*) \Rightarrow h_*(X)\,.$$

(2). *If h is a BP-algebra spectrum, $E^2_{**}(X,h) = \mathrm{Tor}^{h_*(\Omega X)}_{**}(h_*, h_*)$. If Y is another space such that $H_*(\Omega Y, \mathbb{Z}_{(2)})$ is torsion-free, then there is a product pairing*

$E^r_{**}(X,h) \otimes E^r_{**}(Y,h) \to E^r_{**}(X \times Y, h)$ *that is compatible with the algebraic pairing of* Tor *and the cross-product pairing* $h_*(X) \otimes h_*(Y) \to h_*(X \times Y)$.

(3). *Let G be a compact connected Lie group and let h be a BP-algebra spectrum. Then $E^r_{**}(G,h)$ is a spectral sequence of commutative algebras. If E^r is h_*-flat for $r \le r_0$ then it is a spectral sequence of bicommutative Hopf-algebras upto stage r_0.*

Outline of a Proof. Let ΛX be the Moore loop space of X. Then $B(*, \Lambda X, *)$, the geometric bar construction on ΛX [Se], is a simplicial space whose geometric realization is naturally weak homotopy equivalent to X [Ma, *p.* 84]. The spectral sequence of this simplicial space is the bar spectral sequence. As $BP_*(\Omega X)$ is flat over BP_*, the universal coefficient spectral sequence [Ad] implies that $h_*(\Omega X) = h_* \otimes BP(\Omega X)$. It follows from this that the E^1-term is the algebraic bar resolution: $E^1_{**}(X,h) = B(h_*, BP_*(\Omega X), BP_*)$. Note that this gives a canonical and natural E^1 term.

Let h be a BP-algebra theory. The homomorphism $h_* \otimes BP_*(\Omega X) \to h_*(\Omega X)$ is a homomorphism of bialgebras by our standing assumption of quasi-commutativity. Thus

$$\begin{aligned} E^1_{**}(X,h) &= B(h_*, BP_*(\Omega X), BP_*) \\ &\cong h_* \otimes_{BP_*} B(BP_*, BP_*(\Omega X), BP_*) \\ &\cong B(h_*, h_*(\Omega X), h_*). \end{aligned}$$

We have maps

$$B(*, \Omega(X \times Y), *) \xrightarrow{g} B(*, \Omega X \times \Omega Y, *) \xrightarrow{f} B(*, \Omega X, *) \times B(*, \Omega Y, *)$$

induced by the appropriate projections. Now g is a levelwise homotopy equivalence and $|f|$, the geometric realization of f, is a homeomorphism of filtered spaces whose inverse respects the filtrations. This gives our pairing. Note that the homomorphism of the E^1-terms

$$B(h_*, h_*(\Omega X) \otimes h_*(\Omega Y), h_*) \longrightarrow B(h_*, h_*(\Omega X), h_*) \otimes B(h_*, h_*(\Omega Y), h_*),$$

induced by f^{-1}, is chain-homotopic to the shuffle map: This is standard for BP and follows in general by the previous paragraph.

By a classical theorem of Bott [Bt], $H_*(\Omega G, \mathbb{Z})$ is torsion-free if G is a compact connected Lie group. Thus (1) and (2) apply. The algebra structure comes from

$$E^r_{**}(G,h) \otimes E^r_{**}(G,h) \to E^r_{**}(G \times G, h) \to E^r_{**}(G,h),$$

where the last homomorphism is induced by the group product. Since ΩG is homotopy commutative, $BP_*(\Omega G)$ and $h_*(\Omega G) = h_* \otimes BP_*(\Omega G)$ are bicommutative Hopf-algebras. By the previous paragraph, the product on the E^2 term is induced by the algebraically defined product on the E^1-term. The last is just the algebraic bar construction and is commutative upto chain homotopy. Thus E^r_{**} is commutative. A similar argument shows that the pairing $E^r_{**}(G,h)\otimes E^r_{**}(H,h)\to E^r_{**}(G\times H,h)$ is an algebra homomorphism. This implies the last statement of the theorem by a standard argument. ∎

Remarks: If $h=P(k)$, this theorem reflects the fact the Milnor operations act trivially on the canonical E^1-term and so on E^r for all r.

Let h be a BP-algebra spectrum. Note that the edge homomorphism $E^2_{0*}\to h_*(X)$ is just $h_*(pt)\to h_*(X)$. We also have the 'algebraic homology suspension' (see [Ca]; a definition is given in the next paragraph)

$$\widetilde{h}_*(\Omega X)\longrightarrow \mathrm{Tor}^{h_*(\Omega X)}_{1*}(h_*,h_*)=E^2_{1*}(X,h)\ .$$

It is an h_*-module homomorphism whose kernal is the module of decomposables of $\widetilde{h}_*(\Omega X)$, *i.e.* $\{\sum_i x_i\,y_i \mid x_i,y_i\in\widetilde{h_*}(\Omega X)\}$. Composing with the edge homomorphism

$$E^2_{1*}(X,h)\longrightarrow E^\infty_{1*}(X,h)\longrightarrow \widetilde{h}_{*+1}(X)$$

gives the (usual) homology suspension.

Let $x\in\widetilde{h}_*(\Omega X)$. Take any $h_*(\Omega X)$-projective resolution $(R_{\bullet *},d)$ of h_*, with $R_{0*}=h_*(\Omega X)$. By exactness, there is a $y\in R_{1*}$ such that $d(y)=x$. The homology class represented by $1\otimes y\in h_*\otimes_{h_*(\Omega X)}R_{\bullet *}$ is independent of choices and is the suspension of x. (In the algebraic bar resolution, we may pick $[x]$ to play the role of y, whence the last sentence of the previous paragraph. See also [Ka, Section 3].)

4. $BP_*(\Omega SO(2n+1))$ and related matters.

Throughout this section, h will denote a BP-algebra spectrum. As h is complex oriented, there is a formal group law $F(x,y)\in h_*[[x,y]]$ [Qu]. The [2]-series is $[2](x)=F(x,x)$. The $[-1]$-series is the formal group inverse, uniquely defined by $F(x,[-1](x))=0$. As the coefficients of these series exist already in BP_*, we may suppress h from the notation.

Let $Q_n=SO(n+2)/(SO(2)\times SO(n))$ be the generating variety for the homology of $\Omega_0 SO(n+2)$; *i.e.* there is a map $Q_n\to\Omega_0 SO(n+2)$ such that $H_*(Q_n,\mathbb{Z})$ maps monomorphicaly into $H_*(\Omega_0 SO(n+2),\mathbb{Z})$ and the image of the former generates the latter as an algebra [Bt]. Note that the direct limit of Q_n is $\mathbb{C}P^\infty$. Let x be the Conner-Floyd Chern class of the canonical complex line bundle over Q_n. Then $MU\mathbb{Q}^*(Q_{2n-1})=MU\mathbb{Q}^*[x]/(x^{2n})$. Denote the dual basis, with respect to $\{x^i\mid 0\le i<2n\}$, of $MU\mathbb{Q}_*(Q_{2n-1})$ by ${\beta'}_0,\beta_1,\ldots,\beta_{2n-1}$. We identify the elements of $h_*(Q_{2n-1})$ with their images in $h_*(\Omega SO(2n+1))$. Let β_0 be the generator of $\widetilde{MU}_0(\Omega SO(n))$ such that $\beta_0^2=2\beta_0$. For $0\le j<2n$, define $\alpha_j=\sum_{i=0}^{j}c_{j-i}\,\beta_i$ where c_i is the coefficient of x^{i+1} in $[2](x)$. It follows from [Ra1] that $\alpha_j\in MU_*(\Omega SO(2n+1))$ if $j<2n$. [The definition of α_j given above differs slightly from that of [Ra1]: α_j, as defined here, lives in $\widetilde{MU}_*(\Omega SO(2n+1))$ rather than in $MU_*(\Omega_0 SO(2n+1))$.] We will denote the images of the α's and β's in $h_*(\Omega SO(2n+1))$ by the same symbols. These elements are independent of n in the sense that if $n < q$, then $\beta_i \mapsto \beta_i$ for $0 \le i < n$ and $\alpha_j \mapsto \alpha_j$ for $0 \le j < 2n$ under

$$h_*(\Omega SO(2n+1)) \xrightarrow{\text{incl}_*} h_*(\Omega SO(2q+1))\,.$$

In the case $h = BP$, the next proposition follows from the above remarks, the results of [Bt] and the Atiyah-Hirzebruch spectral sequence. See also [Pe, Section I.2] and [Ra1]. The general case follows by the universal coefficient spectral sequence.

We say that an algebra B is simply generated over another algebra A by a possibly finite set $\{x_1, x_2, \dots\}$ if $\{x_{i_1} \dots x_{i_q} \mid i_1 < \cdots < i_q\}$ is a basis for B over A. Fix n. Unless otherwise specified, $m = [n/2]$ and $a = [(n-1)/2]$ (here $[x]$ refers to the integer part of x!).

Proposition 1. *$h_*(\Omega SO(2n+1))$ is simply generated by $\{\beta_i \mid 0 \le i \le a\}$ over*

$$h_*[\{\beta_i \mid a < i < n\} \cup \{\alpha_{2j+1} \mid m \le j < n\}]\,;$$

$h_*(\Omega SO)$ is simply generated by $\{\beta_i \mid 0 \le i\}$.

For $1 \le i \le a$, we can obtain β_i^2 from

$$\left(1 + \sum_{i=1}^{a} \beta_i t^i\right)\left(1 + \sum_{i=1}^{a} \beta_i ([-1]t)^i\right) = 1 \pmod{t^{2a}} \tag{2}$$

[Ra1] (see also [Bk, Proposition 3.3]). By [Mr],

$$[-1]t = t + v_k\, t^{2^k} \pmod{t^{2^{k+1}}} \text{ in } P(k)_*.$$

It follows that $\beta_i^2 = 0$ in $P(k)_*(\Omega SO(2n+1))$ if $2i < 2^k$.

Warning: The generators we use are quite different from the ones used in [Pe], although they have the same reduction to ordinary homology.

The following resolution of h_* over $h_*(\Omega SO(2n+1))$, taken from [Pe, Section I.3], will be used in Section 6 as well as in [Ra2]. Let

$$N = h_*(\Omega SO(2n+1)) \otimes \bigotimes_{i=0}^{m+a} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{a} \Gamma(\gamma_i)\ ,$$

where the bidegrees of $\overline{\beta}_i$, $\overline{\alpha}_{2i-1}$ and γ_i are $(1, 2i+1)$, $(1, 4i-1)$ and $(2, 4i)$ respectively; and $\Gamma(t)$ is the algebra of divided powers on t. Define a differential d^1 of bidegree $(-1, 0)$ by: $d^1 = 0$ on $h_*(\Omega SO(2n+1))$; $d^1(\overline{x}) = x$ for $x = \beta_i$ or α_{2j+1}; and

$$d^1(\gamma_{ks}) = \begin{cases} 2\overline{\beta}_0 \gamma_{0,s-1} & k = 0 \\ \left(\sum b_{ki}\overline{\beta}_i \ + \ \sum d_{kij}\beta_i\overline{\beta}_j\right)\gamma_{k,s-1} & 1 \le k \le a. \end{cases}$$

where γ_{ks} is the s^{th} divided power of γ_k, and the b's and d's are defined by

$$\beta_k^2 = \sum_{0<i} b_{ki}\beta_i \ + \sum_{0<i<j} d_{kij}\beta_i\beta_j \ (1 \le k \le a) \ .$$

Extend d^1 to all of N by requiring it to be a derivation. The proof in [Pe, Section I.3] carries over almost *verbatim* and shows that (N, d^1) is a resolution if $h = BP$. The general case follows because

$$h_*(\Omega SO(2n+1)) = h_* \otimes_{BP_*} BP_*(\Omega SO(2n+1)) \ .$$

We will call $h_* \otimes_{h_*(\Omega SO(2n+1))} N$ the Petrie complex. Given

$$x = \sum_{i=0}^{n-1} a_i\beta_i + \sum_{j=m}^{n-1} b_j\alpha_{2j+1} \in h_*(\Omega SO(2n+1)), \text{ where } a_i, b_j \in h_*,$$

we denote by $\overline{x}$ the element $\sum a_i\overline{\beta}_i + \sum b_j\overline{\alpha}_{2j+1}$ in the Petrie Complex.

We record the following observations for later use, both in this paper and in [Ra2]. Firstly, the definition of the algebraic homology suspension, combined with the Petrie resolution, shows β_i and α_j are mapped to $\overline{\beta}_i$ and $\overline{\alpha}_j$ respectively. Thus the suspension of $x \in \widetilde{h}_*(\Omega SO(2n+1))$ is represented by $\overline{x}$. Secondly, if $n < q$, then there is a canonical homomorphism of the Petrie resolutions over

$$h_*(\Omega SO(2n+1)) \xrightarrow{\text{incl}_*} h_*(\Omega SO(2q+1)) \ .$$

It sends γ_{is} to γ_{is} if $0 \le i \le a$, and $\overline{\beta}_i$ to $\overline{\beta}_i$ if $0 \le i < n$. Also, $\overline{\alpha}_{2j+1}$ goes to $\overline{\alpha}_{2j+1}$ if $q \le 2j+1 < 2n$, and to $\sum_i c_{2j+1-i}\overline{\beta}_i$ if $n \le 2j+1 < q$. By the previous observation, this sum is just $\overline{\alpha}_{2j+1}$ in $E^2_{1*}(SO(2q+1), h)$.

Lemma 3. *If $n < q$, $r < \min(2^{l+1}-1, \ q)$ and $j = r - 2^l + 1 \ge 0$, then*

$$E^2_{1*}(SO(2n+1), P(l)) \xrightarrow{\text{incl}_*} E^2_{1*}(SO(2q+1), P(l))$$

maps $\overline{\alpha}_r$ to $v_l\overline{\beta}_j$.

Proof. Let c_i be the coefficient of x^{i+1} in the $P(l)$ [2]-series. Then $c_i = 0$ if $i < 2^{l+1}-1$ and $i \ne 2^l - 1$; further, $c_{2^l-1} = v_l$. This can be shown, for example, using [Wi, Lemma 3.17]. By the observations above, $\overline{\alpha}_r \mapsto \sum_i c_i\overline{\beta}_{r-i} = v_l\beta_j$. ∎

By the remark following (2), the differential of the Petrie complex is trivial if $n < 2^k$ and $h = P(k)$. In this case we may identify the E^2-term with the Petrie complex. Also

$$\text{(4)} \qquad \text{Tor}_{**}^{P(k)_*(\Omega SO(2n+1))} = \bigotimes_{i=a+1}^{m+a} E(\overline{\beta}_i) \otimes \bigotimes_{i=m}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=0}^{a} \Gamma(\overline{\beta}_i)$$

as *Hopf algebras*, if $n < 2^k$. With these identifications, $\gamma_{is} = \gamma_{2s}(\overline{\beta}_i)$, the $(2s)$-th divided power of $\overline{\beta}_i$.

Remark: The comment in the introduction concerning $h_*(SO(2n+2))$ can be justified as follows: By the results of [Bt] and [Ra1],

$$MU_*(SO(2n+2)) = MU_*(SO(2n+1))[\beta_n - \epsilon]\,,$$

where ϵ is a spherical element of $MU\mathbb{Q}_*(\Omega_0 SO(2n+2))$. Furthermore, $\beta_n - \epsilon$ maps to a generator of $MU_*(\Omega\mathbb{S}^{2n+1})$. It follows that for any BP-algebra spectrum h,

$$\mathrm{Tor}_{**}^{h_*(SO(2n+2))}(h_*, h_*) = \mathrm{Tor}_{**}^{h_*(SO(2n+1))}(h_*, h_*) \otimes E(\overline{\beta_n - \epsilon})\,,$$

where $\overline{\beta_n - \epsilon}$ is the algebraic homology suspension of $\beta_n - \epsilon$. Thus,

$$E_{**}^r(SO(2n+2), h) = E_{**}^r(SO(2n+1), h) \otimes E(\overline{\beta_n - \epsilon})\ \forall\ 2 \le r \le \infty\,.$$

5. The Main Result.

Fix a gound ring of charateristic 2. Let $\Gamma_k(t)$ denote the divided power algebra of height k on t. This is the dual of the primitively generated truncated polynomial algebra $P(x)/(x^{2^k})$. Given elements α and β of bidegrees $(c, 2^k(a+b)-c-1)$ and (a,b) respectively, define a spectral sequence of Hopf algebras by

$$E_{**}^r(\beta,\alpha) = \begin{cases} E(\alpha)\otimes\Gamma(\beta) & r \le 2^k a - c \\ \Gamma_k(\beta) & r > 2^k a - c \end{cases}$$
$$d^r(\gamma_j(\beta)) = \begin{cases} 0 & r < 2^k a - c \text{ or } j < 2^k \\ \alpha\gamma_{j-2^k}(\beta) & r = 2^k a - c \text{ and } j \ge 2^k. \end{cases}$$

Fix n. Let $m = [n/2]$ and $a = [(n-1)/2]$. For $i \le a$, let $j(i)$ be the unique integer, neccessarily odd, such that $(j(i)+1)/(2i+1)$ is a power of 2 and $n \le j(i) < 2n$.

Theorem 1. *The Bar spectral sequence $E_{**}^*(SO, H\mathbb{Z}/2)$ collapses at the E^2 level. If $n < 2^l$, then*

$$E_{**}^*(SO(2n+1), P(l)) = \bigotimes_{i=1}^{m} E(\overline{\beta}_{a+i}) \quad\otimes\quad \bigotimes_{i=0}^{a} E_{**}^*(\overline{\beta}_i, \overline{\alpha}_{j(i)})\,.$$

There are no Hopf algebra extension problems if $n < 2^{l-1}$.

Proof. By 4.4, $E_{**}^2(SO, H\mathbb{Z}/2) = \bigotimes_{i=0}^{\infty}\Gamma(\overline{\beta}_i)$. This has the same Poincare polynomial as $H_*(SO, \mathbb{Z}/2)$. So $d^r = 0$ if $r \ge 2$.

The case of $SO(2n+1)$, n finite, will be broken up into three subcases: $l = \infty$, $n < 2^{l-1}$ and $2^{l-1} \le n < 2^l$. The proofs are similar, but with increasing complexity in details. In all three, one starts by observing that, by 4.4, the E^2-term is as described. Next one shows, by induction on r, that d^r and hence E^{r+1} are as described. This part is similar to [RW, Section 6] and we will only outline the argument.

In the case of $P(\infty) = H\mathbb{Z}/2$, the Bss is automatically a spectral sequence of Hopf algebras. Fix i and let $u = 2(j(i)+1)/(2i+1)$. Suppose that $d^r(\gamma_j(\overline{\beta}_i)) \neq 0$ for some j. Select the smallest such j. Since d^r is a Hopf algebra derivation, $\gamma_j(\overline{\beta}_i)$ must be indecomposable; its differential must be primitive and map to 0 in $E^r_{**}(SO, H\mathbb{Z}/2)$. Thus j is a power of 2, $r = j-1$ and $d^r(\gamma_j(\overline{\beta}_i)) = \overline{\alpha}_{j'}$ for some j' for some odd j' between n and $2n-1$. Dimensional considerations show that $j' = j(i)$ and $r = u-1$. As in [RW, Proof of Lemma 6.9], we can show that

$$d^{u-1}(\gamma_s(\overline{\beta}_i)) - \gamma_{s-u}(\overline{\beta}_i)\overline{\alpha}_{j(i)}$$

is primitive. It must be zero as all the primitives are in E^*_{1*}.

To complete the proof for the case $l = \infty$, we must know that for any i, not all $\gamma_j(\overline{\beta}_i)$ can be permanent cycles. No $\gamma_j(\overline{\beta}_i)$ bounds, because it does not bound in $E^*_{**}(SO, H\mathbb{Z}/2)$. The claim follows as the E^∞-term is finite.

Now let l be finite and $n < 2^{l-1}$. The argument needs two modifications: We use Theorem 3.1 to prove inductively that (E^r, d^r) is a differential Hopf algebra. We determine $d^{u-1}(\gamma_u(\overline{\beta}_i))$ by mapping it into the Bss for $H\mathbb{Z}/2$. Note that $P(l)_i = 0$ if $0 < i < 2(2^l - 1)$ and so

$$E^*_{*t}(SO(2n+1), P(l)) = E^*_{*t}(SO(2n+1), \mathbb{Z}/2)$$

if $t \leq 4n < 2^{l+1} - 2$.

We need the following facts about $H(SO(q+1), \mathbb{Z}/2)$. The reader is referred to [Mo] for the details. Let b_i be the non-trivial element of $H_i(\mathbb{R}P^q, \mathbb{Z}/2)$ for $1 \leq i \leq q$. Map $\mathbb{R}P^q$ into $SO(q+1)$ by sending a line in $\mathbb{R}^{q+1}$ to the reflection in the hyperplane perpendicular to that line composed with the reflection in a fixed hyperplane. Identify b_i with its image in $H_i(SO(q+1), \mathbb{Z}/2)$ under the induced homomorphism. Then $H_i(SO(q+1), \mathbb{Z}/2)$ is the exterior algebra on $b_1, \ldots, b_q$. Furthermore, if r is even and $s = r - 2^l + 1 > 0$, then $Q_{l-1}(b_r) = b_s$ (we are doing *homology* operations!)

Let $2^{l-1} \leq n < 2^l$. In this case, we need to rule out the possibility that $d^{u-1}(\gamma_u(\overline{\beta}_i)) = \overline{\alpha}_{j(i)} + v_l\overline{\beta}_j$ for some i with $j = j(i) - 2^l + 1 \geq 0$. Assume that this occurs. Pick a $q > 2n$. By Lemma 4.3 and the naturality of the Bss, $d^{u-1}(\gamma_u(\overline{\beta}_i)) = 0$ in $E^*_{**}(SO(2q+1), P(l))$. But then $\gamma_u(\overline{\beta}_i)$ is a permanent cycle in the latter for filtration reasons. Take an element in $P(l)_*(SO(2q+1))$ to which $\gamma_u(\overline{\beta}_i)$ converges, and let x be its reduction to $H_*(SO(2q+1), \mathbb{Z}/2)$. Then x is indecomposable and so, in the notation of the previous paragraph, $x = b_{u(2i+1)}$ modulo the decomposables of $\widetilde{H}_*(SO(2q+1), \mathbb{Z}/2)$. Now $Q_l(x) = 0$ because x is in the image of $P(l)_*(SO(2q+1))$. But $u(2i+1) - 2^{l+1} + 1 = 2j+1 > 0$ and $Q_l(b_{u(2i+1)}) = b_{2j+1}$ is indecomposable. This contradiction completes the determination of the terms of the spectral sequence.

Finally, we consider the algebra extension problem. Note that the additive extensions are trivial as the E^∞-term is free over $P(l)_*$. By the structure theory of

Hopf algebras [MM], the Hopf algebra extension will be trivial if $P(l)_*(SO(2n+1))$ is a bicommutative Hopf algebra. Assume that $n < 2^{l-1}$. Using our knowledge of the E^∞-terms, we see that $P(l-1)_*(SO(2n+1))$ maps *onto* $H_*(SO(2n+1), \mathbb{Z}/2)$. By Proposition 2.1, $P(l)_*(SO(2n+1))$ is a cocommutative Hopf algebra.

Using the E^∞-term of the Bss, one can show that

$$\widetilde{P(l)}_i(SO(2n+1)) \longrightarrow \mathbb{Z}/2 \otimes_{P(l)_*} \widetilde{P(l)}_i(SO(2n+1))$$

is an isomorphism if $i \leq 2(2^l - 1)$, and that for all i,

$$\mathbb{Z}/2 \otimes_{P(l)_*} \widetilde{P(l)}_i(SO(2n+1)) \cong \widetilde{H}_i(SO(2n+1), \mathbb{Z}/2)\,.$$

Furthermore, $H_*(SO(2n+1), \mathbb{Z}/2)$ is commutative and the algebra generators of $E^\infty(SO(2n+1), P(l))$ have total degree $\leq 2n$. Putting all this together, we see that $P(l)_*(SO(2n+1))$ is commutative if $n < 2^{l-1}$. ∎

6. An example: $BP_*(Spin(7))$.

In this section we will compute the $BP_*(Spin(7))$ as a BP_*-module. We will also get almost all of the algebra structure.

We start by sketching the analogs of the statements in Section 4. We make the identifications

$$h_*(\Omega Spin(2n+1)) = h_*(\Omega_0 SO(2n+1)) \subset h_*(\Omega SO(2n+1))\ .$$

It follows from the descriptions of these given in [Ra1] that $\alpha_i' = \alpha_i - c_i\beta_0$ is in $\widetilde{BP}_*(\Omega Spin(2n+1))$ if $i < 2n$. Further, the subcomplex

$$N' = h_*(\Omega Spin(2n+1)) \ \otimes \ \bigotimes_{i=1}^{m+a} E(\overline{\beta}_i) \ \otimes \ \bigotimes_{i=m}^{n-1} E(\overline{\alpha}'_{2i+1}) \ \otimes \ \bigotimes_{i=1}^{a} \Gamma(\gamma_i)\ ,$$

of the Petrie resolution over $h_*(\Omega SO(2n+1))$, is a resolution of h_* over $h_*(\Omega Spin(2n+1))$. Recall that $\overline{\alpha}'_{2j+1} = \overline{\alpha}_{2j+1} - c_{2j+1}\,\overline{\beta}_0$. The term 'Petrie complex' will also be applied to $h_* \otimes_{h_*(\Omega Spin(2n+1))} N'$.

As in the proof of Lemma 4.3, we see that, if $i < 2^{l+1} - 2$ and $i \neq 2^l - 1$, then $\overline{\alpha}_i' = \overline{\alpha}_i$ in $E^2_{1*}(SO(2n+1), P(l))$. Thus, for such i,

$$E^r_{**}(Spin(2n+1), P(l)) \xrightarrow{\text{incl}_*} E^r_{**}(SO(2n+1), P(l))$$

takes $\overline{\alpha}_i'$ to $\overline{\alpha}_i$. We can show, by induction on r, that if $n < 2^l$ and $r \leq j(0)$, then the above homomorphism is monomorphic.

As $[-1](t) = -t + v_1\,t^2 \pmod{t^3}$ in $BP_*[[t]]$, $\beta_1^2 = 2\,\beta_2 - v_1\,\beta_1$ in $BP_*(Spin(7))$. Hence $d^1(\gamma_{1i}) = (2\,\overline{\beta}_2 - v_1\,\overline{\beta}_1)\,\gamma_{1,i-1}$ in the Petrie complex for $BP_*(Spin(7))$. Thus, cycles of the latter are

$$E(\overline{\alpha}'_3, \overline{\alpha}'_5) \otimes \left\{ a_0 + b_0\overline{\beta}_1 + c_0\overline{\beta}_2 + \sum_{i \geq 0} (b_i\overline{\beta}_1 + c_i\overline{\beta}_2 + d_i\overline{\beta}_1\overline{\beta}_2)\gamma_{1i} \mid v_1 b_i + 2c_i = 0\,\forall\, i > 0 \right\}.$$

and its boundaries are

$$E(\overline{\alpha}'_3, \overline{\alpha}'_5) \otimes \left\{ \sum_{i\geq 0} (a_i(2\overline{\beta}_2 - v_1\overline{\beta}_1) + d_i\overline{\beta}_1\overline{\beta}_2)\gamma_{1i}) \mid d_i \in (2, v_1) \right\}.$$

Thus the E^2-term is

$$E(\overline{\alpha}'_3, \overline{\alpha}'_5) \otimes \; BP_*\langle 1, \overline{\beta}_1, \overline{\beta}_2, \{\overline{\beta}_1\overline{\beta}_2\gamma_{1i} \,|\, i \geq 0\}\rangle$$

where the second factor is subject to the relations

$$(1) \qquad v_1\overline{\beta}_1 = 2\overline{\beta}_2\ , \qquad a\overline{\beta}_1\overline{\beta}_2\gamma_{1i} = 0 \text{ if } a \in (2, v_1).$$

We can determine d^3 using the fact that

$$P(2)_* \otimes_{BP_*} E^2_{**}(Spin(7), BP) \longrightarrow E^2_{**}(Spin(7), P(2)) \xrightarrow{\text{incl}_*} E^2_{**}(SO(7), P(2))$$

is monomorphic: If $i \geq 2$, then $d^3(\overline{\beta}_1\overline{\beta}_2\gamma_{1i}) = \overline{\beta}_1\overline{\beta}_2\overline{\alpha}'_5\gamma_{1,i-2}$ and d^3 is trivial on the other generators. So the E^4-term is

$$E(\overline{\alpha}'_3) \otimes BP_*\langle 1, \overline{\beta}_1, \overline{\beta}_2, \overline{\alpha}'_5, \overline{\beta}_1\overline{\beta}_2, \overline{\beta}_1\overline{\alpha}'_5, \overline{\beta}_2\overline{\alpha}'_5, \overline{\beta}_1\overline{\beta}_2\gamma_{11}\rangle\,,$$

where the second factor is subject to the relations implied by (1). For degree reasons, all further differentials are trivial.

Next we solve the additive extension problem. The relations, $2\overline{\beta}_2 = v_1\overline{\beta}_1$ and $a\overline{\beta}_1\overline{\beta}_2 = 0$ if $a \in (2, v_1)$, hold in $BP_*(Spin(7))$ for filtration reasons. Let $\overline{\beta}_1\overline{\beta}_2\gamma_{11}$ converge to $x \in BP_*(Spin(7))$. We need to determine only $2x$ and v_1x. For degree and filtration reasons, we must have

$$\begin{aligned} 2x &= f\overline{\beta}_1\overline{\beta}_2 + g\overline{\beta}_1\overline{\alpha}'_3 + h\overline{\beta}_2\overline{\alpha}'_3 + k\overline{\beta}_1\overline{\alpha}'_5\ ; \\ v_1x &= a\overline{\beta}_1\overline{\beta}_2 + b\overline{\beta}_1\overline{\alpha}'_3 + c\overline{\beta}_2\overline{\alpha}'_3 + d\overline{\beta}_1\overline{\alpha}'_5 + e\overline{\beta}_2\overline{\alpha}'_5\ . \end{aligned}$$

Mapping to $P(2)_*(Spin(7))$ we see that $f \in (2, v_1)$. Hence $f\overline{\beta}_1\overline{\beta}_2 = 0$. Since $2\overline{\beta}_2 = v_1\overline{\beta}_1$, we may assume that e is either 0 or 1. Now $a \in BP_8$, $c \in BP_4$ and $d \in BP_2$. So they are divisible by v_1. By changing x if necessary, we may assume that they are 0. Similarly, we can assume that $b = b'v_2$.

Eliminating x from the two equations above and mapping the resulting equation into $BP\mathbb{Q}_*(Spin(7)) = E(\overline{\beta}_1, \overline{\alpha}'_3, \overline{\alpha}'_5)$, where $\overline{\beta}_2 = v_1\overline{\beta}_1/2$, we get

$$(2b'v_2 - gv_1 - hv_1^2/2)\overline{\beta}_1\overline{\alpha}'_3 + (e-k)v_1\overline{\beta}_1\overline{\alpha}'_5 = 0\ .$$

Hence $4b'v_2 = (2g + hv_1)v_1$ and $e = k$. The first equation implies that $b' = 0$, and that for some h', $h = 2h'$ and $g = -h'v_1$. Thus

$$g\overline{\beta}_1\overline{\alpha}'_3 + h\overline{\beta}_2\overline{\alpha}'_3 = h'v_1(2\overline{\beta}_2 - v_1\overline{\beta}_1)\overline{\alpha}'_3 = 0\,.$$

It follows that $2x = e\overline{\beta}_1\overline{\alpha}'_5$ and $v_1 x = e\overline{\beta}_2\overline{\alpha}'_5$, where e is either 0 or 1.

A straightforward computation shows that

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Tor}^{BP_*}_{s*}(\mathbb{Z}/2, E^\infty(Spin(7), BP)) = \begin{cases} 16 & s = 0 \\ 12 & s = 1 \\ 4 & s = 2 \\ 0 & s > 2 \end{cases}.$$

If $e = 0$, then the extension problem would be trivial, and the universal coefficient spectral sequence $\operatorname{Tor}^{BP_*}_{**}(\mathbb{Z}/2, BP_*(Spin(7))) \Rightarrow H_*(Spin(7), \mathbb{Z}/2)$ would imply that $\operatorname{rank}_{\mathbb{Z}/2} H(Spin(7), \mathbb{Z}/2) \geq 24$. This contradicts the known value of 16. So $e = 1$ and we have proved

Proposition 2. *As a BP_*-module, $BP_*(Spin(7))$ is*

$$E(\overline{\alpha}'_3) \otimes BP\langle\, 1, \overline{\beta}_1, \overline{\beta}_2, \overline{\alpha}'_5, x, \overline{\beta}_1\overline{\beta}_2 \,\rangle$$

subject to the relations $2\beta_2 = v_1\beta_1$, $2\beta_1\beta_2 = 0$ and $v_1\beta_1\beta_2 = 0$.

One can find all the products of the generators of $BP_*(Spin(7))$ given above, except for $x\overline{\alpha}'_3$, by using the fact that

$$BP_*(Spin(7)) \to BP\mathbb{Q}_*(Spin(7)) \oplus P(2)_*(Spin(7))$$

is monomorphic. I conjecture that $x\overline{\alpha}'_3 = \overline{\alpha}'_3(x + v_2\overline{\beta}_1\overline{\beta}_2)$ but have been unable to prove it.

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