$\mathit{Spin}(n)$ IS NOT HOMOTOPY NILPOTENT FOR $n \geq 7$.

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Abstract. We show that if $n \geq 7$, then $SO(n)$ and $\mathit{Spin}(n)$ are not homotopy nilpotent. Also $SO(3)$, $SO(4)$ and the exceptional Lie group $G(2)$ are not homotopy nilpotent. This is done by showing that the iterated commutator maps are non-trivial in a suitable Morava K-theory.

0. Introduction.

If X is a finite homotopy associative H-space, then the functor $[\_, X]$ takes its values in the category of groups. [Unless otherwise specified, all spaces referred to will be assumed to have the homotopy type of connected CW-complexes.] One may then ask when this functor takes its values in various subcategories of groups. For example X is homotopy commutative iff $[A, X]$ is abelian for all A. We have the the analogous notions of homotopy nilpotency and homotopy solvability. These properties too can be charaterized in terms of X and its structure maps. We quote the condition for homotopy nilpotency, after setting up some notation: Let μ and σ be the multiplication and the inverse maps of X. Define c_2, the commutator, to be the composite

$$X \times X \xrightarrow{\Delta_{X\times X}} X \times X \times X \times X \xrightarrow{\mathrm{id}\times\mathrm{id}\times\sigma\times\sigma} X \times X \times X \times X \xrightarrow{\mu(\mu\times\mu)} X$$

and define the iterated commutators $c_n : X^n \to X$ inductively by $c_n = c_2(c_{n-1} \times \mathrm{id}_X)$.

Proposition 0.1. [Zb, Lemma 2.6.1] *A finite homotopy associative H-space X is homotopy nilpotent iff c_n is null homotopic for sufficiently large n.*

There is another reason for considering this concept: If A is a finite complex and X is homotopy associative H-space, then $[A, X]$ will be a nilpotent group with nilpotency class at most $\dim A$. One might ask if there is an upper bound for the nilpotency class of $[A, X]$ that is independent of A. If that is the casethen X must be homotopy nilpotent.

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Zabrodsky [Zb, Proposition 2.6.10] proved that the classical Lie groups $SU(n)$, $Sp(n)$ and $SO(2n+1)$ were homotopy solvable. The homotopy nilpotency of $\mathbb{S}^3$ is a classical result that follows from the triviality of the relevant quadruple Whitehead product [Po]. Recently M. J. Hopkins [Hp] found cohomological criteria for a finite H-space to be homotopy nilpotent, and used it to prove that H-spaces with no torsion in homology are homotopy nilpotent. The result is as follows:

Theorem 0.2. [Hp, Theorem 2.1] *Let X be a finite homotopy associative H-space. Then the following conditions are equivalent:*

(1) *X is homotopy nilpotent.*
(2) *$\widetilde{MU}^* * c_n = 0$ for sufficiently large n.*
(3) *For every prime p, $\widetilde{BP}^* * c_n = 0$ for sufficiently large n.*
(4) *For every prime p and positive integer l, $\widetilde{K(l)}_* c_n = 0$ for sufficiently large n.*

Here $K(l)$ is Morava's l-th extraordinary K-theory at the prime p.

Remark: Hopkins states the last condition as "$K(l)_*X$ is a nilpotent Hopf algebra". However, what he actually proves is the version given in the statement of Theorem 0.2. Now, at the prime 2, $K(l)$ is not commutative if $0 < l < \infty$. Hence, the equivalence of Hopkins' condition is problematic.

Hopkins also conjectured that all finite connected homotopy associative H-spaces are homotopy nilpotent. However, it turns out that not even all simply connected Lie groups are homotopy nilpotent.

Theorem 0.3. *If $n \geq 7$, then $Spin(n)$ and $SO(n)$ are not homotopy nilpotent. In addition, $SO(3)$ and $SO(4)$ are not homotopy nilpotent.*

This theorem will be proved by showing that the iterated commutators induce non-trivial homomorphisms in a suitably chosen homology theory. A more detailed outline is as follows: Let $l \geq 1$. There is a periodic homology theory $B(l)$ with a unit $v_l \in B(l)_{2^{l+1}-2}$ and a "Bockstein" operation Q_0 that is a derivation. We will show that there is an element y in $\widetilde{B(l)}_{2^{l+1}-2}SO(2^{l+1}-1)$ such that $y^2 = v_l y$ and $Q_0 y$ is primitive. If $l \geq 2$, then y originates in $\widetilde{B(l)}_* Spin(2^{l+1}-1)$. It will follow that $B(l)_* c_2(Q_0 y \otimes y) = [Q_0 y, y] = v_l Q_0 y$. By induction on s we get

$$B(l)_* c_s(Q_0 y \otimes y \otimes \cdots \otimes y) = v_l^{s-1} Q_0 y\,.$$

It turns out that $Q_0 y$ maps to a non-zero element of $B(l)_* SO(2^{l+2}-4)$. Thus the iterated commutators of $SO(n)$ induce a non-trivial homomorphism in $B(l)_*$-homology if $2^{l+1}-1 \leq n \leq 2^{l+2}-4$.

If $l \geq 2$ and n is of the form $2^{l+2}-3$ or $2^{l+2}-2$, we must replace $Q_0 y$ by a different element. Otherwise the proof is the same.

Since this paper was originally written, N. Yagita [Ya] has proved that for any simply connected compact Lie group G and prime p, G localized at p is homotopy nilpotent if and only if $H_*(G, \mathbb{Z})$ has no p-torsion. This is done by a case by case analysis: $SU(n)$ and $Sp(n)$ have no torsion in homology. This paper handles the case of $Spin(n)$ and $G(2)$. The other exceptional Lie groups are done by Yagita. For $p = 2$, he builds on the case of $G(2)$. For $p = 3$ and $p = 5$, he uses his earlier calculations of the Morava K-theories of the exceptional Lie groups. The second Morava K-theory $K(2)$ is periodic with a unit v_2 of degree $2(p^2 - 1)$. Yagita shows that if G is an exceptional Lie group with p-torsion in homology, then there is an element y in $K(2)_*G$ such that $y^p = v_2 y$ and $Q_0 y \neq 0$. As above, this is enough.

It can be shown that the above result is true for any connected compact Lie group G: Suppose that G is homotopy nilpotent. The universal cover of G is homotopy nilpotent and has the form $\widehat{G} \times \mathbb{R}^n$ where $\widehat{G}$ is compact. By Yagita's result, $\widehat{G}$ has no p-torsion in homology. Suppose that $\pi_1 G$ has p-torsion. Then $\widehat{G}$ has a central element g of order p that lies in the kernal of $\widehat{G} \to G$. Let H be a simple factor of G such that h, the projection of g onto H, is non-trivial. The homotopy nilpotency of G implies that the mod-p complex K-homology of $\widehat{G}/\langle g \rangle$ and of $H/\langle h \rangle$ are commutative. A case by case check of possible H's shows that this is impossible. Hence $\pi_1 G$ has no p-torsion, and so G is p-equivalent, as a space, to $\widehat{G} \times T^n$ where T^n is the n-dimensional torus. The details will appear elsewhere.

1. Preliminaries on Morava K-theories.

Throughout this paper BP will refer to the 2-local theory. For background information on BP and related topics, see [Wi].

It is well known that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ where the degree of v_i is $2(2^i - 1)$ (see, for example, [Qu]). Let $l > 0$. Using the Sullivan-Baas technique ([Su], [Bs]), we can kill $\{v_i \mid i < l\}$ to get $P(l)$, a BP-module theory with coefficient ring $P(l)_* = \mathbb{Z}/2[v_l, v_{l+1}, \dots]$. (This and the next few statements are due to Jack Morava. See [JW] for a source in print.) Inverting v_l gives $B(l) = v_l^{-1} P(l)$. For any space X, $B(l)_* X$ is free as a $B(l)_*$-module [JW]. $P(l)$ and $B(l)$ can be made into BP-module spectra in a canonical manner. There are maps

$$BP \to P(1) \to \cdots \to P(l) \to P(l+1) \to \cdots \to H\mathbb{Z}/2$$

of BP-module spectra. We will let $P(0)$ and $P(\infty)$ denote BP and $H\mathbb{Z}/2$ respectively.

For any l, $P(l) \xrightarrow{v_l} P(l) \to P(l+1)$ is a cofibration sequence of spectra. In particular, the kernal of the homomorphism $P(l)_* X \to P(l+1)_* X$ is $v_l P(l)_* X$.

For $0 \leq i < l$, there are maps $Q_i \colon P(l) \to P(l)$, of degree $2^{i+1} - 1$, that cover the Milnor Bocksteins in ordinary homology. The homology operations induced by these maps will also be denoted by Q_i. These are respected by the maps $P(l) \to P(m)$ with the convention that $Q_i = 0$ on $P(l)$ if $i \geq l$.

We will make use of the following well-known result (see [JW, Section 4]):

Proposition 1.1. *For any space X, and $l > 0$, the following are equivalent:*

1. *If $l < m \leq \infty$, then $P(m)_* X \cong P(m)_* \otimes_{P(l)_*} P(l)_* X$.*
2. *The homomorphism $P(l)_* X \to H_*(X, \mathbb{Z}/2)$ is surjective.*
3. *The Atiyah-Hizrebruch spectral sequence $P(l)_* \otimes H(X, \mathbb{Z}/2) \Rightarrow P(l)_* X$ collapses.*

Making $P(l)$ into ring spectra is more complicated at the prime 2 than at odd primes. A geometric approach is used in [SY], where it is shown that $P(l)$ can be given an associative product with respect to which Q_is are derivation. Würgler ([Wu1], [Wu2]) used a homotopy theoretic approach that gives more information. Products that make $P(l)$ into an algebra spectrum over BP can be identified with primitive elements a of $P(l)^0(P(l) \wedge P(l))$ such that

$$\left(BP \wedge BP \to P(l) \wedge P(l) \xrightarrow{a} P(l)\right) = (BP \wedge BP \to BP \to P(l)) \ .$$

Analysing the module of primitives of $P(l)^*(P(l) \wedge P(l))$ shows that there exactly two such elements. If one is m, then the other is $m' = m + v_l m \circ (Q_{l-1} \wedge Q_{l-1})$. Assuming that m is commutative leads to the false conclusion that $m = m'$. So m is not commutative and $m' = m \circ T$ where $T\colon P(l) \wedge P(l) \to P(l) \wedge P(l)$ is the map that transposes the factors. Arbitarily choose one of the two products as the product for $P(l)$. Then the following statements are true, irrespective of the choices made:

Proposition 1.2. [Wu2, 2.5] *Let X and Y be spectra and $\tau : X \wedge Y \to Y \wedge X$ be the switch map. Then $P(l)_*\tau(x \wedge y) = y \wedge x + v_l(Q_{l-1}y) \wedge (Q_{l-1}x)$ for $x \in P(l)_*X$ and $y \in P(l)_*Y$.*

Lemma 1.3. *For $0 \le i < l$, the Milnor Bocksteins Q_i are derivations of the ring spectrum $P(l)$.*

Lemma 1.4. *For $m > l$, the map $P(l) \to P(m)$ is a map of BP-algebra spectra.*

Give $B(l)$ the product induced by the chosen product for $P(l)$. Then the first two are true for $B(l)$ as well.

Suppose that $P(l)_*X$ and $P(l)_*Y$ are free as $P(l)_*$-modules. Then the $\times$-product $P(l)_*X \otimes P(l)_*Y \to P(l)_*(X \times Y)$ is a module isomorphism allowing us to identify the two modules. This also allows us to make $P(l)_*X$ and $P(l)_*Y$ into coalgebras by the usual approach. But $\times$-product need not be a coalgebra morphism because $P(l)$ is not commutative. Thus, if X is an H-space with $P(l)_*X$ free, then $P(l)_*X$ is both an algebra and a coalgebra, but not necessarily a Hopf algebra. Similar remarks apply to $B(l)$ as well (we need not even worry about freeness). The corrections to be made are given by the next two results.

Proposition 1.5. *Suppose that X and Y are two spaces such that $P(l)_*X$ and $P(l)_*Y$ are free $P(l)_*$-modules. Identify $P(l)_*X \otimes P(l)_*Y$ with $P(l)_*(X \times Y)$ as modules. Then, for $x \in P(l)_*X$ and $y \in P(l)_*Y$,*

$$\begin{aligned}\Delta_{P(l)_*(X\times Y)}(x \otimes y) = \Delta_{P(l)_*X}x \otimes \Delta_{P(l)_*Y}y \\ + v_l(\mathrm{id} \otimes Q_{l-1} \otimes Q_{l-1} \otimes \mathrm{id})(\Delta_{P(l)_*X}x \otimes \Delta_{P(l)_*Y}y)\end{aligned}$$

where $\Delta_{P(l)_*X}$ is the diagonal of $P(l)_*X$ etc.

Proof. Consider FIGURE A in which P stands for $P(l)$, τ transposes the appropriate factors, and θ and ϕ are given by

$$\theta(x_1 \otimes y_1 \otimes x_2 \otimes y_2) = x_1 \otimes y_1 \otimes x_2 \otimes y_2 + v_l x_1 \otimes Q_{l-1} y_1 \otimes Q_{l-1} x_2 \otimes y_2$$
$$\phi(x_1 \otimes y_1 \otimes x_2 \otimes y_2) = x_1 \times y_1 \times x_2 \times y_2 + v_l x_1 \times Q_{l-1} y_1 \times Q_{l-1} x_2 \times y_2$$

$$\begin{tikzcd}
P_*X \otimes P_*Y \arrow[rr, "\Delta_{P_*(X)\otimes P_*(Y)}"] \arrow[dr, "\Delta_{P_*(X)} \otimes \Delta_{P_*(Y)}"] \arrow[dd] & & P_*X \otimes P_*Y \otimes P_*X \otimes P_*Y \arrow[dd] & \\
& P_*X \otimes P_*X \otimes P_*Y \otimes P_*Y \arrow[rr, "1 \otimes \tau \otimes 1"] \arrow[dd] & & P_*X \otimes P_*Y \otimes P_*X \otimes P_*Y \arrow[ul, "\theta"'] \arrow[dd, "\phi"] \\
P_*(X \times Y) \arrow[rr, "P_*(\Delta_{X \times Y})"] \arrow[dr, "P_*(\Delta_X \times \Delta_Y)"'] & & P_*(X \times Y \times X \times Y) \arrow[dr, equal] & \\
& P_*(X \times X \times Y \times Y) \arrow[rr, "P_*(1 \otimes \tau \otimes 1)"'] & & P_*(X \times Y \times X \times Y)
\end{tikzcd}$$

FIGURE A

The unmarked vertical maps, induced by the $\times$-product, are isomorphisms because $P(l)_*X$ and $P(l)_*Y$ are free. The right hand face commutes by the naturality of the $\times$-product. The bottom face commutes by functoriality. The left and back faces commute by the definition of the diagonal. The front commutes by Proposition 1.2. It follows that the top face commutes.

Corollary 1.6. *Suppose that X is an H-space such that $P(l)_*X$ is free as a $P(l)_*$-module. Let x and y be elements of $P(l)_*X$ and Δ be the diagonal of the latter. Then*

$$\Delta(xy) = \Delta(x)\Delta(y) + v_l((\mathrm{id} \otimes Q_{l-1})\Delta(x))((Q_{l-1} \otimes \mathrm{id})\Delta(y))\,.$$

The previous two results also hold for $B(l)$. The same proofs, with the obvious changes, apply.

2. Preliminaries on $P(l)_*SO(n)$.

Let $G_n = SO(n+2)/(SO(2) \times SO(n))$ be the generating variety for the homology of $\Omega_0 SO(n+2)$; *i.e.* there is a map $G_n \to \Omega_0 SO(n+2) = \Omega Spin(n)$ such that $H_*(G_n, \mathbb{Z})$ maps monomorphicaly into $H_*(\Omega_0 SO(n+2), \mathbb{Z})$ and the image of the former generates the latter as an algebra [Bt]. (These were referred to as Q_n by Bott and in our previous papers. The notation has been changed to avoid confusion with the Bocksteins.) G_n has no torsion in homology. The direct limit of G_n is $\mathbb{C}P^\infty$. This gives a canonical complex line bundle on G_n. Let x be its first Conner-Floyd Chern class. Then $MU\mathbb{Q}^* G_{2n-1} = MU\mathbb{Q}[x]/(x^{2n})$, because the same is true of rational cohomology. G_{2n-1} has an almost complex structure and we have an embedding

$$\overline{\mathbb{C}P}^{n-1} = \frac{U(n)}{U(1)\times U(n-1)} \to \frac{SO(2n+1)}{SO(2)\times SO(2n-1)} = G_{2n-1}$$

of almost complex manifolds, where $\overline{\mathbb{C}P}^{n-1}$ is the complex projective $(n-1)$-space with the conjugate of the usual complex structure. Let y be the "Atiyah-Poincaré dual" of $\overline{\mathbb{C}P}^{n-1}$ in G_{2n-1}. Then $x^n = y([2](x)/x)$ in MU^*G_{2n-1}, where $[2](x)$ is the 2-series for MU [R1, Proposition 2.1]. This is proved by calculating intersection numbers and is the crucial lemma of [R1]. Also, $\{1, x, \dots, x^{n-1}, y, yx, \dots, yx^{n-1}\}$ is an MU^*-basis of MU^*G_{2n-1}, for the reduction to integral cohomology is a basis.

Let $\{\beta_0', \beta_1, \dots \beta_{2n-1}\}$ the basis of $MU\mathbb{Q}_*G_{2n-1}$ that is dual to $\{1, x, \dots, x^{2n-1}\}$. The basis dual to $\{1, x, \dots, x^{n-1}, y, yx, \dots, yx^{n-1}\}$ is

$$\{\beta_0'\} \cup \{\beta_i \mid 1 \le i < n\} \cup \{\sum_{j=0}^{i-n} a_i\beta_{i-j} \mid n \le i < 2n\}$$

where a_i is the coefficient of t^{i+1} in $[2](t)$. It follows that $\alpha_i' = \sum_{j=0}^{i-1} a_j\beta_{i-j}$ is integral in the sense that it lies in the image of $MU_*G_{2n-1} \to MU\mathbb{Q}_*G_{2n-1}$. Also β_i is integral if $1 \le i < n$.

Remark: The last fact can be deduced from the fact that $\mathbb{C}P^{n-1} \to G_{2n-1} \to G_\infty = \mathbb{C}P^\infty$ is the usual inclusion (of spaces). The author is not aware of an equally simple proof of the integrality of the α_i'.

Using the fact that the relavant Atiyah-Hizrebruch spectral sequences collapse, we see that MU_*G_{2n-1} injects into $MU_*\Omega SO(2n+1)$. We will identify the former with its image in the latter. Let $\beta_0 \in \widetilde{MU}_0\Omega SO(2n+1)$ be the unique element such that $\beta_0^2 = 2\beta_0$. Define α_i in $\widetilde{MU\mathbb{Q}}_*\Omega SO(2n+1)$, for $i < 2n$, by $\alpha_i = \sum_{j=0}^{i} a_{i-j}\beta_j$. By the previous paragraph, α_i is actually in $\widetilde{MU}_*\Omega SO(2n+1)$.

Let h be an MU-algebra theory. The images of the βs and αs in $h_*\Omega SO(2n+1)$, under the homomorphism induced by $MU \to h$, will be denoted by the same symbols. These elements are independent of n in the sense that if $n < q$, then

$$h_*\Omega SO(2n+1) \xrightarrow{\text{incl}_*} h_*\Omega SO(2q+1)$$

sends β_i to β_i for $0 \le i < n$, and similarly for the αs. For an element x of $\widetilde{h}_*\Omega SO(2n+1)$, $\overline{x}$ will denote the image of x under the homology suspension

$$\widetilde{h}_*\Omega SO(2n+1) \to \widetilde{h}_{*+1}\Sigma\Omega SO(2n+1) \to \widetilde{h}_{*+1}SO(2n+1)\,.$$

Proposition 2.1. *For any BP-algebra theory h, $h_*\Omega_0 SO(2n+1)$, as an h_*-algebra, is generated by β_i, $1 \le i < n$ and α_{2j+1}, $n \le 2j+1 \le 2n-1$.*

This follows from [R1, Theorem 2.3.(2)]. It is proved by comparison with ordinary homology and the Atiyah-Hizrebruch spectral sequence.

Lemma 2.2. *In $P(l)_*\Omega SO(2^{l+1}+3)$, $\beta^2_{2^{l-1}} = v_l\beta_1$.*

This follows from [R1, Theorem 2.3.(6)] and the fact that $[-1](t) = t + v_l t^{2^l - 1}$ (mod t^{2^l}) in $P(l)_*$ (see [R3, proof of Lemma 3.1]).

Let G be a compact connected Lie group. Let h be a BP-algebra theory such that for any $a \in h_*$, $x \in h_*X$ and $y \in h_*Y$, where X and Y are spaces, $(ax) \wedge y = a(x \wedge y) = x \wedge (ay)$ in $h_*(X \wedge Y)$. We will denote the Bar spectral sequence

$$E^2_{**}(G,h) = \mathrm{Tor}^{h_*\Omega G}_{**}(h_*, h_*) \Rightarrow h_*G$$

by $E^*_{**}(G,h)$. This is a spectral sequence of commutative algebras, obtained from the "bar filtration" on $B\Omega G \simeq G$. If $E^r_{**}(G,h)$ is free over h_* for all r, then it is a spectral sequence of bicommutative, biassociative Hopf algebras (see [R2, Theorem 3.1]). The Hopf algebra structure on the E^∞-term is compatible with the algebra and coalgebra structure on h_*G (even if it is not a Hopf algebra).

Note that $E^r_{0*}(G,h) = h_*$ and that $E^2_{1*} = \widetilde{h}_*\Omega G / \left(\widetilde{h}_*\Omega G\right)^2$ is the module of indecomposables of the h_*-algebra $h_*\Omega G$. Also the homology suspension factors as

$$\widetilde{h}_*\Omega G \to E^2_{1*}(G,h) \to E^\infty_{1*}(G,h) \to \widetilde{h}_{1+*}G\,.$$

For the rest of this paper we will fix an $l > 0$. For $0 \leq i < 2^{l-1}$, define $k(i)$ by $2^l \leq 2^{k(i)}(2i+1) < 2^{l+1}$.

Fix a gound ring of charateristic 2. Let $\Gamma_k(t)$ denote the divided power algebra of height k on t. This is the dual of the primitively generated truncated polynomial algebra $P(x)/(x^{2^k})$. The j^{th} divided power of t will be denoted by $\gamma_j(t)$.

We will make use of the following calculations of the bar spectral sequences done in [R2] and [R3]. The E^2-term of the Bar ss is calcuated using a complex introduced by T. Petrie [Pe]. The ss collapses if $l = \infty = n$. Then we use descent on n and l to get Proposition 2.4. Proposition 2.6 is deduced using the map $E^*_{**}(SO(2^{l+1}-1), P(l)) \to E^*_{**}(SO(2n+1), B(l))$.

Proposition 2.3. [R2, Theorem 1.1] *$P(l)_*SO(2^l-1)$ is a bicommutative Hopf algebra, and is isomorphic to*

$$\bigotimes_{i=0}^{2^{l-1}-2} \Gamma_{k(i)}(\overline{\beta}_i)\,.$$

If $0 < i \leq 2^{l-1}-2$ and $1 \leq j < 2^{k(i)}$, then we will denote the images of $\gamma_j(\overline{\beta}_i)$ in $P(l)_*SO(2^{l+1}-1)$ by the same symbol.

Proposition 2.4. [R2, Theorem 1.1 and p. 58] *If $l \leq m$, then*

$$E^\infty_{**}(SO(2^{l+1}-1), P(m)) \cong \bigotimes_{i=0}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)\,;$$

$$E^\infty_{**}(Spin(2^{l+1}-1), P(m)) \cong E(\overline{\alpha}'_{2^l-1}) \otimes \bigotimes_{i=1}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)\,.$$

Corollary 2.5. *Suppose that X is $SO(2^{l+1}-1)$, $Spin(2^{l+1}-1)$ or a finite product of those two. Then $\widetilde{P(l)}_i X \cong \widetilde{P(m)}_i X$ if $l < m$ and $0 \le i \le 2^{l+1}-2$.*

Proof. Use Proposition 2.4, Proposition 1.1 and the fact that $P(m)_j = 0$ if $0 < j < 2^{m+1}-2$.

It follows that there exists a unique element $\widetilde{\gamma}_i$ of $\widetilde{P(l)}_* SO(2^{l+1}-1)$ that reduces to $\gamma_{2^{k(i)}}(\overline{\beta}_i)$ in $P(l+1)_* SO(2^{l+1}-1)$, for $0 \le i \le 2^{l-1}-1$. Note that $\overline{\beta}_i$ has degree $2i+1$ and $\widetilde{\gamma}_i$ has degree $2^{k(i)}(2i+1)$.

Proposition 2.6. [R3, Proposition 3.2] *If $2^l \le n \le 2^{l+1}-2$, then*

$$E^{\infty}_{**}(SO(2n+1), P(l)) \cong \bigotimes_{i=2^l-1}^{n-1} E(\overline{\alpha}_{2i+1}) \otimes \bigotimes_{i=2^l-1}^{n-1} E(\overline{\beta}_i) \otimes \left(\bigotimes_{i=0}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i) \right) \Big/ \left(\overline{\beta}_i \mid 0 \le i \le n-2^l\right) .$$

Lemma 2.7. [R2, *p.* 56] *For any n and any BP-algebra theory h,*

$$E^{*}_{**}(SO(2n+2), h) \cong E^{*}_{**}(SO(2n+1), h) \otimes E(\overline{w}_{2n+1})$$

where $\overline{w}_{2n+1}$ has bidegree $(1, 2n)$.

3. Proof of Theorem 0.3 for $2^{l+1}-1 \le n \le 2^{l+2}-4$.

For typographical convenience, we write y for $\widetilde{\gamma}_{2^{l-1}-1}$ and z for $\overline{\beta}_{2^{l-1}-1}$. Unless otherwise specified, we will be working with $P(l)_* SO(2^{l+1}-1)$. Let Δ be the diagonal of $P(l)_* SO(2^{l+1}-1)$ and σ the homomorphism in $P(l)$-homology induced by $g \mapsto g^{-1}$. Note that if $x \in \widetilde{P(l)}_* SO(2^{l+1}-1)$ and $\Delta(x) = \sum x_i' \otimes x_i''$, then $\sum x_i' \sigma(x_i'') = 0$.

Lemma 3.1. *$Q_j \overline{\beta}_i = 0$, $\Delta(\beta_i) = 1 \otimes \overline{\beta}_i + \overline{\beta}_i \otimes 1$ and $\sigma(\overline{\beta}_i) = \overline{\beta}_i$.*

Proof. Note that $\overline{\beta}$'s originate in $\widetilde{BP}_* \Sigma(G_{2^{l+1}-3} \cup \{pt\})$ and that the latter is free as a BP_* module. This proves the first two equalities. The third now follows as $1\sigma(\overline{\beta}_i) + \overline{\beta}_i \sigma(1) = 0$.

Lemma 3.2. *$\overline{\beta}_i^2 = 0$ and $\overline{\beta}_i \overline{\beta}_j + \overline{\beta}_j \overline{\beta}_i = 0$.*

Proof. Recall that $\overline{\beta}_i$ has filtration 1 in the Bss and in the E^{∞}-term, $\overline{\beta}_i^2 = 0$. Thus $\overline{\beta}_i^2$ has filtration 0 or 1. But $E^{*}_{*,2t+1} = 0$ and $E^{*}_{0*} = P(l)_*$.

Proof of the second claim is similar.

Proposition 3.3. *The module of primitives of $P(l)_*SO(2^{l+1}-1)$ is free with basis $\{\overline{\beta}_i \mid 0 \leq i \leq 2^l - 2\}$.*

Proof. $\overline{\beta}_i$s are primitive by Lemma 3.1 and are linearly independent by Proposition 2.4. Thus it is enough to show that any primitive can be written as a linear combination of $\overline{\beta}_i$. We will prove this by induction on $\dim x$. Note that the claim is vacuous if $\dim x \leq 0$.

Let x be primitive. Then the reduction of x to $P(l+1)_*SO(2^{l+1}-1)$ is primitive. By Proposition 2.3, there exist $a_i \in P(l+1)_*$ such that $x = \sum a_i\overline{\beta}_i$ in $P(l+1)_*SO(2^{l+1}-1)$. Consider these a_i's as elements of $P(l)_*$. Then $u = x - \sum a_i\overline{\beta}_i$ is divisible by v_l because it maps to 0 in $P(l+1)_*SO(2^{l+1}-1)$. As $P(l)_*SO(2^{l+1}-1)$ is free, u/v_l is primitive. By the induction hypothesis, it is a linear combination of $\overline{\beta}_i$.

Lemma 3.4. *$\Delta(y) = 1 \otimes y + z \otimes z + y \otimes 1$, $\sigma(y) = y$ and $Q_i y = \overline{\beta}_{2^l-2^i-1}$ for $0 \leq i < l$.*

Proof. To prove the first claim, note that it holds in $P(l+1)_*SO(2^{l+1}-1)$ and use Corollary 2.5. Lemma 3.2 and the fact that $1\sigma(y) + z\sigma(z) + y\sigma(1) = 0$ proves the second equality.

I claim that $Q_i y \neq 0$: The reduction of y to mod 2 ordinary homology is $\gamma_2(\overline{\beta}_{2^{l-1}-1})$ which is indecomposable by Proposition 2.3. Recall that the module of indecomposables of $H_*(SO(2^{l+1}-1), \mathbb{Z}/2)$ is isomorphic to $H_*(\mathbb{R}P^{2^{l+1}-2}, \mathbb{Z}/2)$ as modules over the Steenrod algebra. Thus y', the unique non-zero element of degree $2^{l+1}-2$ in the latter, corresponds to y. It is well known that $Q_i y' \neq 0$ if $0 \leq i < l$.

As Q_i is a derivation and $Q_i z = 0$, $Q_i y$ is primitive. Its degree is $2^{l+1} - 2^{i+1} - 1$. By Proposition 3.3, the only possibility is $\overline{\beta}_{2^l-2^i-1}$.

Lemma 3.5. *$y^2 = v_l y$.*

Proof. An easy calculation using Corollary 1.6 and the lemmas above gives

$$\begin{aligned}\Delta(y^2) &= \Delta(y)^2 + v_l((\mathrm{id} \otimes Q_{l-1})\Delta(y))(Q_{l-1} \otimes \mathrm{id})(\Delta(y)) \\ &= 1 \otimes y^2 + z \otimes (zy + yz) + (zy + yz) \otimes z + y^2 \otimes 1 + v_l z \otimes z\end{aligned}$$

I claim that $zy + yz$ is either 0 or $v_l z$: Note that in the Bar ss, y and z have filtration 2 and 1 respectively. Now $zy+yz = 0$ in $E^\infty_{3,2^{l+1}+2^l-6}$. By Proposition 2.4, $E^\infty_{pq} = 0$ if q is odd. Hence $zy + yz$ is in $E^\infty_{1,2^{l+1}+2^l-2} = \{0, v_l z\}$.

So $z \otimes (zy + yz) + (zy + yz) \otimes z$ is either 0 or $2v_l z \otimes z = 0$. It follows that $\Delta(y^2 - v_l y) = 1 \otimes (y^2 - v_l y) + (y^2 - v_l y) \otimes 1$. Hence $y^2 - v_l y$ is primitive of even degree. By Proposition 3.3, it must be trivial.

Lemma 3.6. *Suppose that $x \in \widetilde{P(l)}_*SO(2^{l+1}-1)$ is a linear combination of $\overline{\beta}_i$s. Let c_2 be the commutator map of $SO(2^{l+1}-1)$. Then $P(l)_*c_2(x \otimes y) = xy + yx$.*

Proof. Note that $Q_{l-1}x = 0$ by Lemma 3.1. An easy calculation using the definition of c_2, Proposition 1.5, Lemma 3.1 and Lemma 3.4 gives

$$P(l)_*c_2(x \otimes y) = xy + zxz + yx + xy + xzz + xy\,.$$

But $xz^2 = 0$ and $zxz = xzz = 0$ by Lemma 3.2.

Lemma 3.7. *Let c_s be the iterated commutator map of $SO(2^{l+1}-1)$. Then*

$$P(l)_* c_s(\overline{\beta}_{2^l-2} \otimes \widetilde{\gamma}_{2^{l-1}-1} \otimes \cdots \otimes \widetilde{\gamma}_{2^{l-1}-1}) = v_l^{s-1}\overline{\beta}_{2^l-2}$$

Proof. From Lemma 3.5 we get $(Q_0 y)y + y(Q_0 y) = v_l Q_0 y$. By Lemma 3.4, $Q_0 y = \overline{\beta}_{2^l-2}$. Now Lemma 3.6 implies that $P(l)_* c_2(\overline{\beta}_{2^l-2} \otimes y) = v_l \overline{\beta}_{2^l-2}$. This gives the result if $s = 2$. Induction on s completes the proof.

Lemma 3.8. *Let $l \geq 2$. Then there is a unique element $\widetilde{\gamma}' \in P(l)_* Spin(2^{l+1}-1)$ such that*

$$\Delta(\widetilde{\gamma}') = 1 \otimes \widetilde{\gamma}' + \widetilde{\gamma}' \otimes 1 + \overline{\beta}_{2^{l-1}-1} \otimes \overline{\beta}_{2^{l-1}-1}\,.$$

$\widetilde{\gamma}'$ maps to $\widetilde{\gamma}_{2^{l-1}-1}$ in $P(l)_* SO(2^{l+1}-1)$.

Proof. Throughout this proof $n = 2^{l+1}-1$.

By [MM, Theorem 7.11 dualized] and Proposition 2.4,

$$H_*(Spin(n), \mathbb{Z}/2) = E(\overline{\alpha}'_{2^l-1}) \otimes \bigotimes_{i=1}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)$$

as *coalgebras*. Thus, there is an element $y' \in H(Spin(n), \mathbb{Z}/2)$, to which $\gamma_2(\overline{\beta}_{2^{l-1}-1})$ converges, such that $\Delta(y') = 1 \otimes y' + y' \otimes 1 + \overline{\beta}_{2^{l-1}-1} \otimes \overline{\beta}_{2^{l-1}-1}$. This determines y' uniquely as any two choices must differ by an even dimensional primitive.

By Corollary 2.5 there exists a unique element $\widetilde{\gamma}'$ of $P(l)_* Spin(n)$ that maps to y' and that $\Delta(\widetilde{\gamma}')$ is as stated. The image of $\widetilde{\gamma}'$ in $P(l)_* SO(n)$ is $\widetilde{\gamma}_{2^{l-1}-1}$ because the difference is an even dimensional primitive.

Proof of Theorem 0.3 for $n \neq 2^m-3,\ 2^m-2$. Let $s, l \geq 2$ and let f_s be the composition

$$Spin(2^{l+1}-1)^s \xrightarrow{\text{proj}} SO(2^{l+1}-1)^s \xrightarrow{c_s} SO(2^{l+1}-1) \xrightarrow{\text{incl}} SO(2^{l+2}-4)\,.$$

By Lemma 3.7 and Lemma 3.8, $P(l)_* f_s(\overline{\beta}_{2^l-2} \otimes \widetilde{\gamma}' \otimes \cdots \otimes \widetilde{\gamma}') = v_l^{s-1}\overline{\beta}_{2^l-2}$. On the other hand, $\overline{\beta}_{2^l-2} \neq 0$ in $B(l)_* SO(2^{l+2}-4)$ by Proposition 2.6 and Lemma 2.7. So f_s is not null-homotopic for any s.

Suppose that $2^{l+1}-1 \leq n \leq 2^{l+2}-4$. Then f_s can be factored as

$$\begin{aligned} &Spin(2^{l+1}-1)^s \longrightarrow Spin(n)^s \xrightarrow{c_s} Spin(n) \longrightarrow SO(n) \longrightarrow SO(2^{l+2}-4) \\ &= Spin(2^{l+1}-1)^s \longrightarrow SO(2^{l+1}-1)^s \longrightarrow SO(n)^s \xrightarrow{c_s} SO(n) \longrightarrow SO(2^{l+2}-4)\,. \end{aligned}$$

Hence $Spin(n)$ and $SO(n)$ are not homotopy nilpotent.

Arguing similarly, we can show that the map

$$SO(3)^s \xrightarrow{c_s} SO(3) \longrightarrow SO(4) = SO(3)^s \longrightarrow SO(4)^s \xrightarrow{c_s} SO(4)$$

is not null-homotopic for any $s \geq 2$. This proves the second sentence of Theorem 0.3.

Remark. Note that the proof of the lemmas of this section depend only on the the coalgebra structure of $P(l)_*SO(2^{l+1}-1)$, the action of the Bocksteins Q_i on it and the algebra structure of $E^{\infty}_{**}(SO(2^{l+1}-1), P(l))$. The first two are clearly independent of the multiplication on $SO(2^{l+1}-1)$. The last depends only on the algebra structure of $BP_*\Omega SO(2^{l+1}-1)$. But the H-space structure of $\Omega SO(2^{l+1}-1)$ is independent of the product on $SO(2^{l+1}-1)$. The only time we used the usual product was to claim that $Spin(n) \to Spin(m)$ and $SO(n) \to SO(m)$ were H-maps. It follows that if $l \geq 2$, then $Spin(2^{l+1}-1)$ is not homotopy nilpotent with *any* H-space structure.

Suppose that $l \geq 2$ and that $2^{l+1} \leq n < 2^{l+1} + 2^l - 1$. Then it can be shown that $Spin(n)$ is not homotopy nilpotent with any homotopy associative product: Let y denote the image of $\widetilde{\gamma}'$ in $B(l)_*Spin(n)$. The diagonal, the action of Q_i, and the algebra structure of the E^∞-term of the Bar ss are independent of the product. So Lemmas 3.1, 3.2, 3.4 and 3.6 still hold. To determine y^2, in the *new* product, we proceed as follows: By the Bar ss, y^2 is either 0 or $v_l y$. As in the proof of Lemma 3.5, $y^2 - v_l y$ is primitive; but, by Proposition 2.6, $v_l y$ is not (this is where the restriction on n is needed). Lemma 3.7 will then follow.

Corollary 3.9. *The exceptional Lie group $G(2)$ is not homotopy nilpotent.*

Proof. It is well-known that there is a principal fibration $G(2) \to Spin(7) \to \mathbb{S}^7$ that splits at the prime 2. So, $P(2)_iG(2) \to P(2)_iSpin(7)$ is bijective if $i \leq 6$. In particular, we can consider $\overline{\beta}_2$ and $\widetilde{\gamma}'$ as elements of $P(2)_*G(2)$. The corollary now follows from Lemma 3.7.

Remark. Alternatively, we can first show that $\overline{\beta}_1^2$, $\overline{\beta}_2^2$, $[\overline{\beta}_1, \overline{\beta}_2]$ and $(\widetilde{\gamma}')^2 - v_2\widetilde{\gamma}'$, considered as elements of $P(2)_*G(2)$, are even dimensional primitives. These elements must be trivial and we can then deduce that $G(2)$ is not homotopy nilpotent. Note that this argument will apply to *any* H-space strucutre on $G(2)$.

4. Proof of Theorem 0.3 completed.

Let $s, l \geq 2$ and let g_s be the composition

$$Spin(2^{l+1}+3)^s \xrightarrow{\text{proj}} SO(2^{l+1}+3)^s \xrightarrow{c_s} SO(2^{l+1}+3) \xrightarrow{\text{incl}} SO(2^{l+2}-2)\,.$$

The rest of this section is devoted to proving that $B(l)_*g_s \neq 0$. This implies that $Spin(n)$ and $SO(n)$ are not homotopy nilpotent if $2^{l+1}+3 \leq n \leq 2^{l+2}-2$, completing the proof of Theorem 0.3.

Lemma 4.1. *Let y' be the image of $\widetilde{\gamma}'$ in $H_*(Spin(2^{l+1}+3), \mathbb{Z}/2)$. Then $[y', \overline{\beta}_{2^l}] \neq 0$.*

Proof. Throughout this proof $n = 2^{l+1}+3$.

We need some facts conerning the $\mathbb{Z}/2$ cohomology of $SO(n)$ and $Spin(n)$: For $1 \leq i \leq n-1$, let $x_i \in H^*(SO(n), \mathbb{Z}/2)$ be the cohomology suspension of the $(i+1)$st Stiefel-Whitney class and x_i' be the image of x_i in $H^*(Spin(n), \mathbb{Z}/2)$. Then $x_i' = 0$ if i is a power of 2. By [MZ], there is an indecomposable element $u \in$

$H^*(Spin(n),\mathbb{Z}/2)$ such that the latter is generated as an algebra by $u, x'_3, x'_5, \ldots$. Furthermore, u is of degree $2^{l+2}-1$ and

$$\Delta(u) = 1\otimes u + u\otimes 1 + \sum_{j\le 2^{l+1}-1} x'_j \otimes x'_{2^{l+2}-1-j}\,.$$

It is well-known that the primitives of $H^*(SO(n),\mathbb{Z}/2)$ are the x_is. By Proposition 2.3, the indecomposables of $H_*(SO(n),\mathbb{Z}/2)$ are $\{\gamma_{2^s}(\overline{\beta}_j) \mid 0\le j\le 2^l,\ 2^s(2j+1)\le 2^{l+1}+2\}$. Thus

$$\langle \gamma_{2^s}(\overline{\beta}_j), x_i\rangle = \begin{cases} 1 & 2^s(2j+1)=i\,;\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 3.8 $\widetilde{\gamma}'$ maps to $\gamma_2(\overline{\beta}_{2^{l-1}-1})$ in $H_*(SO(2^{l+1}+3),\mathbb{Z}/2)$. Combining this with the previous two paragraphs, we see that

$$\langle [y',\overline{\beta}_{2^l}], u\rangle = \langle y'\otimes\overline{\beta}_{2^l} - \overline{\beta}_{2^l}\otimes y', \Delta(u)\rangle = 1\,.$$

Lemma 4.2. *In $P(l)_*Spin(2^{l+1}+3)$, $[\widetilde{\gamma}', \overline{\beta}_{2^l}] = \overline{\alpha}'_{2^{l+1}-1} + ev_l\overline{\beta}_{2^l}$ where e is either 0 or 1.*

Proof. It follows from Proposition 2.1, Lemma 2.2 and the fact that $E^2_{1*}(G,h)$ is the module of indecomposables of $\widetilde{h}_*\Omega G$, that $E^2_{1*}(Spin(2^{l+1}+3), P(l))$ is generated by

$$\overline{\beta}_0,\ldots,\overline{\beta}_{2^l},\overline{\alpha}'_{2^l+1},\ldots,\overline{\alpha}'_{2^{l+1}+1}$$

as a $P(l)_*$-module and that $v_l\overline{\beta}_1 = 0$. Thus

$$E^2_{1,2^{l+1}-2}(Spin(2^{l+1}+3),P(l)) = \{0, \overline{\alpha}'_{2^{l+1}-1}, v_l\overline{\beta}_{2^l}, \overline{\alpha}'_{2^{l+1}-1}+v_l\overline{\beta}_{2^l}\}\,.$$

It follows from Proposition 2.6 that all these elements survive to the E^∞-term.

In the bar spectral sequence, $\widetilde{\gamma}'$ has filtration 2 and $\overline{\beta}_{2^l}$ has filtration 1. As E^r_{**} is commutative and $E^r_{2,2j+1}=0$, $[\widetilde{\gamma}',\overline{\beta}_{2^l}]$ has filtration 1. It has total degree $2^{l+2}-1$, and by Lemma 4.1, is not divisible by v_l. Only possibilites left are $\overline{\alpha}'_{2^{l+1}-1}$ and $\overline{\alpha}'_{2^{l+1}-1}+v_l\overline{\beta}_{2^l}$.

Lemma 4.3. *In $B(l)_*SO(2^{l+1}+3)$,*

$$B(l)_*c_s(\overline{\beta}_{2^l}\otimes\widetilde{\gamma}_{2^{l-1}-1}\otimes\cdots\otimes\widetilde{\gamma}_{2^{l-1}-1}) = v_l^{s-2}[\overline{\beta}_{2^l},\widetilde{\gamma}_{2^{l-1}-1}]$$

Proof. The proof is by induction on s. For typographical convenience, we will write x and y for $\overline{\beta}_{2^l}$ and $\widetilde{\gamma}_{2^{l-1}-1}$ respectively.

As $Q_{l-1}x=0$, the case $s=2$ follows as in the proof of Lemma 3.6. Using Lemma 4.2, we see that $Q_{l-1}[x,y]=0$ and that $[x,y]$ is primitive. Hence

$$B(l)_*c_2([x,y]\otimes y) = [[x,y],y] = xy^2 - yxy - yxy + y^2x = [x,y^2] = v_l[x,y]$$

where we used Lemma 3.5 (and the fact that we are working in characteristic 2). Thus

$$\begin{aligned} B(l)_* c_{s+1}(x \otimes y \otimes \cdots \otimes y) &= B(l)_* c_2((B(l)_* c_s(x \otimes y \otimes \cdots \otimes y)) \otimes y) \\ &= B(l)_* c_2(v_l^{s-2}[x,y] \otimes y) = v_l^{s-1}[x,y]\,. \end{aligned}$$

Proof that $B(l)_* g_s \neq 0$ for all $s \geq 2$. Note that $\alpha'_{2^{l+1}-1} - \alpha_{2^{l+1}-1}$ is in $BP_*\beta_0$. But by Proposition 2.6, $\overline{\beta}_0 = 0$ in $B(l)_* SO(2^{l+2}+3)$. So, in the latter, $\overline{\alpha}'_{2^{l+1}-1} = \overline{\alpha}_{2^{l+1}-1}$.

Using Proposition 2.6 once more, we see that $\overline{\alpha}_{2^{l+1}-1}$ and $\overline{\beta}_{2^l}$ are linearly independent in $B(l)_* SO(2^{l+2}-3)$. The latter injects into $B(l)_* SO(2^{l+2}-2)$ (Lemma 2.7). Hence $v_l^{s-2}(\overline{\alpha}_{2^{l+1}-1} + e\overline{\beta}_{2^l}) \neq 0$ for any $s \geq 2$ and $e \in \{0,1\}$. Combining this with the previous two lemmas completes the proof.

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