On $K(2)_*SO(9)$

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Abstract

We carry out a detailed study of both connected and periodic second Morava K-theories of $SO(9)$, using the methods of the previous papers.

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1 Introduction and Preliminaries

In earlier papers, the additive structure of the (periodic) Morava K-theories of special orthogonal groups was calculated and some progress was made analyzing the algebra structure. However, this is difficult due the problems in passing between ordinary homology and periodic theories. It might be better to use the connective theories as an intermediary. In this paper, we make a start on this objective by studying $K(2)_*SO(9)$ and its connective version $k(2)_*SO(9)$.

Throughout this paper, we work at the prime 2. In particular, BP will denote the 2-primary Brown-Peterson theory and $k(l)$ and $K(l)$ will denote respecively the connective and periodic lth Morava K-theories at the prime 2. Also, H_*X will denote the ordinary homology of X with $\mathbb{Z}/(2)$ coefficients.

Next, we recall some of the eariler results and set up our notation.

The source for this paragraph is [2]. $G_n = SO(n+2)/(SO(2) \times SO(n))$, the generating variety for the homology of $\Omega_0 SO(n+2)$, has torsion-free homology [1]. Note that $G_\infty = \mathbb{C}P^\infty$. Let x be the image of the standard generator of $MU^*\mathbb{C}P^\infty$ in MU^*G_n. Then $MU\mathbb{Q}^*G_{2n-1} = MU\mathbb{Q}^*[x]/(x^{2n})$. Let

$\{\beta_0', \beta_1, \dots \beta_{2n-1}\}$ be the basis of $MU\mathbb{Q}_* G_{2n-1}$ that is dual to $\{1, x, \dots, x^{2n-1}\}$. Let $\beta_0 \in \widetilde{MU}_0 \Omega SO(2n+1)$ be the unique element such that $\beta_0^2 = 2\beta_0$. Define $\alpha_i = \sum_{j=0}^{i} c_{i-j}\beta_j$ where c_j is the coefficient of t^{j+1} in the MU-[2]-series. If $1 \le i \le 2n-1$ and $1 \le j \le n-1$, then α_i and β_j are in $\widetilde{MU}_* \Omega SO(2n+1)$.

If h is an MU-algbra theory, we will denote the images of the β's and α's in $h_* \Omega SO(2n+1)$ by the same symbols. These elements are independent of n in the sense that if $0 \le i < n < q$ and $j < 2n$, then the map $h_*\Omega SO(2n+1) \to h_* \Omega SO(2q+1)$ induced by inclusion preserves β_i and α_j. The image of u under homology suspension $\widetilde{h}_* \Omega SO(2n+1) \to h_{*+1} SO(2n+1)$ will be denoted by $\overline{u}$.

Fix a ground ring of charateristic 2. Let $\Gamma_k(u)$ denote the divided power algebra of height k on u. This is the dual of the primitively generated truncated polynomial algebra. The full divided power algebra will be denoted by $\Gamma(u)$.

We need some facts concerning the bar spectral sequence; for the details, see [3, Section 3]. If G is a compact connected Lie group and h a BP-algebra theory, then the bar spectral sequence

$$E^2_{**}(G, h) = \mathrm{Tor}^{h_* \Omega G}_{**}(h_*, h_*) \Rightarrow h_* G$$

is a spectral sequence of commutative algebras. If $E^r_{**}(G, h)$ is free over h_* for all r, then it a spectral sequence of bicommutative, biassociative Hopf algebras.

By [3, Theorem 1.1],

$$E^{\infty}_{**}\left(SO(2^{l+1}-1), P(s)\right) = \bigotimes_{i=0}^{2^l-2} \Gamma_{p(i)+1}(\overline{\beta}_i)$$

if $l \le s \le \infty$, and there are no Hopf algebra extension problems if $s > l$. Here $p(i)$ is defined by $2^l \le 2^{p(i)}(2i+1) < 2^{l+1}$, and $\Gamma_k(x)$ is the dual of the truncated primitively generated polynomial algebra $P(x)/(x^{2^k})$.

Hence there are unique elements $\gamma_{ij} \in P(l)_{2j(2i+1)} SO(2^{l+1}-1)$ which project to the $(2j)$th divided power of $\overline{\beta}_i$ in $H_{2j(2i+1)} SO(2^{l+1}-1)$ for $0 \le i \le 2^l - 2$, $1 \le j \le 2^{p(i)-1}$. We will use the same symbols to denote the images of these elements in $h_* SO(N)$ for $N \ge 2^{l+1}$, where h may be $P(l)$, $k(l)$ or $K(l)$.

Let A be an algebra over a commutative ring R. A is said to be simply generated by $g_1, g_2, \dots$ over R if A is a free R-module with basis

$$\{1\} \cup \{g_{i_1} g_{i_2} \dots g_{i_s} \mid i_1 < i_2 < \dots < i_s\}.$$

As we do not assume that A is commutative, the order of the elements may be important.

2 The calculations

Proposition 2.1. *The E^∞-term of the bar spectral sequence converging to $k(2)_*SO(9)$ contains the exterior algebra on $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\beta}_3$, γ_{01}, γ_{02} and γ_{11}, and is generated as module over that exterior algebra by $\overline{\beta}_0$, $\overline{\beta}_0\gamma_{04}$ and $\overline{\alpha}_7$ subject to the relations $v_2\overline{\beta}_0 = 0$, $v_2\overline{\beta}_2\gamma_{04} = 0$.*

Proof. Using the description in [3, *pp.* 54–55], we see that the Petrie complex for $k(2)_*\Omega SO(9)$ is

$$E(\overline{\beta}_0) \otimes \Gamma(\gamma_0) \otimes E(\overline{\beta}_1) \otimes \Gamma(\gamma_1) \otimes E(\overline{\beta}_2, \overline{\beta}_3, \overline{\alpha}_5, \overline{\alpha}_7)$$

with trivial differential. Thus this is the E^2-term of the Bss. For degree reasons, $d^{2k} = 0$ for all k. Also, note that all differentials must vanish on $\overline{\beta}_i$ and $\overline{\alpha}_j$.

We will denote the jth divided power γ_i by γ_{ij}.

Mapping the Bss converging to $P(2)_*SO(7)$ into this E^2-term and using the description of the Bss in [3, Theorem 1.1], we see that the only non-trivial values of d^3 are given by $d^3(\gamma_{1,2i}) = \overline{\alpha}_5\gamma_{1,2i-2}$ for $i \geq 1$. So the E^4-term is

$$E(\overline{\beta}_0) \otimes \Gamma(\gamma_0) \otimes E(\overline{\beta}_1, \gamma_1, \overline{\beta}_2, \overline{\beta}_3, \overline{\alpha}_7)$$

The differential d^5 vanishes on $\Gamma(\gamma_0)$ since it does so in the Bss converging to $P(2)_*SO(7)$. Using the fact that $\alpha_3 = v_2\beta_0$ in $P(2)_*\Omega SO(9)$ and the values of d^7 for $SO(7)$, we deduce that the only non-trivial values of d^7 are given by $d^7(\gamma_{0,4i}) = v_2\overline{\beta}_0\gamma_{0,4i-4}$. It follows that the E^8-term is

$$E(\overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3, \overline{\alpha}_7, \gamma_{01}, \gamma_{02}, \gamma_{11})\langle 1, \overline{\beta}_0\gamma_{0,4i}\rangle \Big/ \{v_2\overline{\beta}_0\gamma_{0,4i} = 0\}$$

For degree reasons, all further differentials are zero on the exterior geneators.

Note that the projection to the Bss converging to $H_*SO(9)$ is injective on the $k(2)_*$-submodule generated by $\{\overline{\beta}_0\gamma_{0,4i}\}$. It follows from the description of the Bss converging to $H_*SO(9)$ (see [3, Theorem 1.1]) that $d^k = 0$ for $8 \leq k < 15$ and that $d^{15}\overline{\beta}_0\gamma_{0,4i} = \overline{\beta}_0\overline{\alpha}_7\gamma_{0,4i-8}$ and for $i \geq 2$. Hence the E^{16}-term is the E^∞-term as described in the statement of the proposition. All further differentials are trivial for degree reasons. □

By [7, Theorem 3.1], theer is a unique element $\zeta_4 \in k(2)_9SO(9)$ whose reduction to mod-2 homology is the nineth divided power of $\overline{\beta}_0$ and which satisfies $v_2\zeta_4 = \overline{\alpha}_7$.

Proposition 2.2. *The elements $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\beta}_3$, ζ_4, γ_{01}, γ_{02} and γ_{11} simply generate a subalgebra of $k(2)_*SO(9)$. As a module over that subalgebra, $k(2)_*SO(9)$ is generated by $\overline{\beta}_0$ subject to the relation $v_2\overline{\beta}_0 = 0$.*

Proof. Combining the above calculation of the Bss with the fact that ζ_4 projects to $\overline{\beta}_0\gamma_{04}$ in $H_*SO(9)$ implies that ζ_4 is repesented by $\overline{\beta}_0\gamma_{04}$ in the Bss. As $\alpha_3 = v_2\beta_0$ in $P(2)_*\Omega SO(9)$ and $\overline{\alpha}_3 = 0$ in $P(2)_*SO(7)$, $v_2\overline{\beta}_0 = 0$ in $k(2)_*SO(9)$. Finally $v_2\zeta_4 = \overline{\alpha}_7$. These relations solve the remaining extension problems in the Bss converging to $k(2)_*SO(9)$. □

The products listed in the next proposition follow from [7, Proposition 4.1] and the results of [6, Section 5].

Proposition 2.3. *The following relations hold in $k(2)_*SO(9)$.*

1. *The subalgebra generated by $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\beta}_3$ and ζ_4 is exterior.*
2. *$\gamma_{01}^2 = 0$, $\gamma_{02}^2 = v_2\gamma_{01}$ and $\gamma_{11}^2 = v_2\gamma_{11}$.*
3. *$[\gamma_{01}, \overline{\beta}_1] = 0$, $[\gamma_{02}, \overline{\beta}_1] = 0$ and $[\gamma_{11}, \overline{\beta}_1] = v_2\overline{\beta}_1$.*
4. *$[\gamma_{01}, \overline{\beta}_2] = 0$, $[\gamma_{02}, \overline{\beta}_2] = v_2\overline{\beta}_1$ and $[\gamma_{11}, \overline{\beta}_2] = v_2\overline{\beta}_2$, and*
5. *$[\gamma_{01}, \gamma_{02}] = 0$, $[\gamma_{01}, \gamma_{11}] = 0$ and $[\gamma_{02}, \gamma_{11}] = 0$.*

Recall that for finite complexes X and Y,

$$(k(l)_*X/v_l\text{-torsion}) \otimes (k(l)_*Y/v_l\text{-torsion}) \to k(l)_*(X \times Y)/v_l\text{-torsion}$$

is an isomorphism. Since $k(l)$ is not commutative, this is not enough to make $k(l)_*X/v_l$-torsion a Hopf algebra when X is an H-space. However, the obstruction to the commutativity of $k(l)$ is divisible by v_l. Adapting the discussion in [5, Section 1], we can show that if X is an H-space, then $(k(l)_*X/v_l\text{-torsion}) \otimes \mathbb{Z}/2$ is a cocommutative Hopf algebra. Alternatively, note that $(k(l)_*X/v_l\text{-torsion}) \otimes \mathbb{Z}/2$ is the E^∞-term of the Bockstein spectral sequence and the E^2-term is a Hopf algebra.

The next lemma is a consequence of Theorem 3.1 and Lemma 3.4 of [7].

Lemma 2.4. *As a Hopf algebra, $(k(2)_*SO(9)/v_2\text{-torsion}) \otimes \mathbb{Z}/2$ is given by $E(\overline{\beta}_2, \overline{\beta}_3, \zeta_4) \otimes \Gamma_2(\gamma_{01}) \otimes \Gamma_2(\overline{\beta}_1)$.*

Lemma 2.5. *The module of primitives of $K(2)_*SO(9)$ is free on the basis $\{\overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3, \zeta_4, \gamma_{01}\}$.*

Proof. The listed elements are independent by Proposition 2.2. Since the homolgy suspension of any element of $BP_*\Omega SO(9)$ is primitive, $\overline{\beta}_i$'s are primitive. So is $\overline{\alpha}_7$ and hence $\zeta_4 = v_2^{-1}\overline{\alpha}_7$ is also primitive. Finally γ_{01} is primitive because $\Delta(\gamma_{01}) = \gamma_{01} \otimes 1 + \overline{\beta}_0 \otimes \overline{\beta}_0 + 1 \otimes \gamma_{01}$ in $k(2)_*SO(7)$ and $\overline{\beta}_0$ is v_2-torsion in $k(2)_*SO(9)$.

Denote by A the $k(2)_*$-module generated by by $\{\overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3, \zeta_4, \gamma_{01}\}$, and by $\overline{\Delta}$ the reduced diagonal $SO(9) \to SO(9) \wedge SO(9)$.

Let x be a primitive of $K(2)_*SO(9)$. We can write $x = v_2^s y$, where y is the image of an element in $k(2)_*SO(9)$ that is not divisible by v_2 and $\overline{\Delta}_* y$ is v_2-torsion. We will show by induction on its degree that y is the sum of an element of A and a v_2-torsion element. Note that this is vacuous if the degree of y is less than two.

The image of y in $(k(2)_*SO(9)/v_2\text{-torsion}) \otimes \mathbb{Z}/2$ is primitive. So by Lemma 2.4, it is in the image of A. Hence we can write $y = y' + v_2^t z + y''$ where $y' \in A$, $t \geq 1$, $z \in k(2)_*SO(9)$ is not divisible by v_2, and y'' is v_2-torsion. Then z is primitive in $K(2)_*SO(9)$. By our induction assumption, $z \in A$. □

Lemma 2.6. *The values of the Milnor primitive Q_1 are given by the following: $Q_1(\overline{\beta}_i) = 0$ for $0 \leq i \leq 3$, $Q_1(\gamma_{01}) = 0$, $Q_1(\gamma_{02}) = \overline{\beta}_0$, $Q_1(\gamma_{11}) = \overline{\beta}_1$, and $Q_1(\zeta_4) = \overline{\beta}_0\overline{\beta}_2$.*

Proof. All but the last statement follow from the known values of Q_1 on $H_*SO(9)$ (see [7, *p*.. 427]).

Now $Q_1(v_2\zeta_4) = Q_1(\overline{\alpha}_7) = 0$ because $\overline{\alpha}_4$ comes from $BP_*SO(9)$. So $Q_1(\zeta_4)$ is v_2-torsion. In $H_*SO(9)$, $Q_1(\zeta_4) = Q_1(\overline{\beta}_0\gamma_{08}) = \overline{\beta}_0(\overline{\beta}_0\gamma_{04} + \overline{\beta}_2)$. □

Proposition 2.7. *In $k(2)_*SO(9)$ $[\gamma_{01}, \overline{\beta}_3] = v_2\overline{\beta}_1$, $[\gamma_{02}, \overline{\beta}_3] = v_2\overline{\beta}_1\gamma_{01} + v_2\overline{\beta}_2$ and $[\gamma_{11}, \overline{\beta}_3] = v_2\overline{\beta}_3$.*

Proof. This proved as in [6, Section 5]. We will indicate the proof of the first. Others are proved in a similar fashio.

First note that these commutators must be divisible by v_2 because $H_*SO(9)$ is commutative. Next, because $Q_1(\overline{\beta}_3) = 0$, $[-, \overline{\beta}_3]$ is both an algebra derivation and a coalgebra derivation. So $x = [\gamma_{01}, \overline{\beta}_3]$ is primitive in $K(2)_*SO(9)$. Degree considerations and Lemma 2.5 imply that x is either 0 or $v_2\overline{\beta}_1$.

If $x = 0$, then $\gamma_{01} \otimes \overline{\beta}_3 - \overline{\beta}_3 \otimes \gamma_{01}$ would be in the image of

$$\widetilde{k}(2)_{*-2}P \to \widetilde{H}_{*-2}P \to \widetilde{H}_*(SO(9) \wedge SO(9)) \cong \widetilde{H}_*SO(9) \otimes \widetilde{H}_*SO(9)$$

where P is the projective plane of $SO(9)$. This contradicts [6, Proposition 4.1] □

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