THE ALGEBRA STRUCTURE OF $K(l)_*SO(2^{l+1}-1)$

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Abstract. We show that if G is a compact connected Lie group that has p-torsion in homology, then G localized at p is not homotopy nilpotent. Thus, a connected Lie group is homotopy nilpotent if and only if it has no torsion in homology.

0. Introduction

If X is a homotopy associative H-space, then the functor $[\_, X]$ takes its values in the category of groups. We can ask when the values of this functor lie in various subcategories of groups. One special case is asking when $[\_, X]$ is always nilpotent. Such X are said to be homotopy nilpotent.

Now, if A is finite, $[A, X]$ is a nilpotent group; but the nilpotency class may depend on the dimension of A. If X is a finite H-space, then $[A, X]$ will be nilpotent for all A precisely when the nilpotency class of $[A, X]$ is bounded above for all finite A.

The above condition has a direct formulation in terms of the structure maps of X: Let μ and σ be the multiplication and the inverse maps of X. Define the commutator c_2 to be the composite

$$X \times X \xrightarrow{\Delta_{X\times X}} X \times X \times X \times X \xrightarrow{\text{id} \times \text{id} \times \sigma \times \sigma} X \times X \times X \times X \xrightarrow{\mu(\mu\times\mu)} X$$

and define the iterated commutators $c_n : X^n \to X$ inductively by $c_n = c_2(c_{n-1} \times \text{id}_X)$.

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Proposition 0.1. *A finite homotopy associative H-space X is homotopy nilpotent iff c_n is null homotopic for sufficiently large n.*

This is standard. See [13, Section 2.6].

For $\mathbb{S}^3$, c_n will be Whitehead products. By calculating c_4. G. J. Porter showed that $\mathbb{S}^3$ is homotopy nilpotent [5].

The first major advance was made by M. J. Hopkins [2]. He showed that a finite H-space X is homotopy nilpotent if and only if for sufficiently large n, c_n's induce trivial homomorphism in complex bordism. This is same as asking that c_n's induce trivial homomorphisms in all Morava K-theories. It follows that all homotopy associative finite H-spaces with no torsion in homology are homotopy nilpotent. Hopkins conjectured that all finite H-spaces were homotopy nilpotent.

In [9], the author showed that $Spin(n)$ is not homotopy nilpotent for $n \geq 7$. The same method applies to G_2, $SO(3)$ and $SO(4)$. N. Yagita [12] showed that the simply connected exceptional Lie groups were not homotopy nilpotent [12] using his earlier calculation of their Morava K-theories.

These results are all local. That is, if G is a simple simply connected Lie group and G has p-torsion in homology, then $G_{(p)}$, the localization of G at p, is not homotopy nilpotent. In this paper we generalize this to the following.

Theorem 0.2. *Let G be a compact connected Lie group. Let p be a prime. Then $G_{(p)}$, the localization of G at p, is homotopy nilpotent if and only if $H_*(G, \mathbb{Z}_{(p)})$ is torsion-free.*

Of course, the "if" part is due to Hopkins.

We refer the reader to [10] for a summary of facts about Morava K-theory that we need.

1. A Reduction

Lemma 1.1. *Let X be an H-space and let $\overline{X}$ be a covering space. Then there is a unique H-space structure on $\overline{X}$ such that the covering projection is a H-map. If X is homotopy nilpotent, then so is $\overline{X}$.*

Proof. The first part is standard. The second follows from the fact that for any connected A, $[A, \overline{X}] \to [A, X]$ is injective.

Fix a prime p. Let G be a compact connected Lie group such that $G_{(p)}$ homotopy nilpotent. Assume that G is not simply connected and is not a torus. The universal cover of G has the form $\widetilde{G} \times \mathbb{R}^n$ where $\widetilde{G}$ is compact. It follows from the lemma above that $\widetilde{G}$ is homotopy nilpotent. By [12] $\widetilde{G}$ has no p-torsion in homology.

The kernel of $\widetilde{G} \times \mathbb{R}^n \to G$ has the form $C \times F$, where $C < \widetilde{G}$ is finite and F is free abelian. First suppose that the order of C is prime to p. Then $\widetilde{G} \simeq_{(p)} \widetilde{G}/C$. By [3, Proposition 3.2], G is homotopy equivalent, as a space, to the product of $\widetilde{G}/C$ and a torus. It follows that G has no p-torsion in homology.

Now suppose that p divides the order of C. Then C contains an element of order p, say g. Then $\widetilde{G}/\langle g\rangle$ is equivalent to a covering group of G, and so is homotopy nilpotent. This contradicts the next lemma:

Lemma 1.2. *Let G a compact connected Lie group with $\pi_1 G = \mathbb{Z}/(p)$. Suppose that $\widetilde{G}$, the universal cover of G has no p-torsion in homology.Then $G_{(p)}$ is not homotopy nilpotent.*

The rest of this paper is devoted to proving this lemma.

2. A Calculation

Lemma 2.1. *Let X be a finite H-space such that $\pi_1 X = \mathbb{Z}/(p)$ and $\widetilde{X}$, the universal cover of X, has no p-torsion in homology. Then there is an element $\tau \in K(1)_2 X$ such that $K(1)_*X$ is a free $K(1)_*\widetilde{X}$ left module with basis $\left\{1, \tau, \ldots, \tau^{p-1}\right\}$ and $\tau^p = -v_1\tau$.*

Proof. There is a fiber sequence $\widetilde{X} \to X \to K(\mathbb{Z}/(p), 1)$ which gives $\Omega X \simeq \Omega\widetilde{X} \times \mathbb{Z}/(p)$ when looped. By our assumption, $H_*(\widetilde{X}, \mathbb{Z}_{(p)})$ is an exterior algebra on odd dimensional generators. So $H_*(\Omega\widetilde{X}, \mathbb{Z}_{(p)})$ is a polynomial algebra on even-dimensional generators. It follows that $BP_*(\Omega\widetilde{X})$ is also a polynomial algebra. Let the generators be $\beta_1, \ldots, \beta_n$. Let β_0 be a generator of any summand of $\widetilde{BP_0\Omega X}$ such that $\beta_0^p = p\beta_0$.

We will make use of the Bar spectral sequence. We refer the reader to [7] and references cited there for details of construction. We will denote the divided power algebra on an element x by $\Gamma(x)$ and the truncated divided power algebra of height s by $\Gamma_s(x)$. These are the duals of the polynomial algebra $P(y)$ and the truncated polynomial algebra $P(y)/(y^{p^s})$ on a primitive generator y. $\gamma_i(x)$ will denote the ith divided power of x.

We will do the case of odd p first. Let h be any BP-algebra theory with $h_0 = \mathbb{Z}/(p)$ and h_* concentrated in even degrees. The Bar ss $E^*_{**}(X, h)$ converging to h_*X has E^2-term

$$\mathrm{Tor}^{h_*\Omega X}_{**}(h_*, h_*) = \Gamma(\tau) \otimes E(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_n)$$

where τ has bidegree $(0, 2)$ and $\overline{\beta}_i$ is the homology suspension of β_i and has bidegree $(2t, 1)$, where $2t \geq 2$ is the dimension of β_i. For dimensional reasons, τ is a permanent cycle, as are $\overline{\beta}_i$. Furthermore, even differentials are zero for degree reasons. If E^r are free h_*-modules for $2 \leq r \leq k$, then E^r is Hopf algebra and d^r a Hopf derivation for $2 \leq r \leq k$. The identification of the E^2-term given above respects the Hopf algebra structure.

A calculation as in [11, proof of 6.9, 6.10] shows that the first non-trivial differential d^r must occur for $r = 2p^s - 1$ for some s. Then $d^r(\gamma_t(\tau)) = \gamma_{t-p^s}(\tau)d^r(\gamma_{p^s}(\tau))$; $d^r(\gamma_{p^s}(\tau))$ is primitive and so must be a linear combination of $\overline{\beta}_i$'s.

The map $\Omega X \to \Omega K(\mathbb{Z}/(p),1)$ induces a homomorphism of Bar ss in $K(1)$-theory. The Bar ss $E^*_{**}(K(\mathbb{Z}/(p),1),K(1))$ is described in [11]. The E^2-term is $\Gamma(\tau)\otimes E(\overline{\beta}_0)$. The only non-trivial differential is given by $d^{2p-1}(\gamma_p(\tau)) = cv_1\overline{\beta}_0$, with $c\neq 0$ in $\mathbb{Z}/(p)$. Hence, in $E^{2p-1}_{**}(X,P(1))$ we must have $d^{2p-1}(\tau) = cv_1\overline{\beta}_0 + \overline{\beta}$, where β is a possibly trivial $\mathbb{Z}/(p)$-linear combination of β_i's.

An easy calculation using [11, Lemma 6.9] shows that if $\beta\neq 0$, then

$$E^{2p}_{**}(X,P(1)) = \Gamma_1(\tau)\otimes E(\overline{\beta}_0,\overline{\beta}_1,\dots,\overline{\beta}_n)/(cv_1\overline{\beta}_0+\overline{\beta}=0)\,.$$

But if $\beta = 0$, then the E^{2p}-term is generated by 1 and $\overline{\beta}_0\gamma_{ps}(\tau)$ as a module over $\Gamma_1(\tau)\otimes E(\overline{\beta}_1,\dots,\overline{\beta}_n)$ subject to the relations $v_1\overline{\beta}_0\gamma_{ps}(\tau)=0$. Note that in the first case, all further differentials are trivial, while in the second further differentials cannot affect elements of total degree at most $2p$.

In both cases, $E^{2p}(X,K(1)) = \Gamma_1(\tau)\otimes E(\overline{\beta}_1,\dots,\overline{\beta}_n)$ because v_1 is invertible in $K(1)$. This proves that $K(1)_*X$ is as stated.

Note that $\tau^p = 0$ in the E^2-term. Hence, in $P(1)_*X$, τ^p is a sum of elements total degree $2p$ and filtration at most $2p-1$. But such elements must be $\mathbb{Z}/(p)$-linear combination of $v_1\tau$ and products of $\overline{\beta}_i$'s and τ. Now $H_i(X,\mathbb{Z}/(p)) \cong H_i(K(\mathbb{Z}/(p),1),\mathbb{Z}/(p))$ for $i\leq 2$. So the reduction of τ to mod-p homology is primitive. Note that $\widetilde{P(1)}_*Y \cong \widetilde{H}_i(Y,\mathbb{Z}/(p))$ for $i\leq 2p-2$ for any connected Y. It follows that the diagonal of X takes τ to $1\times\tau+\tau\times 1$. So τ is primitive in $K(1)_*X$. As $\tau^p = -v_1\tau$ in $K(1)_*K(\mathbb{Z}/(p),1)$ [11], $\tau^p+v_1\tau$ is a primitive element of $K(1)_*X$ that is a linear combination of product of $\overline{\beta}_i$'s and τ. Hence it is trivial.

We now indicate the changes in the proof for $p=2$: the E^2-term is $\Gamma(\overline{\beta}_0)\otimes E(\overline{\beta}_1,\dots,\overline{\beta}_n)$ and $\gamma_s(\tau)$ must be replaced by $\gamma_{2s}(\overline{\beta}_0)$. However, τ is not primitive and we cannot eliminate the possibility that τ^2 contains a product of $\overline{\beta}_i$'s. However, if $i\geq 1$, β_i has dimension $2t\geq 2$ and $\overline{\beta}_i$ has bidegree $(2t,1)$. Thus the only products that can occur in τ^2 are $\overline{\beta}_0\overline{\beta}_i$, with β_i of dimension 2. So we must have $\tau^2 = v_1\tau+\overline{\beta}_0\overline{\sigma}$, where $\overline{\sigma}$ is the $K(1)$-reduction of some element in $P(1)_3\widetilde{X}$.

Comparison with $H_*(K(\mathbb{Z}/(2),1),\mathbb{Z}/2)$ shows that the diagonal of τ is $1\times\tau+\overline{\beta}_0\times\overline{\beta}_0+\tau\times 1$ and that $Q_0\tau=\overline{\beta}_0$. It follows that $[\tau,\overline{\beta}_0] = Q_0(\tau^2) = v_1\overline{\beta}_0$. As in the proof of [9, Lemma 3.5], we see that $\tau^2-v_1\tau$ is primitive in $K(1)_*X$. Hence $\overline{\beta}_0\overline{\sigma}=0$.

Note that, for X as in the lemma, $K(1)_*\widetilde{X}$ injects into $K(1)_*X$. This allows us to consider $K(1)_*\widetilde{X}$ to be a subalgebra of $K(1)_*X$. Note also that $\overline{\beta}_0$ is either 0 or $-v_1^{-1}\overline{\beta}$ for some $\overline{\beta}\in P(1)_{2p-1}\widetilde{X}$, according to whether $d^{2p-1}(\gamma_p(\tau))$ is $\overline{\beta}_0$ or not.

Note that $\overline{\beta}_0$ and τ come from $P(1)_*X$. We will make the choice of τ canonical by making a choice in $P(1)_2K(\mathbb{Z}/(p),1)$, lifting it to $P(1)_2X$ and then reducing to $K(1)_2X$. If it is necessary to avoid confusion, we will use the notation τ_X. Note that if we have a H-map $f: X\to Y$ between two such H-spaces with $f_*:\pi_1X\cong\pi_1Y$, then $f_*(\tau_X)=\tau_Y$.

We will abuse notation by denoting the chosen preimages of $\overline{\beta}_0$ and τ, as well as their reduction to mod-p homology, by the same symbols

3. Proof of Lemma 1.2

Lemma 3.1. *Let $\widetilde{H}$ be a simple simply connected compact Lie group with no p-torsion in homology, h a central element of order p and $H = \widetilde{H}/\langle h\rangle$. Then τ_H is not central in $K(1)_*H$.*

Proof. The only simple simply connected Lie groups with a central element of order p and no p-torsion in homology are $Spin(m)$ with $3 \leq m \leq 6$ and $p = 2$, $Sp(m)$ with $m \geq 2$ and $p = 2$ and $SU(pm)$ with $pm \geq 3$ and p any prime: See, for example, [Mi], especially *p.* 954 and *pp.* 956–961. It follows that H is one of $SO(n)$ for $3 \leq n \leq 6$, the semi-spin group $SS(4)$, $SU(pm)/\left\langle e^{2\pi i/p}\right\rangle$ and $Sp(n)/\langle -1\rangle$. We shall show that in each case τ_H is not central in $K(1)_*H$.

For $SO(3)$ and $SO(4)$, this was done in [9]. Next we consider $SO(5)$. By the results of [7] and [8], $E^2_{**}(SO(5), P(1)) = \Gamma(\overline{\beta}_0) \otimes E(\overline{\beta}_1, \overline{\alpha}_3)$. $d^2 = 0$ and $d^3(\gamma_4(\overline{\beta}_0)) = v_1\overline{\beta}_0$. It follows that the only non-zero element in E^4_{**} of total degree 5 and filtration less than 3 is $v_1\overline{\beta}_1$. Note that τ corresponds to $\gamma_2(\overline{\beta}_0)$. As the Bar ss is commutative, $[\tau, \overline{\beta}_1]$ must have filtration less than 3. Hence, $[\tau, \overline{\beta}_1]$ is either 0 or $v_1\overline{\beta}_1$.

Let P be the projective plane of $SO(5)$. For any multiplicative homology theory h, there is a long exact sequence

$$\ldots \widetilde{h}_{*+2}P \to \widetilde{h}_*(SO(5) \wedge SO(5)) \xrightarrow{\mu_*} \widetilde{h}_*SO(5) \to \widetilde{h}_{*+1}P \ldots$$

where μ_* is induced by multiplication. Thus, if $[\tau, \overline{\beta}_1]$ were 0, $\tau \wedge \overline{\beta}_1 - \overline{\beta}_1 \wedge \tau$ would be in the image of $\widetilde{P(1)}_7P$. But this contradicts [10, Proposition 4.1]. Hence, $[\tau, \overline{\beta}_1] = v_1\overline{\beta}_1$. As $\overline{\beta}_1$ comes from $K(1)_*Spin(5)$, it is non-zero in $K(1)_*SO(5)$ by Lemma 2.1. So τ is not central in $K(1)_*SO(5)$.

As remarked in [8], $K(1)_*SO(5)$ injects into $K(1)_*SO(6)$. This proves our claim for $SO(6)$.

Next we consider $SS(4)$. The universal cover is $Spin(4)$ whose homology is exterior on two generators of dimension 3. Consider the Bar ss in $\mathbb{Z}/(2)$-homology. This is Hopf algebra spectral sequence. Its E^2-term is $\Gamma(\overline{\beta}_0) \otimes E(\overline{\alpha}, \overline{\beta})$ where $\overline{\alpha}$ and $\overline{\beta}$ have bidegree $(2, 1)$. As $SS(4)$ is finite, there must be non-trivial differentials. As in the proof of Lemma 2.1, the first non-trivial differential d^r occurs when $r = 2(2^s) - 1$ and then $d^r(\gamma_{r+1}(\overline{\beta}_0))$ is primitive. The only possibility is $r = 3$ with $d^r(\gamma_4(\overline{\beta}_0)) = a\overline{\alpha} + b\overline{\beta} \neq 0$.

As in the proof of Lemma 2.1, it follows that $d^3(\gamma_4(\overline{\beta}_0)) = v_1\overline{\beta}_0 + a\overline{\alpha} + b\overline{\beta}$ in $E^*_{**}(SS(4), P(1))$. Hence, $\overline{\beta}_0 = -v_1^{-1}(a\overline{\alpha} + b\overline{\beta}) \neq 0$ in $K(1)_*SS(4)$. On the other hand, applying Q_0 to $\tau^2 = -v_1\tau$ gives $[\tau, \overline{\beta}_0] = -v_1\overline{\beta}_0$ in $K(1)_*SS(4)$.

To deal with the case of $Sp(n)/\langle -1\rangle$, we need to recall the details of $SO(3)$ calculation [7]: $E^2_{**}(SO(3), P(1)) = \Gamma(\overline{\beta}_0)\otimes E(\overline{\sigma})$, where $\overline{\sigma}$ is $\overline{\alpha}'_1$ in the notation of [7]. Of course, $\overline{\sigma}$ is a generator of $K(1)_*S^3$. The only non-trivial differential is given by $d^3(\gamma_{4k}(\overline{\beta}_0)) = (\overline{\sigma} + v_1\overline{\beta}_0)\gamma_{4k-4}(\overline{\beta}_0)$. τ corresponds to $\gamma_2(\overline{\beta}_0)$. In $P(1)_*SO(3)$, $[\tau, \overline{\beta}_0] = v_1\overline{\beta}_0$. As $\overline{\sigma} = -v_1\overline{\beta}_0$, $[\tau, \overline{\sigma}] = v_1\overline{\sigma}$.

Let $K = Sp(1)^n/\langle -1\rangle$. $K(1)_*Sp(1)^n$ is the exterior algebra on $\overline{\sigma}_1, \ldots, \overline{\sigma}_n$, where $\overline{\sigma}_i$ is the image of $\overline{\sigma} \in K(1)_*Sp(1)$ under inclusion as the i-th factor. We will calculate $[\tau_K, \overline{\sigma}_1]$. It has bar filtration at most 2; so it is a linear combination of $\overline{\sigma}_i$'s.

The projection of $Sp(1)^n$ onto the i-th factor induces a homomorphism pr_i : $K \to Sp(1)/\langle -1\rangle = SO(3)$. If $i \neq 1$, then $(pr_i)_*[\tau_K, \overline{\sigma}_1] = [(pr_i)_*\tau_K, (pr_i)_*\overline{\sigma}_1] = [\tau_{SO(3)}, 0] = 0$. On the other hand, $(pr_1)_*[\tau_K, \overline{\sigma}_1] = [\tau_{SO(3)}, \overline{\sigma}] = v_1\overline{\sigma}$. It follows that $[\tau_K, \overline{\sigma}_1] = v_1\overline{\sigma}_1$.

The inclusion $Sp(1)^n \subset Sp(n)$ induces $K \to Sp(n)/\langle -1\rangle$. The former takes $\overline{\sigma}_1$ to the three dimensional generator $\overline{\sigma}$ of $K(1)_*Sp(n)$. Hence, $[\tau_H, \overline{\sigma}] = v_1\overline{\sigma}$.

The last case is that of $H = SU(n)/\langle e^{2\pi i/p}\rangle$, with $p \mid n$ and $n \geq 3$. The homology of the universal cover is exterior on $\overline{\beta}_i$, $1 \leq i < n$. Dualizing [1, Main Theorem I(ii)], we see that in mod-p homology $[\tau, \overline{\beta}_1] = \overline{\beta}_2$. The usual Bar ss argument shows that, in $P(1)_*H$, $[\tau, \overline{\beta}_1]$ is $\overline{\beta}_2$ if $p \geq 5$, $\overline{\beta}_2 + sv_1\overline{\beta}_0$ if $p = 3$ and $\overline{\beta}_2 + sv_1\overline{\beta}_1 + tv_1^2\overline{\beta}_0$ if $p = 2$, where $s, t \in \mathbb{Z}/(p)$. As in the proof of Lemma 2.1, $\overline{\beta}_0 = 0$ or $-av_1^{-1}\overline{\beta}_{p-1}$. It follows that if $p \neq 3$, then $[\tau, \overline{\beta}_1] \neq 0$ in $K(1)_*H$. If $p = 3$ and $[\tau, \overline{\beta}_1] = 0$, we must have $\overline{\beta}_0 = -sv_1^{-1}\overline{\beta}_2 \neq 0$ in $K(1)_*H$. But then, applying Q_0 to $\tau^3 = -v_1\tau$, we see that $[\tau, [\tau, \overline{\beta}_0]] = -v_1\overline{\beta}_0$. Thus, in every case, τ_H is not central in $K(1)_*H$.

Proof of Lemma 1.2. Let g be a generator of the kernel of $\widetilde{G} \to G$. Let $\widetilde{H}$ be any simple factor of $\widetilde{G}$ such that the projection of g to $\widetilde{H}$ is non-trivial and put $H = \widetilde{H}/\langle h\rangle$. Obviously, $\widetilde{H}$ has no p-torsion in homology. Furthermore, $K(1)_*G \to K(1)_*H$ is onto and takes τ_G to τ_H. Lemma 3.1 implies that τ_G is not central in $K(1)_*G$.

In characteristic p, the p-fold commutator $[\tau, [\ldots, [\tau, x]\ldots]$ is $[\tau^p, x]$. Hence $[\tau, [\ldots, \overline{\beta}]\ldots] = -[v_l\tau, \overline{\beta}]$. Standard Hopf-algebra calculation shows that

$$(c_n)_*(\tau \times \cdots \times \tau \times \overline{\beta}_i) = [\tau, [\tau \ldots [\tau, \overline{\beta}_i]\ldots]$$

(for $p = 2$, see [10, Lemmas 5.6, 5.7.]). Combining these with the fact that τ is not central, we see that c_n is detected by $K(1)$ for all n.

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