Why technology is indispensable today in the teaching and learning of mathematics ?

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Everywhere there is a debate…

about the use of technology in mathematics teaching.
At ICME 9, the Working Group 11 devoted to technology gathered the greatest number of participants . At the same congress, In his plenary lecture, Pr Fujita stressed that the 4th peak in history of mathematics is the introduction of computers and that mathematics education should take it into account especially in geometry.
For example, in France, there is an emphasis made on the use of calculators and computers at all levels of schooling in the new curricula. The Ministry of Education supports the use of technology in teaching. But there are voices from some teachers and educators, claiming that when using technology, students have just to press keys or buttons and do not longer really think. They state that mathematics is the science requiring only the use of the brain (and of a pencil).

We would like to support here a different view. We consider that mankind has always made use of various technologies. Paper and pencil is also a technology but because it has been prevailing for centuries, it is no longer perceived as such. According to various researchers (Noss & Hoyles 1996, Jones 1999), our second claim is that technology shapes the way

- of doing mathematics
- of learning, i.e. of constructing new knowledge

A very illuminating example is a counting instrument used in this country (Japan), namely the Soroban: several investigations gave evidence of other strategies developed by Japanese children just because of the existence of 5 as reified by an item on the Soroban; Japanese children are able to introduce 5 units (5, 50, 500, …) into their mental calculation.

My purpose is to give some arguments to this debate which come

- from lessons from history of teaching
- from large international data
- from fine grain analysis of students' work with technology.

It is not a new debate

The same kind of debate took place in the seventies about the four operations calculator which would favor the students lazyness as well as in the XIXth century in France when the metal quill replaced the goose quill. Some inspectors and teachers were claiming that introducing the metal quill into school would definitely damage the writing ability of children. It turned out that

- In the 19th century children could eventually start **earlier** writing and doing written computations.
- Introducing the calculators caused a renewal of **mental computation** and of the notions of **approximation**, **rounding**, and **number sense,** which became explicit parts of the curricula in the seventies.

From TIMSS data,

it appears that the achievement of students is positively related both

- to the frequent use of a calculator (weekly or daily use),
- to the use of a calculator during TIMSS tests.

> *"In general the students who reported most calculator use (during the TIMSS tests) were also those who performed best on the tests. It is not clear however whether calculator use assisted performance or whether the more able students were also those who chose to use a calculator most."* (TIMSS report)

From this conclusion, it can be retained that either the calculator helped students achieving better than without or that the students using technology are among the best. In both cases, it is an interesting situation. If technology allows to do better, why to avoid students to use it? If the best students choose to use technology, there must be some reason. The best students know how to choose their tools, or did they become the best because they used regularly technology ?

In the following section, we will show through the example of students' observations that there is evidence of interrelations between the resources used by the students and their knowledge of mathematics:

- Students use the type of resource corresponding to their mathematical view of the problem
- Questions arise from what they obtain by means of technology which may lead to theoretical progress
- Using a new technology may require another use of knowledge and so contribute to learning

A case study: discrete dynamic linear systems in pre-service teacher education.

(This section is also part of a regular lecture given at ICME 9 August 2000 entitled New Technologies as a Bridge between Various Parts of Mathematics for Pre-service Students Teachers)

This example was experimented twice (in years 1997 and 1998) in a course with pre-service teacher education students (fourth university year) having at their disposal a calculator TI 92. The course was aimed at broadening mathematical knowledge of students by giving them the opportunity of using mathematics as a tool for solving problems and of bridging various parts of mathematics. This is why students had at their disposal various tools and technologies. The calculator TI 92 gave them the opportunity to have available in the same device various tools, a CAS (a computer algebra system), a spreadsheet, a plotter, a programming language, a data matrix editor and a geometry environment (a Cabri version adapted to the TI 92). This enhanced the possibility for the students of moving from one model to another. It has to be noted that students were not very familiar with the TI 92. It was the second time they used it.

The problem

In a population two categories of individuals are distinguished : the "Youngs" and the "Olds". The state Pn of the population at moment n is represented by the couple (Y_n, O_n). The population is observed at a regular pace. The period of time between two observations is half of the average life length. At the end of such a period all Youngs become old, all Olds died. The law of evolution of the population is given by

$$Y_{n+1} = a\,Y_n + b\,O_n$$
$$O_{n+1} = c\,Y_n$$

Students were asked to simulate the beginning of the evolution of the population for the following values of a, b and c.

a = 0,5, b= 0,4, c = 0, 9
a = 0,8, b= 0,5, c = 0,6
a = 0,8, b= 0,5, c = 0,4

Then they were asked to predict the asymptotic behaviour of the population and to justify their prediction.

Observation of the work of the students

The work of the students in the 1998 session is described below. It can be divided into several successive phases.

Phase 1: calculation of some first states of the population

Students used the given values of a, b and c. They chose values for Y_0 and O_0. It is interesting to note that they wanted to use realistic values like 1000 or 10000. They often decided for $Y_0 = O_0$.

Some students performed the calculations quasi by hand, i e by only using TI 92 as an ordinary simple calculator and wanted to plot the corresponding points Pn in a coordinate system with the intention to discover a regularity in the sequence of points. This turned out to be long. They considered the state of the population as a point in a plane and had a geometrical view of the problem.

Some other students more familiar with programming recognised that Y_n and O_n are two iterative sequences and programmed them. They could get different values for different initial choices of the population. They could easily and rapidly guess the destiny of populations according to the different values of (a, b, c) : extinction, stability, explosion.

But at this stage a question was raised by several students: to what extent does the evolution of the population depend on the initial conditions ? It is interesting to note that students generally thought that the absolute values of Y_0 and of O_0. did not matter but that the proportion Y_0/O_0. could affect the evolution of the population.

Some students (but very few) decided to use a specific functionality of the TI 92: the sequence environment. They tabulated the ten or twenty first values of the double sequence. They asked for the manual of the calculator and found the way to manage it. This led them to the same type of conclusion as the former students.

Phase 2: elicitation of the linear application of R^2 onto R^2

The emergence of question of the influence of the initial state led students to stop calculating and to consider the application transforming (Y_n , O_n) into (Y_{n+1}, O_{n+1}). Some of them recognised a linear application (denoted here by T) and that the problem is to find the behaviour of T^n for the big values of n. It took a while for them to formulate that a good method for finding T is to look for its eigenvalues. The "real" context of the problem prevented the students to recognise a classical mathematical task they learned to perform during the previous years.

Phase 3: using the data matrix editor of the TI 92

When the linear application was recognised and its matrix identified, some students asked whether the TI 92 enables to calculate directly any power of a matrix. After the teacher introduced them to the data matrix editor, they rapidly obtained T^{10} or T^{100} for the given values (a,b,c). They could conclude that the calculator gave experimental evidence of the extinction, explosion or stability independently of the initial state of the population.

Example: A case of extinction with a = 0,5 ; b = 0,4 ; c = 0,9

Matrix of T^{10}		Matrix of T^{100}		Matrix of T^{500}	
.24 125	.107 253	.000018	.000008	$9.1528\cdot10^{-24}$	$4.06791\cdot10^{-24}$
.24132	.107358	.000018	.000008	$9.1528\cdot10^{-24}$	$4.06791\cdot10^{-24}$

Example: A case of explosion with a = 0,8 ; b = 0,5; c = 0,6

Matrix of T^{10}		Matrix of T^{100}		Matrix of T^{500}	
1.68822	.782055	1484.45	688.372	$1,80\,557\cdot10^{16}$	$8,3728\cdot10^{15}$
.939438	.43564	826.46	383.055	$1,00474\cdot10^{16}$	$4,65918\cdot10^{15}$

In the previous phase a theoretical question emerged from experimenting in the computer environment. In this phase, the reverse phenomenon took place. From a theoretical progress in the solving process emerges the need for specific facilities of the calculator environment.

Phase 4: a theoretical solution

It was the appropriate time to ask students to explain theoretically the possible behaviours of the population and to express algebraically conditions related to each of the possible behaviours. Students were able to calculate the eigenvalues λ_1 and λ_2 as solutions of the characteristic equation :

$$\lambda^2 - a\,\lambda - bc = 0$$

They called λ_1 the positive solution and λ_2 the negative one and noticed that $|\lambda_1| > |\lambda_2|$. They were able to express the expression of state n of the population in function of state 0 in the system of eigenvectors :

$$Y'_n = (\lambda_1)^n\, Y'_0$$
$$O'_n = (\lambda_2)^n\, O'_0.$$

They could deduce that the behaviour of the population depends on the position of the eigenvalues with respect to 1. But they were unable to find the way to express from the equation an algebraic condition on a, b and c equivalent to $\lambda_1 > 1$. The teacher had to explain to them that the problem was equivalent to the problem of finding the sign of the polynomial $\lambda^2 - a\lambda - bc$ for $\lambda = 1$. Actually they did not recognise the classical problem of positioning a number with respect to the roots of a second degree equation because it was not exactly worded in this way. They could thus find out that $\lambda_1 > 1$ is equivalent to $a + bc > 1$.

A student was not convinced that if $\lambda_1 > 1$, the population would explode; he stated that the absolute value of λ_2 could also be smaller than 1 and then O'_n could tend to 0 as n tends to infinity. This question was left open since no student was able to refute the statement or to prove the statement. It was not possible to vary continuously a, b and c in the data matrix editor in order to empirically analyse the behaviour of the power n of the matrix for big values of n. This gave a good motivation for the teacher to propose to design a geometrical model which allows a continuous variation of a, b and c with the geometry application of the TI 92.

Phase 5: construction of a geometrical model

The vector (Y_n, O_n) defining state n of the population is represented as a vector with origin O and coordinates in the default system of the geometry environment. The linear transformation T is given by two vectors v1 and v2 with origin O which are the images through T of the unit vectors of the default system. The coordinates of vectors v1 and v2 are (a,c) and (b,0) which can be displayed. It is possible to drag the vertices of v1 and v2 and so to vary continuously a, b and c. (Fig 1 & 2).

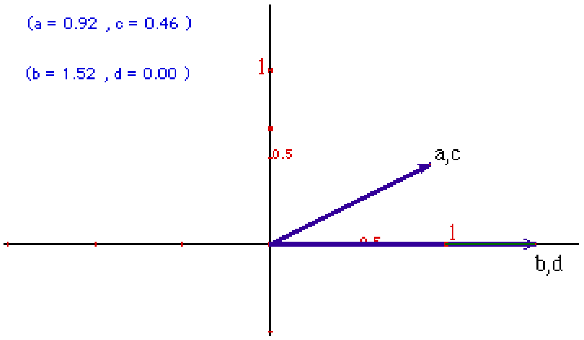


Figure 1

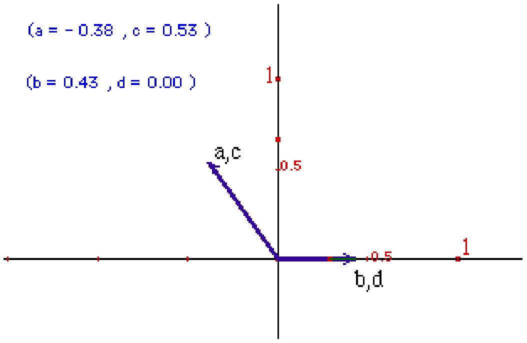


Figure 2

In a first step, students had to construct geometrically the image T(v) of any vector v with coordinates (x,y). It means that they had to translate the relation $T(v) = x \cdot v1 + y \cdot v2$, x and y being real numbers, into geometrical terms. This can be done using parallel lines preserving proportional lengths or by using the measurement transfer possibility of Cabri. This geometrical translation turned out not to be easy for them (Fig 3).

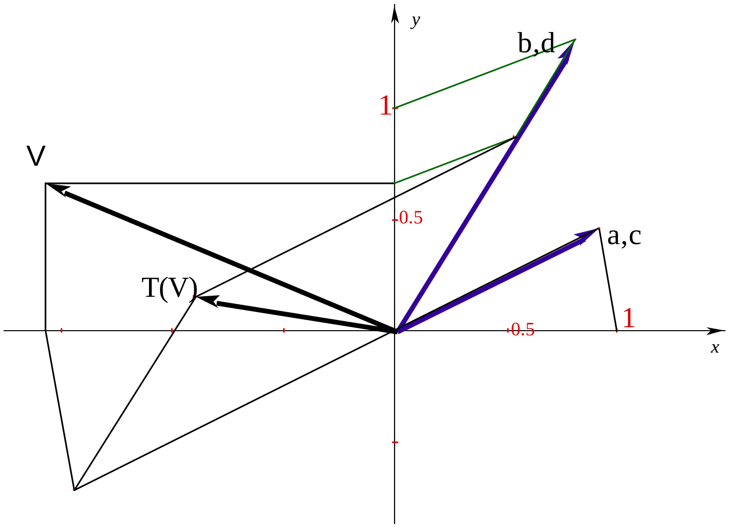


Figure 3

They saved the construction of T(v) from v as a macro-construction which appeared so in the menus as a tool called T.

When they managed to construct the image T(v) for any v, they were asked to find the eigenvectors of T by dragging v. Again it was not clear for several students how to recognise geometrically an eigenvector. They were not able to translate geometrically the condition T(v) = k·v, k element of R.

An episode on a conceptual difficulty attached to the notation $T^n(v)$

In a second step, they were asked to construct several iterated images $T^2(v)$, $T^3(v)$, $T^4(v)$, etc. by using T. It has to be noted that some students were not able to obtain $T^2(v)$ because they operated T a second time on v and not on T(v). It is interesting to note that the geometrical operation revealed that they did not understand $T^2(v)$ as T operating on T(v) but that they were misled by the algebraic notation and understood T^2 as T operating twice on v.

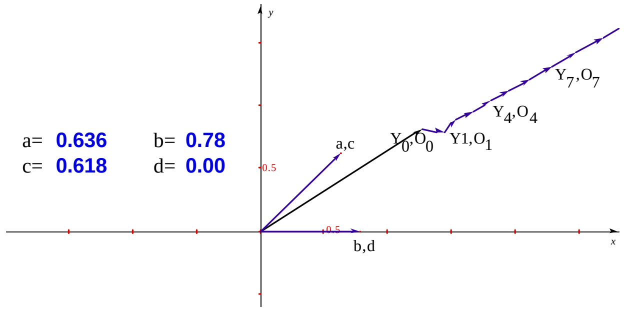
The calculator let them guess that something was wrong because it was for them as if it did nothing when applying T the second time on v; T(v) already existing, a second image T(v) was constructed by the environment coinciding with the first T(v). They called the teacher saying that the calculator did not work properly, it produced nothing. The teacher was able to show by pointing with the pointer that several objects were coinciding (ambiguity message “Which object?” displayed by the environment).

This episode is of interest for two reasons. Firstly, the calculator acted as a window on the conceptual difficulties of the students which could have remained invisible. Secondly, feedback given by the computer cannot always be interpreted by the user. Decoding feedback requires knowledge. In this case, it is because students explained to the teacher that they operated T twice on v that the teacher understood that they obtained superimposed images and could show them evidence of this. Students being sure that they should obtain another image T^2 (v) could not guess that the calculator constructed the same image. It is interesting also to note the inner contradiction of students. T applied a second time on the same v should produce something different. Students acted as if the calculator knew that it was a second time.

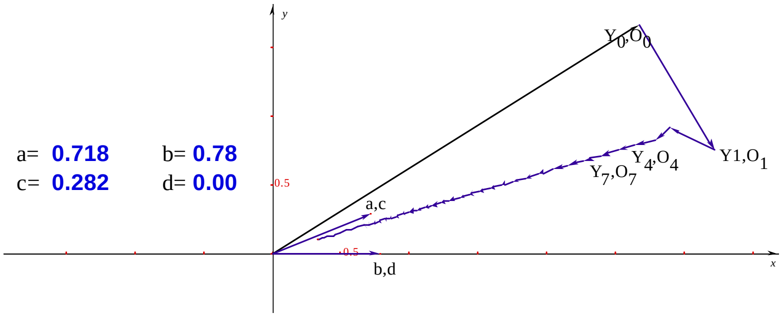
Phase 6: visualisation of the behaviours

Once several iterated images have been constructed, students could drag v1 and v2 and could confirm that they obtain explosion even with a continuous variation of a, b and c in case of a + bc > 1. What is more interesting is that they discovered two phenomena they did not anticipate.

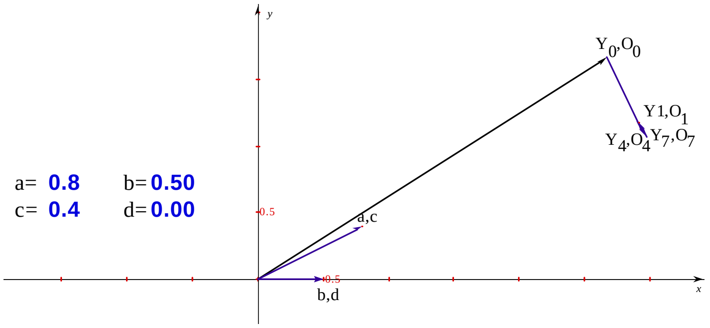
- The great sensitivity of the type of evolution to the variation of a, b and c.
- It seemed that very rapidly (for all n greater than a small number, n > 2 or n > 3) the iterated images are almost collinear, in both cases explosion or extinction (Fig 11). When a + bc = 1, the images also very rapidly converge to the same position, giving a strong evidence of the stability of the population (Fig 4).



Asymptotic collinearity in case of explosion



Asymptotic collinearity in case of extinction



Stability of the population

Figure 4

The collinearity seemed to stay even when varying v1 and v2, i e a, b and c. This raised a new question : why do all images T^n (v) (for n greater than a small number) tend to be collinear ?

Phase 7: back to theory

This observation compelled to study algebraically the asymptotic behaviour of T^n (v). Students did it by determining a system of eigenvectors and found after formal calculation that when n tends to infinity, the direction of vector T^n (v) tends to be the direction of the eigenvector attached to eigenvalue λ_1. Simultaneously the question which was left open was solved. If $\lambda_1 > 1$, Y_n and O_n tend both to infinity for all λ_2.

Phase 8: back to the numerical model

The teacher raised a new question resulting from this theoretical study. The linear asymptotic behaviour of the sequence T^n (v) was not directly visible on the numerical model. Is it possible to check it on numerical values of the matrix of T^n ? and how ? It took a while for the students to reply by proposing to check the proportionality of the rows of matrix T^n for n not too small.

Discussion

It has first to be noted that students spontaneously started working at a numerical level before moving to the theoretical level. They could construct a meaning to the problem only after having tried numerical values. It is after being faced with numerical calculations that they had the question of dependency of the initial values which led them to consider the theoretical problem. So it was important that students had appropriate tools for calculating rapidly and not wasting all their time on calculations.

So the use of computer environments motivated the search for theoretical solutions. In the development of the problem solving process, the recourse to theory occurred in two kinds of circumstances: either after empirical trials or experiments with the intention to justify observed phenomena or after questions stemming from the observations were raised. Theory provides the means for explaining why phenomena empirically observed through experimentation occur, or to overcome some doubt or open question coming from observation.

The disposal of a powerful calculator with several environments allowed to tackle not only a complex problem with long calculations but also a general problem. This is certainly one of the strengths of the computing environments to allow an empirical study of a problem depending on parameters, since it allows a variation of the parameters.

The recourse to several environments fostering the construction of different types of models was certainly a catalyst for this back and forth move between experimentation and theory. Numerical phenomena called to question the problem from a theoretical point of view. One question remained unsolved (case $\lambda_1 > 1$, explosion and/or extinction?). Because of the absence of possible continuous variation of parameters in a numerical model, a geometrical model might appear as useful. It turned out that this model revealed new visual phenomena calling again for a theoretical explanation.

Environments provided by a calculator like the TI 92 offer complementary aspects: numerical models gave an idea of the size and of the speed of the evolution of the population for big values of n but could not easily give account for properties like collinearity. Geometrical models on the other hand do not easily provide T^n (v) for big values of n but allow a continuous variation of parameters, and offer global visualisation of phenomena that the discrete variation in numerical models did not. The sensitivity of the evolution to a small change in the value of the parameters close to the equilibrium situation where a + bc = 1 is easily visible on the geometrical model. It can be checked in a second step on the numerical model in order to get a numerical estimation of the rate of change but a variation of 10^{-2} of the value of a would not have been tried immediately on the numerical model. Implicitly it was assumed that the implied change of the state of the population would be also very small.

The move between several models was not easy for students. Difficulties emerged. The episode of T^n showed that expressing an algebraic operation in another medium may require an understanding that was not constructed by the students (even of fourth university year). Students used the algebraic notation without having to decompose T^n (v) as T acting on T^{n-1} (v). The software required to decompose T^n in a sequence of successive applications of T (that students did) and to consider successively each image as the argument of the next application of T (what they did not do). In a way the *semiotic mediation* required by the computer offered opportunity for learning about composition of functions.

The move between various settings may be a source of construction of meaning. Hillel and Sierpinska followed by several authors (quoted by Dorier 1998, p 209) mentioned the crucial role of flexibility in the case of the learning of linear algebra: flexibility between modes of reasoning and languages including geometrical representations (Dreyfus & Eisenberg 1996, Tall 1996, p 297, Waits & Demana 1993). Students have a deeper mathematical view due to this interplay between different settings (Douady 1986). The joint choice of the population problem and of the multiple software calculator offered good conditions for fostering the use of both modes synthetic-geometrical and analytic-arithmetical, whose combination is essential in linear algebra as claimed by Sierpinska et al.

In the whole session the role of the teacher was critical. He had to prompt students to construct a geometrical model, and generally he played an important role in organising the move between various models or settings (for example for prompting phases 5 and 8). This move is not at all spontaneous for students, and one outcome of the session is certainly that students had to look at the same properties from different points of view and with different means of expression: two recurrent sequences and linear transformation of a vector space, linear combination of vectors and geometrical projection, points on a line and proportionality of lines of a matrix.

The teacher also gave some technical indications on how to use some applications of the TI 92, to help interpret feedback given by the environment. Using computer environments requires two types of knowledge: a technical knowledge but also a good amount of mathematical knowledge when using the environment and interpreting feedback. We consider that a joint learning (of the instrument and of mathematics) may take place if both experimentation in the environment and interpretation of phenomena offered by the environment are possible for students either on their own or thanks to teacher interventions. We could say, using a Vygotskian metaphor, that the combination of the task and the environment must belong to a zone of proximal experimentation and interpretation for the student.

The role of new technologies in the learning of cognitive flexibility

These interrelations between theoretical knowledge and use of the tool become more and more important since technologies currently embed more and more knowledge. They are cognitive technologies. The development of friendly interfaces allows to get rid of syntax problems external to mathematics in the communication with the machine. Knowledge of the instrument becomes more interrelated with mathematical knowledge.

Potentialities of cognitive technologies

Pea (1985) claimed that cognitive tools are not only amplifiers but that they are also reorganisers of cognitive systems. Dörfler (1993) identified several ways in which the introduction of a tool may contribute to a reorganisation; among them he mentioned the reframing of constructs through changing the forms of representation employed and the system of operations admitted. We propose to extend this claim by stating that the co-ordinate use of different tools contributes to the construction of relations between independent knowledge items and may build a basis for structuring them into a coherent organisation.

We advocate for the important role of dynamic geometry software in this interplay for several reasons which are illustrated by the above example:

- Software offer a global visualisation of phenomena which enrich the mental imagery of students, some of these phenomena are more likely to be noticed by students than algebraic or numerical phenomena which require a more analytical apprehension. They may then raise the need for an explanation or a justification students can elaborate in calculus or algebra. In a word they provide imagery for algebra and calculus and they are a source of questions to algebra and calculus;
- Usually students are reluctant to use geometry at university level since for them very often geometry is identified to the taught geometry of secondary school without connections with other parts of mathematics; dynamic geometry environments through their powerful graphic and direct manipulation possibilities are tools which can be manipulated easily (they do not need the use of a specific programming language and do not require a long introduction). The disposal of such environments facilitates the recourse to geometrical representations students would not do in a paper and pencil environment.
- They allow a continuous variation of parameters and contributes hence to the study of general problems and not only of specific situations. This feature seems to be very important for advanced mathematics: students must learn to cope with a general problem by playing with data and considering particular or limit cases but conversely they must also be able to consider a specific situation as a case of a more general one. The drag mode is a powerful instrument of reification of this generalisation process.

Teachers teaching with technology in the 21st century

There are many reasons for integrating technology into teaching

- a societal reason: in a world of increasing technology in which young people use Internet and mobile phone, play at game stations, the teaching of mathematics cannot ignore new technology; it would be an additional reason for our students to believe that mathematics is something old fashioned, definitely outside the real life;

- there are technologies useful for mathematics and for teaching mathematics which allow students to visualise mathematical phenomena, to make connections , to perform experimentation, in a word to really do mathematics as experts do. This ability before the era of technology was restricted to gifted students who were able to imagine in their head the mathematical objects and relations, to play with them in thought. The possibility of real manipulation allowed by technology offers an access to mathematics to more students.

It would be dangerous to let believe that the teacher has a restricted role when using technology. Designing situations integrating technology which favours a deeper learning have to be set up and it takes time to find the most appropriate. Teachers must change the tasks they gave. The teacher should intervene at critical moments in the work of the students, for example as in the above example to favour the interplay between settings, he must ask questions prompting a deep interpretation of observed phenomena and generalisation.

This is why we need to know more about strategies of students in this type of situations by collecting data from empirical studies and analysing the type of complexity they are faced with. This is also why we need teacher education devoted to integrating technology into teaching. The teachers of today and tomorrow must be able to cope with situations in which technology deepens knowledge and leads students to ask challenging mathematical questions.

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